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CONNECTION BETWEEN ASYMPTOTIC PROPERTIES AND ZEROS OF SOLUTIONS OF $y'' = q(t)\,y$

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1.1. Consider a differential equation

$$y'' = q(t)\,y, \qquad q \in C^{\circ}[a, b), \qquad b \leqq \infty, \tag{q}$$

where $C^n[a, b)$ (n being a non-negative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on $[a, b)$. Let y be a non-trivial solution of (q) vanishing at $t \in [a, b)$. If $\varphi(t)$ is the first zero of y lying on the right of t, then φ is called the basic central dispersion of the 1st kind (briefly, dispersion).

The properties of dispersions can be found in [1]. If (q) is an oscillatory ($t \to b_-$) differential equation on $[a, b)$ (i.e. every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$), then the dispersion has these properties:

$$\begin{aligned} &1.\ \varphi(t) \in C^3[a, b)\\ &2.\ \varphi'(t) > 0 \text{ on } [a, b)\\ &3.\ \varphi(t) > t \text{ on } [a, b)\\ &4.\ \lim_{t \to b_-} \varphi(t) = b. \end{aligned} \tag{1}$$

Let φ_n be the n-th iterate of the dispersion φ; then φ_n has the same properties (1) and

$$y^2(\varphi_n(t)) = \varphi_n'(t)\, y^2(t), \qquad t \in [a, b), \tag{2}$$

(see [1], § 13).

A solution y of (q) belongs to $L^p[a, b)$ if

$$\int_a^b |\, y(t)\,|^p \,\mathrm{d}t < \infty, \qquad p > 0.$$

1.2. We shall need another property of dispersions:

Let φ be the dispersion of an oscillatory ($t \to b_-$) differential equation (q), $q \in C^{\circ}[a, b)$. Then for $t_0 \in [a, b)$, $t \geqq t_0$ there exist numbers n, x such that

$$t = \varphi_n(x), \qquad x \in [t_0, \varphi(t_0)). \tag{3}$$

1.3. Results being derived in [2], [3] give us a certain review about the relation among the dispersion of (q) and the behaviour of solutions of (q) on $[a, b)$. Some results from [2], [3], [4] are summed up in the following Theorem (see [2], Theorem 4, [3], Theorem 3, [5] p. 6).

Theorem 1. 1. *Let* (q), $q \in C^{\circ}[a, b)$, $b \leqq \infty$ *be an oscillatory* $(t \to b_-)$ *differential equation and let* φ *be its dispersion. Let* $t_0 \in [a, b)$.

A. *If* (*i*) $\varphi'(t) \leqq \text{const} < 1$ *on* $[t_0, b)$,

or (*ii*) $\varphi'(t) \leqq 1$ *on* $[t_0, b)$,

or (*iii*) $\varphi'(t) \geqq \text{const} > 1$ *on* $[t_0, b)$,

then (*i*) $b < \infty$ *and every solution of* (q) *tends to zero for* $t \to b_-$,

or (*ii*) *every solution of* (q) *is bounded on* $[t_0, b)$,

or (*iii*) $b = \infty$ *and every non-trivial solution of* (q) *is unbounded on* $[t_0, b)$, *respectively.*

B. *Every solution of* (q) *is bounded on* $[t_0, b)$ *if, and only if a constant* N *exists such that*

(4) $$\varphi'_n(x) \leqq N$$

for $x \in [t_0, \varphi(t_0))$ *and all integers* n.

C. *Every solution of* (q) *belongs to* $L^p[t_0, b)$, $p \geqq 1$ *if, and only if*

(5) $$\sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi'_n(t)]^{1+p/2} \, dt < \infty.$$

2. *If oscillatory* $(t \to b_-)$ *differential equations* (q), $(\bar{q})$, q, $\bar{q} \in C^{\circ}[a, b)$ *have the same dispersion, then the statement "Every solution is bounded for* $t \to b_-$" *holds either for both* (q) *and* $(\bar{q})$ *or for neither of them.*

Remark. The necessity of (5) was proved by the author of [3] in his Seminar in Matematický ústav ČSAV Brno. See the remark in [3], too.

