Archivum Mathematicum

Iulian Popovici; Radu Iordanescu; Adriana Turtoi

Some maximal graduation classes and their Riemannian properties

Archivum Mathematicum, Vol. 7 (1971), No. 1, 31--46

Persistent URL: http://dml.cz/dmlcz/104737

Scripta Fac. Sci. nat. UJEP Brunensis
Arch. math. 1, VII: 31—46, 1971

SOME MAXIMAL GRADUATION CLASSES AND THEIR RIEMANNIAN PROPERTIES

IULIAN POPOVICI, RADU IORDĂNESCU AND ADRIANA TURTOI

(Received October 17, 1969)

I. INTRODUCTION

We study the notion of maximal graduation (the graduation with one-dimensional summants) [18]. We use as auxiliary notion the generalized Clifford algebras introduced by Yamazaki [26]. The construction method of simple maximal graduations which we shall develop points out the following aspects:

A. The generalization of classical theory of spinors replacing the spinors with 2^ν components [4], [20] by spinors with a finite arbitrary number of components.

B. The generalization of 6-dimensional unification of Dirac matrices made by A. Popovici [12], [13].

C. The generalization of well known Pauli relations.

D. The connection among the associative, Lie and Jordan case.

E. The writing, for the structure of considered algebras, in a convenient way for several applications.

The obtained results get a geometrical content using the Vranceanu's method [24] of association to a real algebra a space with affine constant connexion. We can prove, e.g., that the autoparallel curves associated to simple Jordan real forms of type A are obtained in two steps: first we construct the geodesics associated to group algebras, then we apply them certain linear transformation groups which generalize Poincare's group.

We shall expose in detail this fact in the case of the existence of metrics.

II. GENERALIZED CLIFFORD ALGEBRAS

Let K_n be the algebra of all $n \times n$ —matrices over a field K and let V_n be an n-dimensional vector space.

If we fix a basis in V_n, K_n is identified with algebra of linear operators on V_n.

Let $K(p)$ be a field which contains a primitive p^{th} root ω_p of unity. If p is odd and $K(p)$ has the characteristic $\neq 2$, then $K(p)$ is also $K(2p)$ and we write: $K(p) = K(2p)$.

Let Z_p be a cyclic group of period p and let Z_p^n be the direct sum of n copies of Z_p. If we take a system of generators $\varrho_1, \ldots, \varrho_n$ of Z_p^n, their elements have the form:

$$\alpha = \sum_i \alpha_i \varrho_i, \quad \beta = \sum \beta_i \varrho_i, \ldots$$

where $\alpha_i, \beta_i, \ldots$ are the quotient classes modulo p.

Let $\{e_\alpha\}_{\alpha \in Z_p^n}$ be a basis of V_{p^n} over $K(p)$. We denote $e_i = e_{\varrho_i}$ $(i = 1, 2, \ldots, n)$.

Proposition 1. *There exists a unique associative algebra structure of V_{p^n} with unity e_0 which verifies the following conditions:*

$$e_i^p = e_0,\ e_j e_i = \omega_p e_i e_j\ (i, j = 1, \ldots, n;\ i < j) \tag{1}$$

$$e_\alpha = e_1^{\alpha_1} e_2^{\alpha_2} \ldots e_n^{\alpha_n} \qquad (\alpha \in Z_p^n). \tag{2}$$

The structure of this algebra is given by:

$$e_\alpha e_\beta = \omega_p^{\sum\limits_{i>j} \alpha_i \beta_j} e_{\alpha+\beta}. \tag{3}$$

Definition 1. *The algebra defined by* (1) *and* (2) *is called generalized Clifford algebra and we denote it by A_p^n. The algebra A_p^2 (which is denoted by A_p) is called generalized Pauli algebra.*

For the algebra A_p, (3) becomes:

$$e_\alpha e_\beta = \omega_p^{\alpha_2 \beta_1} e_{\alpha+\beta}. \tag{3'}$$

The classical Clifford algebras [4], [20] are of the form A_2^n. Yamazaki has shown [6] that A_p^n is simple and central for n even and has a p-dimensional center for n odd.

This fact was given explicitely by I. Popovici and C. Gheorghe [14], [15] and later by A. O. Morris [10] in the following manner:

Theorem 1. *The algebra A_p is isomorphic to $K(p)_p$. The algebra $A_p^{2\nu}$ over $K(2p)$ is isomorphic to $K(2p)_p^\nu$. The algebra $A_p^{2\nu+1}$ over $K(2p)$ is isomorphic to a direct sum of p copies of $K(2p)_{p^\nu}$.*

Proof. Let $\{v_a\}_{a \in Z_p}$ be a basis of V_p over $K(p)$. For the linear operators

$$f_1 : v_a \to \omega_p^a va, \qquad f_2 : v_a \to v_{a-1} \tag{4}$$

(1) holds for $n = 2$. Then the maps $e_i \to f_i (i = 1, 2)$ define an homomorphism $\varphi : A_p \to K(p)_p$ which is an isomorphism because A_p is simple. On the other hand, a straightforward calculus shows that $A_p^{2\nu}$ over $K(2p)$ is isomorphic to the tensorial product of ν copies of A_p. The first two assertions of theorem are proved.

Now, the center of $A_p^{2\nu+1}$ has a basis $\{w_a\}_{a \in Z_p}$ for which

$$w_a^2 = w_a;\ w_a w_b = 0 \qquad (a \neq b), \qquad \sum_{a \in Z_p} w_a = e_0. \tag{5}$$

The relations (5) give homomorphisms $\psi_a : A_p^{2\nu+1} \to A_p^{2\nu+1}$ by

$$\psi_a : x \to w_a x.$$

On the other hand there exists a non-trivial homomorphism $\psi : A_p^{2\nu} \to A_p^{2\nu+1}$. Then the relations (5) give the following split in direct sum

$$A_p^{2\nu+1} = \bigoplus \psi_a \psi(A_p^{2\nu}), \tag{5'}$$

all summants being isomorphic to $A_p^{2\nu}$.

Remark. Theorem 1 is a natural generalization of the well known results about the classical Clifford algebras. The proof of this theorem is also a natural generalization of the classical case, replacing -1 by a primitive root ω_p.

A. O. Morris gave [11] the structure of A_p^n in the case when $K(p)$ is not $K(2p)$ ($\sqrt{\omega_p} \notin K(p)$). We do not insist about this case because it is not necessary for the theory of maximal graduations which we shall develop.

