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CONTRIBUTION TO THE MONOTONICITY OF THE SEQUENCE OF ZERO POINTS OF INTEGRALS OF THE DIFFERENTIAL EQUATION $y'' + g(t)y = 0$ WITH REGARD TO THE BASIS $[\alpha, \beta]$

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In papers [1] and [3] there have been deduced simple sufficient conditions for the monotonicity of the sequence of zero points of an arbitrary integral of the differential equation

$$y'' + q(t)\, y = 0 \tag{q}$$

in the interval $I = (a, b)$, where $a < b$, $a \in E_1$, $b \in \bar{E}_1 = E_1 \cup (\infty)$.

In paper [6] J. Vosmanský have deduced simple sufficient conditions for the monotonicity of the sequence of extremants of an arbitrary integral of the differential equation (q) in the interval I. By "extremant" of the function $y(t) \in C_2(I)$ we understand any number $\bar{t} \in I$, in which the function $y(t)$ acquires an extreme value (local sharply).

The subject of this paper is to investigate these questions so that above-mentioned results will be their special case.

In paper [2] M. Laitoch have introduced a notion of the solution of the differential equation (q) with regard to the basis $[\alpha, \beta]$. By this solution we understand a function $\alpha u + \beta u'$, where $u(t)$ is a solution of equation (q), and α, β are arbitrary but fixly chosen constants with the property $\alpha^2 + \beta^2 > 0$. In paper [2] it is also introduced the first accompanying equation (Q) towards the differential equation (q) with regard to the basis $[\alpha, \beta]$ in the form

$$y'' + Q(t)\, y = 0, \tag{Q}$$

where

$$Q(t) = q + \frac{\alpha\beta q'}{\alpha^2 + \beta^2 q} + \frac{1}{2}\,\frac{\beta^2 q''}{\alpha^2 + \beta^2 q} - \frac{3}{4}\,\frac{\beta^4 q'^2}{(\alpha^2 + \beta^2 q)^2}$$

by assumption that $q(t) \in C_2(I)$, $q(t) > 0$ for any $t \in I$ (I is above — mentioned interval). In paper [2] there is proved that for every integral $u(t)$ of equation (q) the function

$$U(t) = \frac{\alpha u + \beta u'}{\sqrt{\alpha^2 + \beta^2 q}}$$

is an integral of equation (Q) and, conversely, for every integral $U(t)$ of equation (Q) there exists an integral $u(t)$ of equation (q) such that there holds:

$$\frac{\alpha u + \beta u'}{\sqrt{\alpha^2 + \beta^2 q}} = U(t).$$

Agreement 1: In further investigations we shall assume that the carrier $q(t)$ of equation (q) belongs at least to the class $C_2(I)$ and that $q(t) > 0$ for any $t \in I$.

A function $F(t)$ is said to be of class $M_n(a, b)$ or monotone of order n in the interval $I = (a, b)$ (see [1] or [6]), if it has $n \geqq 0$ continuous derivatives $F^{(o)}, F'', \ldots, F^{(n)}$, complying with the relation

$$(-1)^j F^{(j)}(t) \geqq 0 \text{ for } t \in (a, b) \qquad j = 0, 1, 2, \ldots, n.$$

If the previous inequality is fulfilled for $j = 0, 1, 2, \ldots$, then the function $F(t)$ will said to be completely monotone in (a, b) and we shall denote it by $M_\infty(a, b)$. M_n will stand, as abbreviation, for $M_n(0, \infty)$.

Let $\{t_k\}$ denote the sequence and $\triangle^n t_k$ the n-th differences of the sequence $\{t_k\}$, so that

$$\triangle^0 t_k = t_k; \ \triangle t_k = t_{k+1} - t_k; \ \ldots; \ \triangle^n t_k = \triangle^{n-1} t_{k+1} - \triangle^{n-1} t_k,$$

where $k = 1, 2, \ldots, n = 1, 2, \ldots$. The sequence $\{t_k\}$ will be said to be monotone of order n, if

$$(-1)^j \triangle^j t_k \geqq 0 \qquad k = 0, 1, 2, \ldots, j = 1, 2, \ldots, n.$$

In the case $n = \infty$ the sequence $\{t_k\}$ will be said to be completely monotone.