2.1. Let (q) be an oscillatory $(t \to b_-)$ differential equation. Let φ_n be the n-th iterate of its dispersion φ and let y be a non-trivial solution of (q). Let $t_0 \in [a, b)$ and $t \geqq t_0$. According to (3) numbers n, x exist such that

$$t = \varphi_n(x), \qquad x \in [t_0, \varphi(t_0)).$$

Thus it follows from (2) that

(6) $$y^2(t) = \varphi'_n(x) \cdot y^2(x).$$

As y^2 is a continuous function on $[t_0, \varphi(t_0)]$ there exist a constant $M > 0$ and a number x_0 such that

$$0 \leqq y^2(x) \leqq M, \qquad y^2(x_0) = M, \qquad x_0 \in [t_0, \varphi(t_0)].$$

Finally according to (6) we have

(7) $$\begin{aligned} 0 &\leqq y^2(t) \leqq M\varphi'_n(x) \\ y^2(\varphi_n(x_0)) &= M\varphi'_n(x_0). \end{aligned}$$

We shall utilize these facts for the proof of the following

Theorem 2. *Let* φ *be the dispersion of an oscillatory* $(t \to b_-)$ *differential equation* (q), $q \in C^{\circ}[a, b)$ *and* $t_0 \in [a, b)$. *Then*

a) *Every solution of* (q) *tends to zero for* $t \to b_-$ *if, and only if*

$$\lim_{n\to\infty} \varphi_n'(x) = 0 \tag{8}$$

uniformly for $x \in [t_0, \varphi(t_0))$.

b) *If the sequence* $\{\varphi_n'(x)\}$ *is unbounded at least for two different values* $x_1, x_2 \in [t_0, \varphi(t_0))$ *then every non-trivial solution of* (q) *is unbounded.*

c) *If the sequence* $\{\varphi_n'(x)\}$ *does not tend to zero for* $n \to \infty$ *at least for two different values* $x_1, x_2 \in [t_0, \varphi(t_0))$ *then no non-trivial solution of* (q) *tends to zero for* $t \to b_-$.

Proof. a) Let every solution of (q) tends to zero for $t \to b_-$. Let $t_1, t_2 \in (t_0, \varphi(t_0))$, $t_1 < t_2$ and let y_1, y_2 be linearly independent solutions of (q) such that $y_1(t_1) = 0$, $y_2(t_2) = 0$. Then

$$y_1^2(x) \neq 0 \quad \text{for} \quad x \in \left[\frac{t_1 + t_2}{2}, \varphi(t_0)\right]$$

$$y_2^2(x) \neq 0 \quad \text{for} \quad x \in \left[t_0, \frac{t_1 + t_2}{2}\right].$$

We have also

$$\begin{aligned} &\min_{x\in\left[\frac{t_1+t_2}{2},\varphi(t_0)\right]} y_1^2(x) = M_1 > 0 \\ &\min_{x\in\left[t_0,\frac{t_1+t_2}{2}\right]} y_2^2(x) = M_2 > 0 \end{aligned} \tag{9}$$

It follows from the assumptions that for arbitrary $\varepsilon > 0$ there exists $t_3 < b$ such that

$$\begin{aligned} &y_1^2(t) < \varepsilon \,.\, M_1, \qquad t_3 \leqq t < b. \\ &y_2^2(t) < \varepsilon \,.\, M_2 \end{aligned}$$

According to (1), (3) a constant N_0 exists that for $n > N_0$ and $x \in [t_0, \varphi(t_0))$ we have $\varphi_n(x) > t_3$. But it follows from (6) that

$$\varphi_n'(x) = y_1^2(\varphi_n(x))/y_1^2(x) \leqq M_1\varepsilon/M_1 = \varepsilon, \qquad x \in \left[\frac{t_1 + t_2}{2}, \varphi(t_0)\right)$$

$$\varphi_n'(x) = y_2^2(\varphi_n(x))/y_2^2(x) \leqq M_2\varepsilon/M_2 = \varepsilon, \qquad x \in \left[t_0, \frac{t_1 + t_2}{2}\right).$$

We can see that for arbitrary $\varepsilon > 0$ a constant $N_0(\varepsilon)$ exists such that for $n > N_0$, $x \in [t_0, \varphi(t_0))$ we have $\varphi_n'(x) < \varepsilon$ and this is the condition (8).

Let (8) be valid and let y be an arbitrary non-trivial solution of (q). Then for arbitrary $\varepsilon > 0$ an index $N_0(\varepsilon)$ exists such that

$$\varphi_n'(x) < \frac{\varepsilon}{M}, \qquad x \in [t_0, \varphi(t_0)), \qquad M = \max_{x\in[t_0,\varphi(t_0)]} y^2(x) > 0, \qquad n > N_0.$$

Then according to (6) we have

$$y^2(t) \leqq \frac{\varepsilon}{M} M = \varepsilon, \qquad t \geqq \varphi_{N_0+1}'(t_0)$$

and the theorem is valid in this case.

b) Let $x_1 \neq x_2$ and the sequences $\{\varphi_n'(x_1)\}$, $\{\varphi_n'(x_2)\}$ are unbounded. Let y be an arbitrary solution of (q). If $y(x_1) \neq 0$, then by means of (6) the sequence $\{y(\varphi_n(x_1))\}$ is unbounded. If $y(x_1) = 0$, then $y(x_2) \neq 0$ and by virtue of (6) the sequence $\{y(\varphi_n(x_2))\}$ is unbounded and the theorem is proved in this case.

c) This case can be proved by the same method as in b).