Relations (1) and (2) give [26]:

$$\left(\sum_i \lambda_i e_i\right)^p = \sum_i \lambda_i^p e_0 , \qquad (\lambda_i \in K(p)). \tag{6}$$

Remark. The algebras A_p^n are not particular cases of formal theory of Clifford algebras [2], [4]. It is necessary a generalization based on the replacing of quadratic forms by multilinear forms; this fact is indicated by (6).

III. ASSOCIATIVE SIMPLE MAXIMALE GRADUATIONS

We consider three types of finite dimensional algebras over a field K, namely: associative, Lie and Jordan algebras.

Definition 2. *Let A be an algebra over K, Γ an abelian group, B a basis of A and $\Theta : \Gamma \times \Gamma \to K$ a function.*

The triplet $\mathscr{G} = \{\Gamma, B, \Theta\}$ is an maximal Γ-graduation (or maximal graduation) of A if exists a bijection $\Gamma \to B (\alpha \to e_\alpha)$ such that

$$e_\alpha e_\beta = \Theta(\alpha, \beta)\, e_{\alpha+\beta}, \qquad (\alpha, \beta \in \Gamma). \tag{7}$$

If A is simple or semi-simple, $\mathscr{G}$ will be named simple or semi-simple respectively.

We remark that (3) define a maximal Z_p^2-graduation for the generalized Clifford algebra A_p^n. By theorem 1 this maximal graduation is simple for n even and semi-simple for n odd.

We agree that maximal graduation (3) to be defined over algebra $K(2pp^\nu)$ if $n = 2\nu$ and $K(p) = K(2p)$.

Definition 3. *The maximal graduation $\mathscr{G}' = \{\Gamma, B', \Theta'\}$ of A (where $B' = \{e'_\alpha\}_{\alpha \in \Gamma}$) is isomorphic to $\mathscr{G}$ if $\Theta = \Theta'$, i.e. if $e_\alpha \to e'_\alpha$ is an isomorphism of A.*

For instance, the operators (4) and the operators:

$$f_1' : v_a \to \omega_p^{-a} v_a \qquad f_2' : v_a \to v_{a+1} \qquad (a \in Z_p)$$

define, by proposition 1 and theorem 1, two isomorphic maximal graduations of type (3') of $K(p)_p$.

Let $\mathscr{M}$ be the set of all maximal Γ-graduations of A. Every system of scalars $k_\alpha \in K (k_\alpha \neq 0)$ defines an operator $N : \mathscr{M} \to \mathscr{M}$ so

$$e'_\alpha = k_\alpha e_\alpha, \qquad (\alpha \in \Gamma) \tag{7'}$$

If $k_\alpha = k$ for every $\alpha \in \Gamma$, we write $N(\mathscr{G}) = k\mathscr{G}$.

Let $\mathscr{M}'$ be the set of all maximal Γ-graduations of A (Γ, Γ' isomorphic). Every isomorphism $h : \Gamma' \to \Gamma$ defines an operator $H : \mathscr{M} \to \mathscr{M}'$ so:

$$e'_\alpha = e_{h(\alpha)} \qquad (\alpha \in \Gamma') \tag{7''}$$

Clearly $\mathscr{G}' = N \circ H(\mathscr{G})$ implies $\mathscr{G}' = H \circ N'(\mathscr{G})$ where N' has the same form as N. Now we introduce an equivalence relation in the set of maximal graduations of A.

Definition 4. *Two maximal graduations $\mathscr{G}$ and $\mathscr{G}'$ of A are equivalent ($\mathscr{G} \sim \mathscr{G}'$) if exists an operator N of type* (7′) *and an operator H of type* (7″) *such that the maximal graduations $N \circ H(\mathscr{G})$ and $\mathscr{G}'$ are isomorphic.* If the coefficients k_α of (7″) belong to a subset K' of K we say that $\mathscr{G}$ and $\mathscr{G}'$ are K'-equivalent. We consider two cases: a) $K' = \varepsilon = \{+1, -1\}$, if K is an arbitrary field, b) $K' = R$ (real field) if $K = C$ (complex field).

We notice that the maximal graduation (3) is independent of ω_p up to a transformation (7″). We shall characterize the set of maximal graduations of an algebra A using the above equivalence relation.

If A has unity e, then e_0 and e are collinear. By the transformation (7′) we may choose $e = e_0$, that is

(8) $$\Theta(\alpha, 0) = \Theta(0, \alpha) = 1, \qquad (\alpha \in \Gamma).$$

In the following we always suppose that (8) *holds.*

This fact implies $k_0 = 1$ in (7′).

The algebra A is associative if

(9) $$\Theta(\alpha, \beta)\,\Theta(\alpha + \beta, \gamma) = \Theta(\beta, \gamma)\,\Theta(\alpha, \beta + \gamma), \qquad (\alpha, \beta, \gamma \in \Gamma).$$

Let $\mathscr{G}_i = \{\Gamma_i, B_i, \Theta_i\}$ be a maximal graduation of an associative algebra A_i over $K (i = 1, 2)$. We define $\Theta : \Gamma_1 \times \Gamma_2 \to K$ as follows

$$\Theta((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \Theta(\alpha_1, \beta_1)\,\Theta(\alpha_2, \beta_2), \qquad (\alpha_i, \beta_i \in \Gamma_i).$$

Then the algebra $A_1 \otimes A_2$ has the maximal graduation $\mathscr{G}_1 \otimes \mathscr{G}_2 = \{\Gamma_1 \times \Gamma_2, B_1 \otimes B_2, \Theta\}$ which we call *tensorial product of* $\mathscr{G}_1$ with $\mathscr{G}_2$.

Inductively one can define the tensorial product of a finite number of maximal graduations. We denote by $\mathscr{G}^{[\nu]}$ the tensorial power ν of $\mathscr{G}$.

Let $\mathscr{A}_p$ be the maximal graduation (3′) of A_p or $K(p)_p$. It is easily proved that:

Proposition 2. *The maximal graduation* (3) *of generalized Clifford algebra $A_p^{2\nu}(n = 2\nu)$ over $K(2p)$ is equivalent to $\mathscr{A}_p^{[\nu]}$.*

As we know, every finite abelian group G_n with n elements has a representation of the form:

(10) $$G_n \approx Z_{q_1} \times Z_{q_2} \times \ldots \times Z_{q_r} \qquad (q_1 \cdot q_2 \cdot \ldots \cdot q_r = n)$$

where

(10′) $$q_i = p_i^{m_i} \qquad (p_i \text{ — primes}, i = 1, 2, \ldots, r)$$

The numbers r and q_i are invariants for G_n.