Lemma 1: *Let $g(t)$ be N-times differentiate function in the interval I fulfilling the condition*

(1) $$(-1)^{n+1} g^{(n)}(t) \geqq 0 \qquad n = 1, 2, \ldots, N; \ t \in I.$$

Let $\varphi(t)$ be N-times differentiate function on the set $g(I)$, fulfilling

(2) $$(-1)^n \varphi^{(n)}(t) \geqq 0 \qquad n = 1, 2, \ldots, N; \ t \in g(I).$$

Then there holds

(3) $$(-1)^n D_t^n \varphi[g(t)] \geqq 0 \qquad n = 1, 2. \ldots, N; t \in I,$$

where D_t^n denotes the n-th derivative by t. If there holds $g'(t) > 0$ and a sharp inequality in (2) or if there holds $\varphi'(t) < 0$ and a sharp inequality in (1), so there holds a sharp inequality in (3).

Proof: (see paper [5] pg. 1241).

Lemma 2: *Let an agreement 1 hold. Let $q'(t) \in M_n(a, b)$, $n \geqq 2$. Let $\alpha\beta \leqq 0$. Then or the carrier $Q(t)$ of equation (Q) there holds*

$$Q'(t) \in M_{n-2}(a, b)$$

and in the case $b = \infty$ there holds $Q(\infty) = q(\infty)$.

Proof: Let us consider a carrier $Q(t)$ of equation (Q):

$$Q(t) = q + \frac{\alpha\beta q'}{\alpha^2 + \beta^2 q} + \frac{1}{2}\frac{\beta^2 q''}{\alpha^2 + \beta^2 q} - \frac{3}{4}\frac{\beta^4 q'^2}{(\alpha^2 + \beta^2 q)^2}$$

If in lemma 1 we substitute $g(t) = q(t)$ and $\varphi(t) = \dfrac{1}{t}$, then it is obvious that $\dfrac{1}{q(t)}$ and also $\dfrac{1}{\alpha^2 + \beta^2 q}$ belong to the class $M_{n+1}(a, b)$. The function $\dfrac{\beta^2 q'}{\alpha^2 + \beta^2 q^2}$ belongs to the class $M_n(a, b)$ because the sum and the product of two functions of the class $M_n(a, b)$ is a function belonging again to the class $M_n(a, b)$ (see [7] pg. 194). Therefore by using lemma 1 we have that

$$D_t^1\left[-\frac{3}{4}\frac{(\beta^2 q')^2}{(\alpha^2+\beta^2 q)^2}\right] \text{ belongs to the class } M_{n-1}(a, b),$$

$$D_t^1\left[\frac{1}{2}\frac{q''}{\alpha^2+\beta^2 q}\right] \text{ belongs to the class } M_{n-2}(a, b), \text{ and finally}$$

$$D_t^1\left[\frac{\alpha\beta q'}{\alpha^2+\beta^2 q}\right] \text{ belongs to the class } M_{n-1}(a, b) \text{ if}$$

$\alpha\beta \leqq 0$. Therefore $Q'(t) \in M_{n-2}(a, b)$.

To prove $Q(\infty) = q(\infty)$ we consider two cases:

a) let $q'(\infty) = 0$. Since $q' \in M_n(a, \infty)$, it is obvious that $q''(\infty) = 0, \ldots, q^{(n+1)}(\infty) = 0$ and therefore $Q(\infty) = q(\infty)$,

b) let $q'(\infty) = c > 0$. Since $q' \in M_n(a, \infty)$, then $c < \infty$, $q(\infty) = \infty$. $q''(\infty) = 0$ hold and therefore $Q(\infty) = q(\infty)$.

Agreement 2: Let us denote $M_k = \int_{t_k}^{t_{k+1}} \left|\frac{\alpha y + \beta y'}{\sqrt{\alpha^2+\beta^2 q}}\right|^\lambda dt$, for $\lambda > -1$, where $\{t_k\}$ signifies the sequence of zero points of the function $\alpha y + \beta y'$, where $y(t)$ is an integral of equation (q). Let $\{\bar{t}_k\}$ denote the sequence of zero points of the function $\alpha\bar{y} + \beta\bar{y}'$, where $\bar{y}(t)$ is an arbitrary integral of equation (q), which can but needn't be identical with $y(t)$.