The next Theorem follows from Theorem 2.

Theorem 3. *If oscillatory* $(t \to b_-)$ *differential equations* (q), $(\bar{q})$, $q, \bar{q} \in C^\circ[a, b)$, *have the same dispersion then the statement*

"Every solution tends to zero for $t \to b_-$*"*

holds either for both (q) *and* $(\bar{q})$ *or for neither of them.*

2.2. Theorem 2 will be the starting point of our considerations. Further we shall examine a differential equation

$$y'' = q(t)\, y, \quad q \in C^\circ[a, \infty), \tag{q}$$

(q) being an oscillatory $(t \to \infty)$ differential equation.

2.3. The case if (q) is an oscillatory equation on $[a, b)$, $b < \infty$ can be reduced to the case 2.2. by means of the following transformations:

$$t = b - \frac{1}{x}, \qquad y(t) = Z(x), \qquad v(x) = xZ(x). \tag{10}$$

The equation (q) will be transformed into the equation

$$v'' = \frac{q\left(b - \frac{1}{x}\right)}{x^4} v, \qquad x \in \left[\frac{1}{b-a}, \infty\right).$$

We can see form (10) that the oscillatory behaviour is invariant and the solution of (q) is

$$y(t) = (b - t)\, v\left(\frac{1}{b-t}\right).$$

3.1. In this paragraph we shall study the behaviour of solutions of (q) when $t \to \infty$.

First let us prove the following

Lemma 1. *Let* (q), $(\bar{q})$ *be oscillatory* $(t \to \infty)$ *differential equations,* $q, \bar{q} \in C^\circ[a, \infty)$. *Let* φ_n *and* $\bar{\varphi}_n$ *be the n-th iterate of dispersions of* (q) *and* $(\bar{q})$, *respectively,* $t_0, t_1 \in [a, \infty)$, $t_0 \leqq t_1$. *If*

$$\varphi(t) \geqq \bar{\varphi}(t), \qquad t \in [t_1, \infty), \tag{11}$$

then there exist integers n, m such that

$$\varphi_{n+k}(t) > \bar{\varphi}_{m+k}(t), \qquad t \in [t_0, \infty) \tag{12}$$

for every non-negative integer k.

Proof. According to (1) and (11) there exist integers n, m such that

$$\varphi_n(t) > \bar{\varphi}_m(t), \quad \bar{\varphi}_m(t) > t_1, \qquad t \in [t_0, \infty). \tag{13}$$

We shall prove Lemma by induction. For $k = 0$ Lemma is valid according to (13). Let the statement be valid for $k \leqq l$. Then according to (11) and because $\varphi, \bar{\varphi}$ are increasing functions we have

$$\varphi_{n+l+1}(t) - \bar{\varphi}_{m+l+1} = \varphi(\varphi_{n+l}(t)) - \bar{\varphi}(\bar{\varphi}_{m+l}(t)) \geqq$$
$$\geqq \bar{\varphi}(\varphi_{n+l}(t)) - \bar{\varphi}(\bar{\varphi}_{m+l}(t)) > 0.$$

Thus the statement is valid for $k = l + 1$ and Lemma is proved.

Remark. If we assume that $\varphi, \bar{\varphi} \in C^{\circ}[a, \infty)$, $\varphi, \bar{\varphi} > t$, $\bar{\varphi}$ increasing instead of $\varphi, \bar{\varphi}$ to be dispersions, the Lemma is also valid. We utilized namely only these properties of φ.

Lemma 2. *Let φ_n be the n-th iterate of the function φ, $\varphi \in C^1[a, \infty)$ and let $t_0 \in [a, \infty)$.*

a) *If* $\varphi'(t) \geqq 1$ for $t \geqq t_0$,

then $\varphi_n(t) \geqq t + n[\varphi(t) - t]$, $t \in [t_0, \infty)$.

b) *If* $\varphi'(t) \leqq 1$ for $t \geqq t_0$,

then $\varphi_n(t) \leqq t + n[\varphi(t) - t]$, $t \in [t_0, \infty)$.