By theorem 1, the algebra $K(n)_n$ has the following maximal $G_n \times G_n$-graduation

(11) $$\mathscr{G}_n = \mathscr{A}_{q_1} \otimes \mathscr{A}_{q_2} \otimes \ldots \otimes \mathscr{A}_{q_r}.$$

Two maximal graduations (11) are equivalent iff their graduation groups are isomorphic.

Let K be an algebraic closed field whose characteristic is not a divisor of n. Then $K = K(n)$ and the maximal graduation $\mathscr{G}_n$ of K_n has a sense.

Theorem 2. *Let $\mathscr{G}$ be a maximal graduation of K_n. If the field K is algebraic closed and its characteristic is not a divisor of $2n$, then $\mathscr{G}$ is a maximal $G_n \times G_n$-graduation equivalent to a maximal graduation $\mathscr{G}_n$ given by* (11).

Proof. Let $\mathscr{G} = \{\Gamma, B, \Theta\}$ where $B = \{e_\alpha\}_{\alpha \in \Gamma}$ and

$$\Gamma = Z_{q_1} \times Z_{q_2} \times \ldots \times Z_{q_s}. \tag{12}$$

By choosing one generator ϱ_i of Z_{q_i} $(i = 1, 2, \ldots, s)$, the elements of Γ are

$$\alpha = \sum_i \alpha_i \varrho_i, \qquad \beta = \sum_i \beta_i \varrho_i, \ldots$$

where $\alpha_i, \beta_i, \ldots$ are quotient classes modulo p. We denote $e_i = e_{\varrho_i}$. The algebra K_n being simple, we have $\Theta(\alpha, \beta) \neq 0$ for all $\alpha, \beta \in \Gamma$. Then we take (up to a transformation (7')):

$$e_i^{q_i} = e_0\,; \; e_j e_i = \omega_{q_i}^{\lambda_{ij}} e_i e_j = \omega_{q_j}^{\mu_{ij}} e_i e_j \qquad (i, j = 1, 2, \ldots, s, i \neq j) \tag{1'}$$

$$e_\alpha = e_1^{\alpha_1} \,.\, e_2^{\alpha_2} \ldots e_s^{\alpha_s}, \tag{2'}$$

the form of Θ being obtained by generalization of (3).

Let $\Gamma = \Gamma_1 \times \Gamma_2$, where all the elements of Γ_1 have the period p and those of Γ_2 have the period q, p and q being relatively prime. Then (1') implies $\mathscr{G} = \mathscr{G}_1 \otimes \mathscr{G}_2$, $\mathscr{G}_i$ being maximal Γ_i-graduations. Thus, we may suppose in (12) $q_i = p^{m_i}$ (p-prime) and $q_1 \geqq q_2 \geqq \ldots \geqq q_s$. Using the fact that $\mathscr{G}$ is simple we prove that $q_1 = q_2$ and $(\lambda_{12}, {}_{q_1}) = 1$.

We can also (by a convenient transformation (7'')) obtain

$$\lambda_{1i} = \lambda_{2i} = \mu_{1i} = \mu_{2i} = 1 \qquad (i > 2) \text{ and hence } \mathscr{G} \sim \mathscr{A}_{q1} \otimes \mathscr{G}'.$$

We prove by induction that $\mathscr{G}$ is of the form (11). The detailed proof is given in [17].

By theorem 2 two maximal graduations of K_n are equivalent iff their groups are isomorphic. The graduation group defines uniquely (up to an equivalence) the corresponding maximal graduation.

Remark. It is known that any simple associative algebra over K (algebraic closed) has the form K_n. The theorem 2 gives all simple associative maximal graduations over an algebraic closed field of characteristic zero. Under the same hypotheses one can construct all semi-simple associative maximal graduations

In [6] it is determined simple associative maximal graduations over the quaternion field Q. An algebra over Q is, by definition, a quaternion extension of a real algebra. Therefore the matrix algebras Q_n define the class of all simple associative algebras over Q. In the definition of a maximal graduation $\mathscr{G} = \{\Gamma, B, \Theta\}$ over Q it is supposed that Θ commutes with the elements of B. In the definition of N given by (7') it is supposed that k_α commute by Θ and also by the elements of B. These additional hypotheses make natural the notions of maximal graduations and equivalence.

We establish the existence of the maximal graduations (3') of Q_p for any primitive root of the unity $\omega_p \in Q$. For $p > 2$ these graduations are in $1 - 1$ correspondence, up to an equivalence, with the complex projective line P and we write $\mathscr{A}_p(z)$; $\mathscr{A}_2(z)$ does not depend on z. We obtain the following result:

Theorem 3. *Any maximal graduation $\mathscr{G}$ of Q_n is a maximal $G_n \times G_n$-graduation. If G_n is of the form* (11), *then*

$$\mathscr{G} \sim \mathscr{A}_{q_1}(z) \otimes \mathscr{A}_{q_2}(z) \otimes \ldots \otimes \mathscr{A}_{q_r}(z) \qquad (z \in P). \tag{12'}$$

Therefore, the group uniquely determine the maximal graduation iff all its elements has the period 2. Otherewise, to every abelian group $G_n \times G_n$ corresponds a set of simple associative maximal graduations which is in $1 - 1$ correspondence with the complex projective line.

IV. SIMPLE SPECIAL MAXIMAL GRADUATION

To every associative algebra A over K (with characteristic $\neq 2$) corresponds one Lie algebra A_L and one Jordan algebra A_J [8], [9] (called special algebras) using the following multiplication lows

$$\begin{cases} [a, b] = ab - ba \\ a \bullet b = \frac{1}{2}(ab + ba) \end{cases} \qquad (a, b \in A), \tag{13}$$

where juxtaposition denotes the multiplication in A.