Then the following theorem holds:

Theorem: *Let $q(t) > 0$ for $t \in (0, \infty)$ and $q' \in M_n$. Let $\alpha\beta \leqq 0$. Then one has*

$$(-1)^i \triangle^i M_k \geqq 0,\ i = 0, 1, \ldots, n-2;\ k = 1, 2, \ldots$$

and, in particular

$$(-1)^i \triangle^{i+1} t_k \geqq 0\ i = 0, 1, 2, \ldots, n-2;\ k = 1, 2, \ldots$$

consequently, the sequence of the differences of successive zero points of any functions $\alpha y + \beta y'$, where $y(t) \in (q)$ is in the interval $(0, \infty)$ monotone of order $n-2$. Furthermore, if one has $t_1 > \bar{t}_1$, then

$$(-1)^i \triangle^i (t_k - \bar{t}_k) \geqq 0\ i = 0,1, 2, \ldots, n-2;\ k = 1, 2, \ldots \tag{4}$$

Proof: Let $y(t)$ be an arbitrary solution of the differential equation (q). Then the function $\frac{\alpha y + \beta y'}{\sqrt{\alpha^2+\beta^2 q'}}$ satisfies differential equation (Q). By lemma 2 $Q'(t) \in M_{n-2}(a, b)$ and $Q(\infty) > 0$.

Ph. Hartman has shown in [1] that under these assumptions there exist integrals Y_1, Y_2 of equation (Q) such that, for $W = Y_1^2 + Y_2^2$ there holds $W \in M_{n-2}$. Then by [4], for $N_k = \int_{T_k}^{T_{k+1}} |Y(t)|^\lambda dt$, $k = 1, 2, \ldots$, where $\{T_k\}$ signifies the sequence of zero points of the integral $Y(t)$ of equation (Q), we can write

$$(-1)^i \triangle^i N_k \geqq 0 \; i = 0, 1, \ldots, n-2; \; k = 1, 2, \ldots$$

and, if $T_1 > \bar{T}_1$, then

$$(-1)^i \triangle^i (T_k - \bar{T}_k) \geqq 0 \; i = 0, 1, \ldots, n-2; \; k = 1, 2, \ldots$$

Since $Y\sqrt{\alpha^2 + \beta^2 q} = \alpha y + \beta y'$, we have $\{T_k\} = \{t_k\}$, where $\{t_k\}$ denotes the sequence mentioned in agreement 2; there holds

$$N_k = \int_{t_k}^{t_{k+1}} \left| \frac{\alpha y + \beta y'}{\sqrt{\alpha^2 + \beta^2 q}} \right|^{\lambda} dt = M_k$$

and the theorem is being proved.

From the preceding theorem and theorem 2, 1 from [3] the following corollary follows:

Corollary: *If $q(t)$ is not a constant and $q'(t)$ is completely monotone in the interval $(0, \infty)$, then*

$$(-1)^i \triangle^i M_k > 0 \text{ for all } i, k$$

and for that reason, the sequence of zero points of an arbitrary function $\alpha y + \beta y'$, where $y(t) \in (q)$, is completely monotone in $(0, \infty)$. The inequalities (4) may then be written in a stronger form:

$$(-1)^i \triangle^i (t_k - \bar{t}_k) > 0 \text{ for all } i, k.$$

Remark 1: If in preceding considerations we choose $\alpha = 1, \beta = 0$, then we get those results from paper [1] for the monotonicity of the sequence of zeros of an arbitrary integral of equation (q). If we choose $\alpha = 0$, $\beta = 1$, then we obtain those results from [6] for the monotonicity of the sequence of extremants of an arbitrary integral of equation (q).

Remark 2: It is easy to find out that the functions $\alpha u + \beta u'$; $\alpha v + \beta v'$, where (u, v) is a basis of equation (q), form a fundamental system of solutions of trinomical differential equation

$$(5) \qquad y'' - \frac{\beta^2 q'}{\alpha^2 + \beta^2 q} y' + \left(q + \frac{\alpha\beta q'}{\alpha^2 + \beta^2 q}\right) y = 0,$$

where $q(t)$ is a carrier of equation (q). From the proof of the preceding theorem it is obvious that under its assumptions, resp. under assumptions of corollary we obtain conditions for the monotonicity of the sequence of zero points of an arbitrary integral of equation (5).

Example: Let $\alpha = 1$, $\beta = -1$, and $q(t) = 1 - e^{-x}$. Then $q'(t) = e^{-x} \in M\infty$. Therefore by remark 2 the sequence of zero points of an arbitrary integral of equation

$$y'' + \frac{e^{-x}}{2 - e^{-x}} y' + \left(1 - e^{-x} - \frac{e^{-x}}{2 - e^{-x}}\right) y = 0$$

is completely monotone in the interval $(0, \infty)$.

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