Proof. Let us define: $\varphi_0(t) \equiv t$.

a) We have

$$\varphi_i(t) - \varphi_{i-1}(t) = \varphi'(\xi)[\varphi_{i-1}(t) - \varphi_{i-2}(t)] \geqq \varphi_{i-1}(t) - \varphi_{i-2}(t),$$

(14) $$\xi \in (\varphi_{i-2}(t), \varphi_{i-1}(t)), \qquad i = 2, 3, \ldots$$

When we sum up the inequalities (14) for $i = 2, 3, \ldots, j$, then

$$\varphi_j(t) - \varphi_1(t) \geqq \varphi_{j-1}(t) - \varphi_0(t)$$

(15) $$\varphi_j(t) - \varphi_{j-1}(t) \geqq \varphi_1(t) - \varphi_0(t).$$

Let us sum up (15) for $j = 1, \ldots, n$. Then finally

$$\varphi_n(t) - \varphi_0(t) \geqq n[\varphi_1(t) - \varphi_0(t)]$$

and the statement of Lemma is proved.

b) We can prove this case in the same way as a).

3.2. Now some comparison theorems will be given.

Theorem 4. *Let (q), $(\bar{q})$ be oscillatory $(t \to \infty)$ differential equations, $q, \bar{q} \in C^{\circ}[a, \infty)$ and φ and $\bar{\varphi}$ dispersions of (q), $(\bar{q})$, respectively. Let $t_0, t_1 \in [a, \infty)$, $t_0 \leqq t_1$. Let*

$$\varphi'(t) \geqq \bar{\varphi}'(t), \qquad t \in [t_0, \infty),$$
$$\varphi(t) \geqq \bar{\varphi}(t), \qquad t \in [t_1, \infty),$$

and at least one of the functions φ', $\bar{\varphi}'$ be non-decreasing. If every solution of (q) tends to zero for $t \to \infty$, then every solution of $(\bar{q})$ tends to zero for $t \to \infty$, too.

Proof. From the assumptions of the theorem and according to Lemma 1 we have: There exist integers n_0, m_0 such that

$$\varphi_{n0+k}(x) > \bar{\varphi}_{m0+k}(x), \qquad x \in [t_0, \infty),$$
$$k = 1, 2, 3, \ldots.$$

If φ' is a non-decreasing function, $n > m_0$, then

$$\bar{\varphi}_n'(x) = \prod_{i=0}^{n-1} \bar{\varphi}(\bar{\varphi}_i(x)) \leqq \prod_{i=0}^{n-1} \varphi'(\bar{\varphi}_i(x)) \leqq$$

$$\leqq \prod_{i=0}^{m_0-1} \varphi'(\bar{\varphi}_i(x)) \cdot \prod_{i=n_0}^{n-m_0+n_0-1} \varphi'(\varphi_i(x)) = K(x)\, \varphi_{n+n_0-m_0}'(x),$$

where

$$K(x) = \prod_{i=0}^{m_0-1} \varphi'(\bar{\varphi}_i(x))/\varphi_{n_0}'(x) > 0.$$

Similarly if $\bar{\varphi}'$ is a non-decreasing function, $n > m_0$, then

$$\bar{\varphi}_n'(x) = \prod_{i=0}^{n-1} \bar{\varphi}'(\bar{\varphi}_i(x)) \leqq \prod_{i=n_0}^{n-m_0+n_0-1} \bar{\varphi}'(\varphi_i(x)) \cdot \prod_{i=0}^{m_0-1} \bar{\varphi}'(\bar{\varphi}_i(x)) \leqq$$

$$\leqq \bar{\varphi}_{m_0}'(x) \cdot \prod_{i=n_0}^{n-m_0+n_0-1} \varphi'(\varphi_i(x)) \leqq K(x) \cdot \varphi_{n+n_0-m_0}'(x).$$

Let us put: $c = \max\{\varphi(t_0),\ \bar{\varphi}(t_0)\}$.
Finally we can see that in both cases

(16)
$$\bar{\varphi}_n'(x) \leqq K_1 \varphi_{n+n_0-m_0}'(x),\ x \in [t_0,\ c]$$
$$0 < K_1 = \max_{x \in [t_0,\ C]} K(x) < \infty\,.$$

If every solution of (q) tends to zero for $t \to \infty$, then according to (16) and Theorem 2 every solution of $(\bar{q})$ tends to zero for $t \to \infty$.

3.3. **Theorem 5.** *Let (q) be an oscillatory $(t \to \infty)$ differential equation, $q \in C^\circ\,[a,\ \infty)$ and let φ be its dispersion, $t_0 \in [a,\ \infty]$. Let*

$$1 \leqq \varphi'(t) \leqq \psi(t), \qquad t \in [t_0,\ \infty),$$

$$\psi(t) \in C^\circ[a,\ \infty), \qquad \lim_{t\to\infty} \psi(t) = 1$$

and at least one of the functions φ', ψ be non-increasing. If the series

(17)
$$\sum_{i=0}^{\infty} \{\psi(t_0 + ih) - 1\} < \infty, \qquad h = \varphi(t_0) - t_0$$

then every solution of (q) is bounded on $[t_0,\ \infty)$.