Definition 5. *Let $\mathscr{G} = \{\Gamma, B, \Theta\}, B = \{e_\alpha\}_{\alpha \in \Gamma}$, be a maximal graduation of A .The relations*

$$\begin{cases} [e_\alpha, e_\beta] = [\Theta(\alpha, \beta) - \Theta(\beta, \alpha)]\, e_{\alpha+\beta} \\ e_\alpha \bullet e_\beta = \frac{1}{2}[\Theta(\alpha, \beta) + \Theta(\beta, \alpha)]\, e_{\alpha+\beta}, \end{cases} \tag{13'}$$

define a maximal graduation $\mathscr{G}_L$ of A_L and a maximal graduation $\mathscr{G}_J$ of A_J respectively.

We call $\mathscr{G}_L$ and $\mathscr{G}_J$ *special maximal graduations*. $\mathscr{G}_L$ or $\mathscr{G}_J$ are called *simple* when A_L or A_J are simple.

In the following we construct all special maximal graduations of Lie and Jordan simple real forms of type A: precisely, the Lie algebras will be supposed simple up to an one-dimensional center. We consider separately the type A_{I} and A_{II}.

The Jordan simple real forms of type A_{I} are $R_{nJ} (n > 1)$ and $(R_n \otimes Q)_J$ $(n > 0)$ [9] and the Lie simple real forms of the same type are, up to an one-dimensional center, $R_{nL}(n > 1)$ and $(R_n \otimes Q)_L$ $(n > 0)$ [8].

Remark. The unities $1, i, j, k$ of real quaternion algebra Q define a maximal Z_2^2-graduation $\mathscr{B}$, if we take

$$e_0 = 1, \quad e_{(1,0)} = i, \quad e_{(0,1)} = j, \quad e_{(1,1)} = k. \tag{14}$$

$\mathscr{B}$ may be extended to algebra C_2 and we have $\mathscr{A}_2 \sim \mathscr{B}$, but this is not an R-equivalence.

Theorem 4. *Any simple maximal special Jordan graduation of type A_{I} is equivalent to $\mathscr{A}_{2J}^{[\nu]}$ $(\nu \geqq 1)$ or to $(\mathscr{A}_2^{[\nu]} \otimes \mathscr{B})_J (\nu \geqq 0)$. Any simple maximal special Lie graduation (up to an one-dimensional center) of type A_{I} is equivalent to $\mathscr{A}_{2L}^{[\nu]}$ $(\nu \geqq 1)$ or to $(\mathscr{A}_2^{[\nu]} \otimes \otimes \mathscr{B})_L$ $(\nu \geqq 0)$.*

Proof. Let $\mathscr{G} = \{\Gamma, B, \Theta\}$ be a maximal graduation of R_n or $R_n \otimes Q$, where $B = \{e_\alpha\}_{\alpha \in \Gamma}$ and Γ is given by (12). The complex extension of $\mathscr{G}$ is simple and, by (7′) we obtain a maximal graduation of the form (1′), (2′).

Moreover, $\omega_t = -1$ because $\mathscr{G}$ is real.

By theorem 2 it follows the R-equivalence

$$\mathscr{G} \sim \mathscr{A}_2^{[a]} \otimes \mathscr{B}^{[b]} \otimes \mathscr{C}^{[c]} \tag{15}$$

where $\mathscr{C}$ is the following maximal Z_2^2-graduation:

$$e_{(1,0)}^2 = -e_{(0,1)}^2 = e_0, \qquad e_{(1,0)}e_{(0,1)} + e_{(0,1)}e_{(1,0)} = 0, \tag{14'}$$
$$e_{(1,1)} = e_{(1,0)}e_{(0,1)}$$

The automorphism of Z_2^2:

$$(1,0) \to (1,1),\ (0,1) \to (0,1) \tag{16}$$

and the automorphism of Z_2^4:

$$\begin{gathered}(1,0,0,0) \to (1,0,1,1);\ (0,1,0,0) \to (0,1,1,1)\\ (0,0,1,0) \to (1,1,1,0);\ (0,0,0,1) \to (1,1,0,1)\end{gathered} \tag{16'}$$

define two operators (7″) which show that the pairs $\mathscr{B}, \mathscr{C}$ and $\mathscr{A}_2 \otimes \mathscr{B}, \mathscr{B} \otimes \mathscr{B}$ contain ε-equivalent maximal graduations. So, the theorem 4 is a consequence of (15).

Thus, the problem of finding special maximal graduations of type A_{I} is completely solved.

As we know, Lie and Jordan simple real forms of type A_{II} are not special algebras as in the case A_{I}. That impose the extension of the notion of special maximal graduation. On the other hand, we shall introduce the notion of normed maximal graduation which we shall use later on.

Definition 6. *Let $\mathscr{G} = \{\Gamma, B, \Theta\}$ be a maximal graduation of an associative algebra A over the complex field C. If $\Theta(\alpha, \beta) + \Theta(\beta, \alpha) \in R$ for all $\alpha, \beta \in \Gamma$, then $\mathscr{G}_J$ is a maximal graduation of a real form of A_J. If $\Theta(\alpha, \beta) - \Theta(\beta, \alpha) \in R$ for all $\alpha, \beta \in \Gamma$, then $\mathscr{G}_L$ is a maximal graduation of a real form of A_L. We call $\mathscr{G}_J$ and $\mathscr{G}_L$ special maximal graduations.*

It is clearly for $K = C$ that the notion given by definition 5 is a particular case. This fact justify the preserving of the name.

Definition 7. *Let $\mathscr{G} = \{\Gamma, B, \Theta\}$ be a maximal graduation of an associative algebra A over K with unity. We say that $\mathscr{G}$ is normed if*

$$\Theta(\alpha, \beta)\,\Theta(\beta, \alpha) = 1, \qquad (\alpha, \beta \in \Gamma). \tag{17}$$

Remark. It is easily proved for $K = C$ that function Θ of a normed maximal graduation G is unimodular. Then maximal graduations $\mathscr{G}_J$ and ${}^i\mathscr{G}_L (i^2 = -1)$ are special in the sens of definition 6. We shall give explicitly the connection between these two notions in the case $A = C_n$.

We denote by $S_p = \begin{pmatrix} -I_p & 0 \\ 0 & I_{n-p} \end{pmatrix}$ (I_p — unity matrix of order p).

The relations

$$S_pX^*S_p = X, \qquad S_pX^*S_p = -X, \qquad (X \in C_p), \tag{18}$$

where X^* is the adjoint of X, define a real form $J(n, p)$ of C_{nJ} and a real form $L(n, p)$ of C_{nL} respectively. We say that they are real forms of type A_{II} [8], [9]. If we wish to obtain all real forms of type A_{II} which are not isomorphic, we take $p = 0, 1, 2, \ldots$ $[n/2]$. $J(n, p)$ is simple and $L(n, p)$ too, if we omite its one-dimensional center. Therefore, we have $L(n, p) \approx L'(n, p) \times R$ where $L'(n, p)$ is simple. On the other hand, any involution π of indices $1, 2, \ldots, n$ define an antiinvolution $X \to X^*$ of C_n by

$$(19) \qquad (X^+)^a_b = (X^+)^{-\pi(a)}_{\pi(b)} \quad , \quad (a, b = 1, 2, \ldots, n)$$

where $(X^+)^a_b$ is the element of matrix X^+ on a-line and b-row.

The relations

$$(18') \qquad X - X^+ = 0, \qquad X + X^+ = 0, \qquad (X \in C_n),$$

define a real form $J(n, \pi)$ of C_{nJ} and a real form $L(n, \mathscr{G})$ of C_{nL} respectively.

We have [18]:

Proposition 3. *If the involution π has n-$2p$ invariant indices, then $J(n, p) \approx J(n, \pi)$ and $L(n, p) \approx L(n, \pi)$. Thus $J(n, \pi) \approx J(n, \pi')$ or $L(n, \pi) \approx L(n, \pi')$ iff the involutions π and π' are equivalent (if they have the same number of invariant indices). We also have $L(n, \pi) \approx L'(n, \pi) \times R$ where $L'(n, \pi)$ is simple.*

Proposition 4. *Let $\mathscr{G}$ be a maximal graduation of C_n. If $\mathscr{G}_J$ is special of type A_{II}, then $\mathscr{G} = N(\mathscr{G}')$ where $\mathscr{G}'$ is normed and N is a real operator* (7'). *If $\mathscr{G}_L$ is special of type A_{II}, then $\mathscr{G} = N(\mathscr{G}')$ where $\mathscr{G}'$ is normed and N is an imaginary operator* (7').

Proof. We consider only Jordan case, the proof of Lie case being analogous. Let $\mathscr{G} = \{\Gamma, B, \Theta\}$, where $B = \{E_\alpha\}_{\alpha \in \Gamma}$. We can do a real trnasformation (7') so that Θ becomes unimodular. On the other hand, by proposition 3 we can suppose $E^+_\alpha = E_\alpha$.

Then the relations

$$E_\alpha E_\beta = \Theta(\alpha, \beta)\, E_{\alpha+\beta} \text{ and } E^+_\beta E^+_\alpha = \Theta^{-1}(\alpha, \beta)\, E^+_{\alpha+\beta}$$

imply (17).

Therefore, the determination of special maximal graduations of type A_{II} can be made by determining of complex simple normed maximal graduations, up to an R-equivalence.

The study of simple normed maximal graduations can be made for the more general field. We shall give the canonical forms of these graduations up to an ε-equivalence. We show then that in the case of complex field, the study of R-equivalence is reduced to the study of ε-equivalence.

V. SIMPLE NORMED MAXIMAL GRADUATIONS

First of all we give two general properties of normed maximal graduations over arbitrary fields [17].

Proposition 5. *The function Θ of a normed maximal Γ-graduation verifies the following two conditions:*

$$(20) \qquad \Theta(\alpha, -\alpha) = \varepsilon_\alpha \qquad (\varepsilon_\alpha = \varepsilon_{-\alpha} = \pm 1,\ \alpha \in \Gamma)$$

$$(20') \qquad \Theta(-\alpha, -\beta) = \varepsilon_\alpha \varepsilon_\beta \varepsilon_{\alpha+\beta}\, \Theta(\alpha, \beta) \qquad (\alpha, \beta \in \Gamma)$$

Theorem 5. *The basis $\{E_\alpha\}$ of a normed maximal Γ-graduation of K_n verifies the following relation of Pauli's type*:

$$\sum_{\alpha\in\Gamma} \varepsilon_\alpha (E_\alpha)^b_a (E_{-\alpha})^d_c = n\delta^d_a\delta^b_c, \tag{21}$$

under the hypothesis that the characteristic of K is not a divisor of n.

It results immediately that the matrices E_0 have the nulltrace for $\alpha \neq 0$.

Now we construct two normed maximal graduations of $K(2p)_p$ which are obtained from (3′) by a transformation N given by (7′).

We designe by

$$E_\alpha = k_\alpha e_\alpha, \qquad (\alpha \in Z^2_p), \tag{22}$$

where e_α are given by (1) and (2) with $n = 2$.

If p is odd, we take:

$$k_\alpha = \omega_p^{\frac{(p+1)\alpha_1\alpha_2}{2}} \tag{23}$$

If p is even we associate to every $\alpha \in Z^2_p$ the supplementary component α_3 given by: $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let be α'_s, $\alpha'_s \in \alpha_s$, $0 \leq \alpha'_s < p$, $(s = 1, 2, 3)$. Here we define the coefficients k_α of (22) by

$$k_\alpha = \omega_{2p}^{\sum_{s>t} \alpha'_s\alpha'_t + \alpha'^2_3} \tag{23}$$

where $s, t = 1, 2, 3$ and $\omega^2_{2p} = \omega_p$.

We designe by:

$$\mathscr{A}^0_p = N(\mathscr{A}_p) = \{Z^2_p, B, \Theta\}.$$

We have

$$\Theta(\alpha, \beta) = \begin{cases} \omega_p^{\frac{(p+1)(\alpha_2\beta_1-\alpha_1\beta_2)}{2}} & (p \text{ odd}) \\ \omega_{2p}^{\sum_{s>t}(\alpha'_s\beta'_t-\alpha'_t\beta'_s)} & (p \text{ even}) \end{cases} \tag{24}$$

For p even we designe $\mathscr{A}^1_p = N'(\mathscr{A}^0_p)$, where the operator N' is given by

$$F_\alpha = \omega_{2p}^{\alpha'_1} E_\alpha, \qquad (\alpha \in Z^2_p). \tag{22′}$$