Proof. Let $x_1 > x_2$; then

$$\varphi(x_1) - \varphi(x_2) = \varphi'(\xi)\,(x_1 - x_2) \geqq x_1 - x_2, \qquad \xi \in (x_2,\ x_1),$$

(18)
$$\varphi(x_1) - x_1 \geqq \varphi(x_2) - x_2$$

and from this

(19)
$$h_1 \equiv \varphi(x) - x \geqq \varphi(t_0) - t_0 = h, \qquad x \in [t_0,\ \varphi(t_0)).$$

By Lemma 2 and (19) we have:

a) If ψ is a non-increasing function, then we have for $x \in [t_0,\ \varphi(t_0))$:

$$\varphi_n'(x) = \prod_{i=0}^{n-1} \varphi'(\varphi_i(x)) \leqq \prod_{i=0}^{n-1} \psi(\varphi_i(x)) \leqq \prod_{i=0}^{n-1} \psi(x + ih_1) \leqq$$

$$\leqq \prod_{i=0}^{\infty} \psi(t_0 + ih).$$

b) If φ' is a non-increasing function, then we have for $x \in [t_0, \varphi(t_0))$:

$$\varphi_n'(x) = \prod_{i=0}^{n-1} \varphi'(\varphi_i(x)) \leqq \prod_{i=0}^{n-1} \varphi'(h + ix) \leqq \prod_{i=0}^{\infty} \varphi(t_0 + ih)$$

$$\leqq \prod_{i=0}^{\infty} \psi(t_0 + ih).$$

Thus we can see that

$$\varphi_n'(x) \leqq \prod_{i=0}^{\infty} \psi(t_0 + ih), \qquad x \in [t_0, \varphi(t_0)). \tag{20}$$

But it is known ([5], 0.255) that the continued product (20) converges if and only if the infinite series $\sum_{i=0}^{\infty} \{\psi(t_0 + ih) - 1\}$ converges. From this and from the assumptions the inequality

$$\varphi_n'(x) \leqq K < \infty, \qquad x \in [t_0, \varphi(t_0)),$$

s valid and the theorem follows from Theorem 1.

Remark. *The condition* (17) *from Theorem 5 is valid if*

$$\lim_{n\to\infty} n\left[\frac{\psi(t_0 + nh) - 1}{\psi(t_0 + h + nh) - 1} - 1\right] > 1. \tag{21}$$

This statement follows from Raabe's convergence test for the series (17).

3.4. Now let us take notice of some particular functions. In Theorem 5 we can put:

$$\psi = 1 + \frac{c}{t^{1+\varepsilon}}, \qquad \varepsilon > 0, c > 0, t \in [t_0, \infty), t_0 > 0.$$

Then

$$\lim_{n\to\infty} n\left[\left(\frac{t_0 + nh + h}{t_0 + nh}\right)^{1+\varepsilon} - 1\right] = \lim_{n\to\infty} \frac{(1+\varepsilon)\left(\dfrac{t_0 + nh + h}{t_0 + nh}\right)^{\varepsilon} \dfrac{-h^2}{(t_0 + nh)^2}}{-\dfrac{1}{n^2}}$$

$$= \lim_{n\to\infty} (1 + \varepsilon)\left(\frac{t_0 + nh + h}{t_0 + nh}\right)^{\varepsilon} \frac{n^2h^2}{(t_0 + nh)^2} = (1 + \varepsilon) > 1$$

and we can see that the inequality (21) is valid.

Theorem 6. *Let* (q), $q \in C^{\circ}[a, \infty)$ *be an oscillatory* $(t \to \infty)$ *differential equation and* φ *its dispersion. Let* $t_0 \in [a, \infty)$ *and*

$$1 \leqq \varphi'(t) \leqq 1 + \frac{c}{t^{1+\varepsilon}}, \qquad t \in [t_0, \infty), \qquad t_0 > 0, \qquad \varepsilon > 0, \qquad c \geqq 0,$$

c, ε *are arbitrary constants. Then every solution of* (q) *is bounded on* $[t_0, \infty)$.

Theorem 7. *Let* (q), $q \in C^{\circ}[a, \infty)$ *be an oscillatory* $(t \to \infty)$ *differential equation and* φ *its dispersion. Let* $t_0 \in [a, \infty)$ *and*

$$\varphi'(t) \leqq 1 - \frac{C}{t}, \qquad t \in [t_0, \infty), \; 0 < C < t_0.$$

Then every solution of (q) *tends to zero for* $t \to \infty$.