Propostion 6. *The maximal graduations $\mathscr{A}^0_p$ and $\mathscr{A}^1_p$ of $K(2p)_p$ are normed. The constants ε_α defined by* (20) *are*

$$\varepsilon_\alpha = 1 \qquad \textit{for } \mathscr{A}^0_p \tag{25}$$

$$\varepsilon_\alpha = \begin{cases} 1 & (\alpha_1 = 0) \\ -1 & (\alpha_1 \neq 0) \end{cases} \qquad \textit{for } \mathscr{A}^1_p, \textit{ respectively.} \tag{25′}$$

Suppose that in formulas (10), (10′) of definition of abelian group G_n we have:

$$p_1 = p_2 = \ldots = p_s = 2; \qquad p_i \neq 2 \text{ for } i > s, \tag{26}$$

and that the sequence $q_1, q_2, \ldots, q_s$ contains t different terms $q'_1 > q'_2 > \ldots > q'_t$. If s_j terms are equal to q_j, we can write

$$q_1 = \ldots = q_{s_1} = q'_1 > q_{s_1+1} = \ldots = q_{s_1+s_2} = q'_2 > \ldots \\ \ldots > q_{s-s_t+1} = \ldots = q_s = q'_t \qquad (s_1 + \ldots + s_t = s) \tag{26′}$$

The invariant t of the group G_n is zero iff n is odd.

We associate to every integer $j = 0, 1, \ldots, t$ the following normed maximal $G_n \times G_n$ — graduation of $K(2p)p$:

(27) $$\mathscr{G}^j_n = \mathscr{A}^{x_1}_{q_1} \otimes \ldots \otimes \mathscr{A}^{x_s}_{q_s} \otimes \mathscr{A}^0_{q_{s+1}} \otimes \ldots \otimes \mathscr{A}^0_{q_r}$$

where

$$x_i = \begin{cases} 1 & \text{if } i = s_1 + \ldots + s_{j-1} + 1 \\ 0 & \text{otherwise.} \end{cases}$$

For instance, if $G_n = Z_4^2 \times Z_2^3 \times Z_3$, then $t = 2$ and maximal graduations (27), (27′) are:

$$\mathscr{G}^0_n = \mathscr{A}^{0[2]}_4 \otimes \mathscr{A}^{0[3]}_2 \otimes \mathscr{A}^0_3,$$

$$\mathscr{G}^1_n = \mathscr{A}^1_4 \otimes \mathscr{A}^0_4 \otimes \mathscr{A}^{0[3]}_2 \otimes \mathscr{A}^0_3,$$

$$\mathscr{G}^2_n = \mathscr{A}^{0[2]}_4 \otimes \mathscr{A}^1_2 \otimes \mathscr{A}^{0[2]}_2 \otimes \mathscr{A}^0_3.$$

Maximal graduations $\mathscr{G}^j_n$ and $\mathscr{G}^{j'}_n$ are ε-equivalent iff $j = j'$.

Theorem 6. *Let K be an algebraic closed field whose characteristic is not divisor of $2n$. Then any normed maximal graduation of K_n is ε-equivalent to a $\mathscr{G}^j_n$. Two normed maximal graduations of C_n are R-equivalent iff they are ε-equivalent.*

For the proof of this theorem we take in consideration the fact that any normed maximal graduation of K_n is, by theorem 2 of the form $N(\mathscr{G}_n)$, where $\mathscr{G}_n$ is given by (11) and N is an operator (7′), then (20) and (20′) give a special form to k_α, which reveals the equivalence between $N(\mathscr{G}_n)$ and $\mathscr{G}_n$.

The theorem 6 determine *all* simple normed maximal graduations over an algebraic closed field with characteristic zero.

By proposition 4 and theorem 6, it results that simple special maximal graduations of type A_{II} can be find among the maximal graduations of the form $\mathscr{G}^j_{nJ}$ and $\mathscr{G}^j_{nL}$ ($i^2 = -1$). We show that all those graduations are of type A_{II}. For that we u se the notion of unitary maximal graduation.

Definition 8. *Let $\mathscr{G}$ be a normed maximal $G_n \times G_n$-graduation of C_n with basis $\{E_\alpha\}$. We say that $\mathscr{G}$ is unitary of the first kind if*

(28) $$E^*_\alpha = \varepsilon_\alpha E_{-\alpha} \qquad (\alpha \in G_n \times G_n),$$

where ε_α are defined by (20). *We say that $\mathscr{G}$ is unitary of the second kind if there exists an involution π of the indices* 1, 2, ..., *n such that*

(28′) $$E^+_\alpha = E_\alpha,$$

where the map $X \to X^+$ is the antiinvolution of C_n given by (19).

The first kind of graduation coincide with the second if $G_n = Z_2^k$, $\varepsilon_\alpha = 1$ and π is identity involution.

It is easily proved that the set of all unitary maximal graduations of the first kind of C_n isomorphic to a given unitary maximal graduation of the first kind, is the unitary complex projective group PU_n in $n - 1$ independent variables.

In the case of unitary maximal graduations of the second kind, we consider the Lie group $\mathscr{L}(n, \pi)$ associated to the algebra $L(n, \pi)$, defined by

(29) $$SS^+ = \varepsilon I_n, \ [\varepsilon = \pm 1, S \in GL(n, C)].$$

If the number q of invariant indices by π is zero we have both $\varepsilon = \pm 1$. Otherwise $\varepsilon = 1$. The group $\mathscr{L}(n, \pi)$ is connex if $q \neq 0$ and has two connex components if $q = 0$.

Let c be the center of $\mathscr{L}(n, \pi)$ (which coincides with the multiplicative group of unimodular complex numbers) and $\mathscr{L}'(n, \pi) = \mathscr{L}(n, \pi)/c$.

If π is identity involution, then $\mathscr{L}'(n, \pi)$ coincides with PU_n.

Proposition 7. *Let $\mathscr{G}$ be an unitary maximal $G_n \times G_n$-graduation of the second kind of C_n constructed by using the involution π and let $n - 2p$ be the number of invariant indices by π. Then the set of unitary maximal graduations of the second kind isomorphic to $\mathscr{G}$ and which correspond to the same involution π is given by the group $\mathscr{L}'(n, \pi)$; the set of all unitary maximal graduations of the second kind isomorphic to $\mathscr{G}$ is a manifold V given by $1.\ 3.\ 5 \ldots (2p - 1)\ C_n^{2p}$ disjoint copies of $\mathscr{L}'(n, \pi)$. If π is not identity involution and $n > 2$, then V is not a Lie group.*

We note that the unitary maximal graduations of the second kind define, by (28′) and proposition 3, special maximal graduations of type A_{II}. This fact is connected with the existence of canonical forms (27), (27′) of the simple normed maximal graduations, according to the following result:

Theorem 7. *Let G_n be an abelian group with n elements. Let t be the invariant of G_n given by* (26), (26′). *Then any normed maximal graduation $\mathscr{G}_n^j$ $(j = 0, 1, \ldots, t)$ is unitary both of the first and second kind, up to an isomorphism. If $j = 0$ then the corresponding involution π has the form: $\pi : a \to -a (a \in G_n)$. If $j \neq 0$, then π has not invariant indices.*

The proof is based on the fact that the operators f_1 and f_2 defined by (4) are invariant by antiinvolutions $X \to X^+$ for which $\pi : a \to -a$ or $\pi : a \to -a - 1$, $(a \in Z_p)$.

From proposition 4 and theorem 6 follows:

Theorem 8. *Any Jordan or Lie special maximal graduation of type A_{II} is a maximal $G_n \times G_n$-graduation which is equivalent to a $\mathscr{G}_{nJ}^j$ or to an $i\mathscr{G}_{nL}^j$ $(i^2 = -1)$ $(j = 0, \ldots, t)$, respectively. The corresponding real forms are:*

$$\text{a) } J\left(n, \left[\frac{n}{2}\right]\right), \quad L\left(n, \left[\frac{n}{2}\right]\right) \quad \textit{for } n \textit{ odd},$$

$$\text{b) } J\left(n, \left[\frac{n}{2}\right]\right), \quad L\left(n, \left[\frac{n}{2}\right]\right) \quad \textit{for } n \textit{ even and } j \neq 0,$$

$$\text{c) } J(n, p), \quad L(n, p) \quad \textit{for } n \textit{ even and } j = 0,$$

where $n - 2p$ is the number of the elements of period 2 (zero, too) of G_n.

Therefore the graduation group $G_n \times G_n$ uniquely determines the Lie or Jordan special maximal graduation iff n is odd (case a). For n even (cases b) and c)) to every abelian group $G_n \times G_n$ correspond $t + 1$ such graduations, t being the invariant of G_n given by (26′). The only Lie or Jordan unitary real forms of type A which admit special maximal graduations correspond to $n = 2^m$ and their special graduations are uniquely determined.

Suppose now $n = 2^m$ and let $\nu(p)$ be the number of Lie or Jordan maximal graduations corresponding to the case c) for given p and n. We remark that:

$$p = 2^{m-1} - 2^{s-1},$$

where s is the minimum number of generators of G_{2^m}. A straightforward calculus gives:

$$\nu(2^{m-1}-1)=1\ ,\quad \nu(2^{m-1}-2)=\left[\frac{m}{2}\right]$$

$$\nu(2^{m-1}-4)=\begin{cases}3k^2 & m=6k\\ 3k^2+k & m=6k+1\\ 3k^2+2k & m=6k+2\\ 3k^2+3k+1 & m=6k+3\\ 3k^2+4k+1 & m=6k+4\\ 3k^2+5k+2 & m=6k+5\end{cases}$$

The calculus of $\nu(p)$ for an arbitrary s is difficult. Similar considerations may be made also in the case b).

VI. EXISTENCE AND CONSTRUCTION OF METRICS

In this section we give some geometrical results which are obtained by association to any real algebra a space with affine connexion [24]. This fact gives a geometrical aspect to certain algebraic results.

Clearly there exists duality between Jordan real forms and Lie real forms of type A as follows:

$$R_{nJ} \rightleftarrows R_{nL}$$

$$(R_n \otimes Q)_J \rightleftarrows (R_n \otimes Q)_L$$

$$J(n,p) \rightleftarrows L(n,p).$$

Let $A \rightleftarrows A'$ be two such dual algebras and let $\{e_k\}$, $\{e'_k\}$ be two their dual bases.
If A and A' are of type A_{I}, then $e_k = e'_k$.
If A and A' are of type A_{II}, then $e_k = ie'_k$ ($i^2 = -1$).
Let Γ^i_{jk}, Γ'^i_{jk} be the structure constants in the choosen basis.

Proposition 8. *Γ^i_{jk} and Γ'^i_{jk} being the components of two affine constant connexions and Γ^i_{jkl} the curvature tensor, we have*

$$\Gamma^i_{jkl} = \pm\frac{1}{4}\Gamma'^i_{jp}\Gamma'^p_{kl} \qquad \begin{matrix}(-\textit{for the type } A_{\mathrm{I}})\\ (+\textit{for the type } A_{\mathrm{II}})\end{matrix} \tag{30}$$

Therefore, the curvature tensor associated to a Jordan real form of type A may be calculated by means of the connexion associated to dual Lie real form.

We apply this result in the case of Jordan simple special maximal graduations of type A.

Theorem 9. *The only Jordan special maximal graduations of type A which have metrics are*

$$\mathscr{A}^0_{2J},\ \mathscr{A}^1_{2J},\ \mathscr{A}_{2J},\ \mathscr{B}_J,$$

the corresponding metrics being, up to a constant factor:

$$\begin{aligned} ds^2 &= e^{-2x}(dx^2 - dy^2 - dz^2 - dt^2)\\ ds^2 &= e^{-2x}(dx^2 + dy^2 + dz^2 - dt^2)\\ ds^2 &= e^{-2x}(dx^2 + dy^2 - dz^2 - dt^2)\\ ds^2 &= e^{-2x}(dx^2 + dy^2 + dz^2 + dt^2)\end{aligned} \tag{31}$$

The detailed proof may be found in [19].

We remark the fact that the uniqueness of metrics is valid though the systems of linear operators associated to curvature tensors are complete reducible systems, as follows from proposition 8.

Therefore the metrics (31) give the possiblity of generalization of the Teleman's theorem [21].

We mention that in the case of this theorem the metric is a definitive one and the system of operators associated to curvature tensor is irreducible.

Let V_4, V_4', V_4'', V_4''' be the Riemannian spaces associated to metrics (31). These spaces cover the full class of non-isotropic Wagner 4-dimensional spaces with $p = -1$.

A. Turtoi has proved that the simple Jordan algebras of type D cover the full class of non-isotropic Wagner spaces with $p = -1$.