Proof. By the application of Lemma 2 we have

$$\lim_{n\to\infty} \varphi_n'(x) = \lim_{n\to\infty} \prod_{i=0}^{n-1} \varphi'(\varphi_i(x)) = \exp\left\{\lim_{n\to\infty} \sum_{i=0}^{n-1} \ln(\varphi'(\varphi_i(x)))\right\} \leqq$$

$$\leqq \exp\left\{\lim_{n\to\infty} \sum_{i=0}^{n-1} \ln\left(1 - \frac{C}{x + ih}\right)\right\} \leqq \exp\left\{\lim_{n\to\infty} \sum_{i=0}^{n-1} \ln\left(1 - \frac{C}{t_1 + ih}\right)\right\}$$

where

$$t_1 = \varphi(t_0), \qquad h = \max_{x \in [t_0, \varphi(t_0)]} (\varphi(x) - x) > 0.$$

As

$$\sum_{i=0}^{\infty} \ln\left(1 - \frac{C}{t_1 + ih}\right) \leqq \int_0^{\infty} \ln\left(1 - \frac{C}{t_1 + yh}\right) dy = -\infty$$

we can see that $\varphi_n'(x)$ converges uniformly to zero for $x \in [t_0, \varphi(t_0))$ and the theorem is also valid according to Theorem 2.

4.1. This paragraph will deal with the relation between the dispersion of (q) and the property of every solution of (q) belonging to $L^p[a, \infty)$, $p \geqq 1$. The Theorem 1 gives the necessary and sufficient condition for every solution to belong to $L^p[a, \infty)$, $p \geqq 1$. We can simplify the condition for the monotone functions.

Theorem 8. *Let* (q) *be an oscillatory* $(t \to \infty)$ *differential equation,* $q \in C^{\circ}[a, \infty)$, $t_0 \in [a, \infty)$. *Let* φ_n *be the n-th iterate of the dispersion* φ *of* (q) *and let* φ' *be a monotone function on* $[t_0, \infty)$. *Then every solution of* (q) *belongs to* $L^p[t_0, \infty)$, $p \geqq 1$ *if, and only if*

$$\sum_{n=0}^{\infty} [\varphi_n'(t_0)]^{1+\frac{p}{2}} < \infty. \tag{26}$$

Proof. Let φ' be a non-decreasing function. Then for $x \in [t_0, \varphi(t_0))$ we have:

$$\varphi_n'(x) = \prod_{i=0}^{n-1} \varphi'(\varphi_i(x)) \leqq \prod_{i=0}^{n-1} \varphi'(\varphi_i(\varphi(t_0))) = \varphi_{n+1}'(t_0)/\varphi'(t_0) \tag{27}$$

$$\varphi_n'(x) = \prod_{i=0}^{n-1} \varphi'(\varphi_i(x)) \geqq \prod_{i=0}^{n-1} \varphi'(\varphi_i(t_0)) = \varphi_n'(t_0). \tag{28}$$

a) Let every solution belongs to $L^p[t_0, \infty)$, then the formula (5) is valid and according to (28) we have

$$\sum_{n=0}^{\infty} [\varphi_n'(t_0)]^{1+\frac{p}{2}} = \frac{\sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi_n'(t_0)]^{1+\frac{p}{2}} dt}{\varphi(t_0) - t_0} \leqq \frac{\sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi_n'(t)]^{1+\frac{p}{2}} dt}{\varphi(t_0) - t_0} < \infty$$

and we can see that the condition (26) is valid.

b) Let (26) be valid. According to (27)

$$\sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi_n'(t)]^{1+\frac{p}{2}} \, dt \leqq \frac{\varphi(t_0) - t_0}{[\varphi'(t_0)]^{1+p/2}} \sum_{n=0}^{\infty} [\varphi_{n+1}'(t_0)]^{1+p/2} < \infty.$$

Thus (5) from the Theorem 1 is valid and by means of Theorem 1 every solution of (q) belongs to $L^p[t_0, \infty)$.

If φ' is a non-increasing function then the proof is similar.

4.2. Now some kind of comparison theorems will be given.

Theorem 9. *Let* (q), $(\bar{q})$ *be oscillatory* $(t \to \infty)$ *differential equations,* $q, \bar{q} \in C^c[a, \infty)$, $t_0 \in [a, \infty)$. Let φ *and* $\bar{\varphi}$ *be dispersions of* (q) *and* $(\bar{q})$, *respectively.*
Let

$$\varphi'(t) \geqq \bar{\varphi}(t), \qquad \varphi(t) \geqq \bar{\varphi}(t), \qquad t \in [t_0, \infty)$$

and at least one of the functions φ', $\bar{\varphi}'$ *be non-decreasing. If every solution of* (q) *belongs to* $L^p[t_0, \infty)$, $p \geqq 1$, *then every solution of* $(\bar{q})$ *belongs to* $L^p[t_0, \infty)$, $p \geqq 1$.

Proof. It follows from the assumptions of the theorem that the following formula is valid (by induction):

$$\varphi_n(t) > \bar{\varphi}_n(t), \qquad t \geqq t_0.$$

Let φ' be a non-decreasing function on $[t_0, \infty)$. Then

$$\sum_{n=1}^{\infty} \int_{t_0}^{\bar{\varphi}(t_0)} [\bar{\varphi}_n'(t)]^{1+p/2} \, dt = \sum_{n=1}^{\infty} \int_{t_0}^{\bar{\varphi}(t_0)} [\prod_{i=0}^{n-1} \bar{\varphi}'(\bar{\varphi}_i(t))]^{1+p/2} \, dt \leqq$$

$$\leqq \sum_{n=1}^{\infty} \int_{t_0}^{\bar{\varphi}(t_0)} [\prod_{i=0}^{n-1} \varphi'(\bar{\varphi}_i(t))]^{1+p/2} \, dt \leqq \sum_{n=1}^{\infty} \int_{t_0}^{\bar{\varphi}(t_0)} [\prod_{i=1}^{n} \varphi'(\bar{\varphi}_i(t_0))]^{1+p/2} \, dt$$

$$\leqq \frac{\bar{\varphi}(t_0) - t_0}{[\varphi'(t_0)]^{1+p/2}} \sum_{n=2}^{\infty} [\varphi_n'(t_0)]^{1+p/2}.$$

Hence the following formula is valid:

$$\sum_{n=0}^{\infty} \int_{t_0}^{\bar{\varphi}(t_0)} [\bar{\varphi}'(t)]^{1+p/2} \, dt \leqq K_1 \sum_{n=0}^{\infty} [\varphi'_n(t_0)[^{1+p/2}, \tag{29}$$

$$0 < K_1 < \infty.$$

Let $\bar{\varphi}'$ be a non-decreasing function on $[t_0, \infty)$. We have similarly:

$$\sum_{n=1}^{\infty} [\bar{\varphi}_n'(t_0)]^{1+p/2} \leqq \sum_{n=1}^{\infty} \left[\prod_{i=0}^{n-1} \bar{\varphi}'(\bar{\varphi}_i(t_0)) \right]^{1+p/2} \leqq$$

$$\leqq \frac{1}{\varphi(t_0) - t_0} \prod_{n=1}^{\infty} \int_{t_0}^{\varphi(t_0)} \prod_{i=0}^{n-1} [\bar{\varphi}'(\varphi_i(t))]^{1+p/2} \, dt \leqq \frac{1}{\varphi(t_0) - t_0} \sum_{n=1}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi_n'(t)]^{1+p/2} \, dt.$$

Thus we can see that

$$\sum_{n=0}^{\infty} [\bar{\varphi}_n'(t_0)]^{1+p/2} \leqq K_2 \cdot \sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi_n'(t)]^{1+p/2} \, dt, \tag{30}$$

$$0 < K_2 < \infty.$$

Let φ' be non-decreasing. Let every solution of (q) belong to $L^p[t_0, \infty)$. Then it follows from Theorem 8 that the right side of (29) converges and, according to Theorem 1, the statement is valid.

If $\bar{\varphi}'$ is non-decreasing then the situation is analogous if we use the inequality (30).

Theorem 10. *Let φ be the dispersion of an oscillatory $(t \to \infty)$ differential equation (q), $q \in C^\circ[a, \infty)$, $\varphi'(t) \geqq 1$. Then there exists a solution of (q), not belonging to $L^p[a, \infty)$, $p \geqq 1$.*

Proof.

$$\sum_{n=0}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi_n'(t)]^{1+p/2}\, dt \geqq \sum_{n=1}^{\infty} \int_{t_0}^{\varphi(t_0)} \Big[\prod_{i=0}^{n-1} \varphi'(\varphi_i(t))\Big]^{1+p/2} dt \geqq \sum_{n=1}^{\infty} [\varphi(t_0) - t_0]^{1+p/2} = \infty$$

and the theorem is valid according to Theorem 1.

Remark. Theorems 1 and 10 solve cases when $\varphi' \geqq 1$ or $\varphi' \leqq \text{const} < 1$. The following theorem can be applicable to the case when $\varphi' \leqq 1$ and gives us some sufficient condition for every solution to belong to $L^p[a, \infty)$, $p \geqq 1$.

Theorem 11. *Let (q) be an oscillatory $(t \to \infty)$ differential equation, $q \in C^\circ[a, \infty)$, $t_0 \in [a, \infty)$ and let φ be its dispersion. Let*

$$\varphi'(t) \leqq \psi(t), \qquad t \in [t_0, \infty),$$

$$\psi \in C^1[a, \infty), \qquad \lim_{t\to\infty} \psi(t) = 1,$$

ψ non-decreasing. If

$$\lim_{n\to\infty} n^2 \psi'(t_0 + h + nh) \geqq \varepsilon > 0, \qquad h = \varphi(t_0) - t_0,$$

then every solution of (q) belongs to $L^p[t_0, \infty)$ for $p > 2/(\varepsilon \,.\, h) - 2$, $p \geqq 1$.

Proof. Let $t_1 > t_2$. Then

$$\begin{aligned} \varphi(t_1) - \varphi(t_2) = \varphi'(\xi)\,(t_1 - t_2) \leqq t_1 - t_2,\ \xi \in (t_2, t_1), \\ \varphi(t_1) - t_1 \leqq \varphi(t_2) - t_2. \end{aligned} \tag{31}$$

By the application of Lemma 2 and (31) we have

$$\sum_{n=1}^{\infty} \int_{t_0}^{\varphi(t_0)} [\varphi'_n(t)]^{1+p/2}\, dt \leqq \sum_{n=1}^{\infty} \int_{t_0}^{\varphi(t_0)} \prod_{i=1}^{n} [\psi(\varphi_i(t_0))]^{1+p/2}\, dt \leqq$$

$$\leqq \sum_{n=1}^{\infty} \int_{t_0}^{\varphi(t_0)} \Big[\prod_{i=1}^{n} \psi(t_0 + ih)\Big]^{1+p/2} dt = (\varphi(t_0) - t_0) \sum_{n=1}^{\infty} \Big[\prod_{i=1}^{n} \psi(t_0 + ih)\Big]^{1+p/2} < \infty$$

for

$$\lim_{n\to\infty} n \left[\left(\frac{\prod_{i=1}^{n} \psi(t_0 + ih)}{\prod_{i=1}^{n+1} \psi(t_0 + ih)}\right)^{1+p/2} - 1\right] = \lim_{n\to\infty} \frac{\dfrac{1}{[\psi(t_0 + h + nh)]^{1+p/2}} - 1}{\dfrac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{n^2 \psi'(t_0 + h + nh) \,.\, h(1 + p/2)}{[\psi(t_0 + h + nh)]^{2+p/2}} \geqq \varepsilon \,.\, h(1 + p/2) > 1$$

and according to Raabe's convergence test the infinite series converges and the theorem follows from Theorem 1.

4.3. Now let us notice some particular functions which could be utilized in the comparison theorems.

a) Let

$$\psi(t) = 1 - \frac{c}{t^{1-\varepsilon}}, \qquad \varepsilon \in (0, 1), \qquad c > 0,$$

then

$$\lim_{n\to\infty} n^2 \frac{(1-\varepsilon)\, C}{[t_0 + h + nh]^{2-\varepsilon}} = \infty, \qquad h = \varphi(t_0) - t_0$$

and we can see that ψ fulfils all conditions in Theorem 11. Hence we have the following

Theorem 12. *Let* (q) *be an oscillatory* $(t \to \infty)$ *differential equation,* $q \in C^\circ[a, \infty)$, $t_0 \in [a, \infty)$ *and let*

$$\varphi'(t) \leqq 1 - \frac{C}{t^{1-\varepsilon}}, \qquad C > 0, \qquad \varepsilon \in (0, 1), \quad t \geqq t_0, t_0^{1-\varepsilon} > C$$

where C, ε *are arbitrary constants. Then every solution of* (q) *belongs to* $L^p[t_0, \infty)$, $p \geqq 1$.

b) Let

$$\psi(t) = 1 - \frac{C}{t}. \qquad C > 0.$$

Then

$$\lim_{n\to\infty} n^2 \frac{C}{(t_0 + h + nh)^2} = C/h^2 > 0$$

and according to Theorem 11 we have:

Theorem 13. *Let the dispersion* φ *of an oscillatory* $(t \to \infty)$ *differential equation* (q), $q \in C^\circ[a, \infty)$ *fulfil the condition*

$$\varphi'(t) \leqq 1 - \frac{C}{t}, \qquad t \in [t_0, \infty), \qquad t_0 \geqq a, \qquad t_0 > C > 0,$$

where $C > 0$ *is an arbitrary constant. Then every solution of* (q) *belongs to* $L^p[t_0, \infty)$ *for* $p \geqq 1, p > 2 \cdot \frac{\varphi(t_0) - t_0}{C} - 2$.

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