G. Vranceanu [24] has established the following result:

The metric

$$ds^2 = e^{-2x^1}[(dx^1)^2 + \dots + (dx^n)^2] \tag{32}$$

has negative plane curvatures. For $n = 4$ and $n = 6$ the generalized curvature of these metric is zero.

Remark. The spaces V_4, V_4' and V_4'' correspond to the case of nondefinite metric (32). The space V_4 has negative plane curvatures, while the spaces V_4' and V_4'' have not this property. The spaces V_4, V_4' and V_4'' have the generalized curvature zero, too.

We give a simple method for the construction of geodesics of the spaces V_4, V_4', V_4'' and V_4'''.

Let T be the translation group in x, y, z, t;

let O be the rotation group in y, z, t;

let O' be the proper transformation group which leaves invariant the quadratic form: $y^2 + z^2 - t^2$;

let O'' be the proper transformation group which leaves invariant the quadratic form: $y^2 - z^2 - t^2$;

let S be the rotation group in y, z;

let S' be the rotation group in z, t.

Now we consider the following curves:

$$C_1 : \begin{cases} x = \ln \operatorname{ch} t \\ y = z = 0 \end{cases} \qquad C_4 : \begin{cases} x = \ln \cos y \\ z = t = 0 \end{cases}$$

$$C_2 : \begin{cases} x = \ln \operatorname{sh} t \\ y = z = 0 \end{cases} \qquad C_5 : \begin{cases} y = t = e^{x} \\ z = 0 \end{cases}$$

$$C_3 : \begin{cases} x = \ln (-\operatorname{sh} t) \\ y = z = 0 \end{cases} \qquad C_6 : \begin{cases} y = t = e^{-x} \\ z = 0 \end{cases}$$

and the lines

$$D_1 : \begin{cases} x = t \\ y = z = 0 \end{cases} \qquad D_2 : \begin{cases} x = -t \\ y = z = 0 \end{cases} \qquad D_3 : \begin{cases} y = t \\ x = z = 0 \end{cases}$$

See fig. 1 and 2.

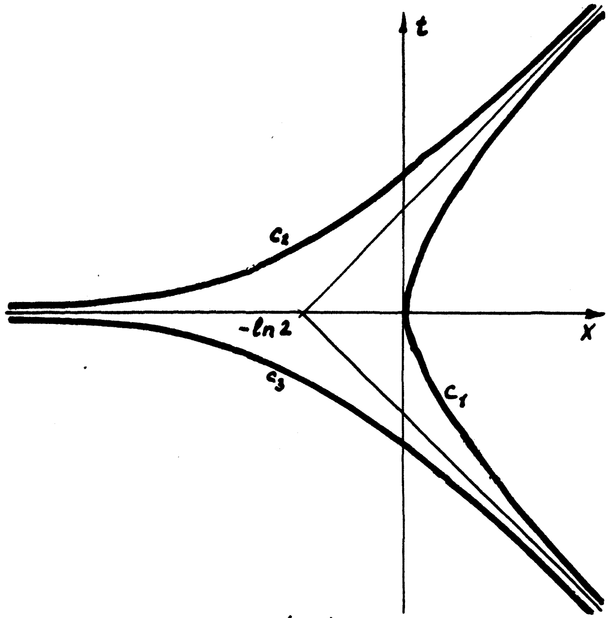

fig. 1

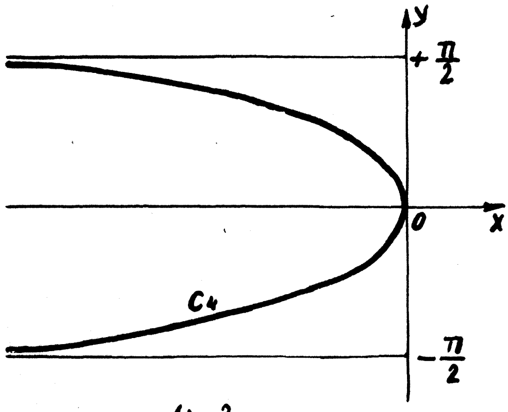

fig. 2

We consider the following families of curves:

$\mathscr{F}_1$ — the set of all parallel lines with Ox

$\mathscr{F}_2 = OT(C_1, C_2, C_3, D_1, D_2)$

$\mathscr{F}_3 = OT(C_4)$

$\mathscr{F}_4 = O'T(C_1, C_2, C_3, D_1, D_2)$

$\mathscr{F}_5 = O'T(C_4)$

$\mathscr{F}_6 = O''T(C_1, C_2, C_3, D_1, D_2)$

$\mathscr{F}_7 = O'''T(C_4)$

$\mathscr{F}_8 = ST(C_5, C_6, D_3)$

$\mathscr{F}_9 = S'T(C_5, C_6, D_3)$

Theorem 22. *The geodesics of the spaces V_4, V_4', V_4'' and V_4''' are: $\mathscr{F}_1 \cup \mathscr{F}_2$, $\mathscr{F}_1 \cup \cup \mathscr{F}_4 \cup \mathscr{F}_5 \cup \mathscr{F}_8$, $\mathscr{F}_1 \cup {}^6\mathscr{F} \cup \mathscr{F}_7 \cup \mathscr{F}_9$ and $\mathscr{F}_1 \cup \mathscr{F}_3$ respectively.*

The proof of this theorem is based on the fact that O is the stability group for V_4 and V_4''', O' is the stability group for V_4' and O'' is the stability group for V_4''. This remark reduces the construction of geodesics to the case of local euclidian spaces associated to complex number algebra, dual number algebra and parabolic number algebra [16].

The projection of an arbitrary curve of $\mathscr{F}_1, \ldots, \mathscr{F}_9$ on the hyperplane $x = 0$ is a line. Hence $V_4, \ldots, V_4'''$ are subprojective spaces of order 2.

Through 2 points of V_4 passes always *only one* geodesic.

Through 2 points P and Q of V_4''' passes one geodesic iff the euclidian distance between the projection of P and Q on hyperplane $x = 0$ is less than π.

F. Amato [1] has obtained the auto-parallel curves of the connexion associated to the quaternion algebra ommiting the fact that these curves are the geodesics of V_4'''.

As for the godesics of V_4', the families $\mathscr{F}_4$ and $\mathscr{F}_5$ have dual properties. The hyperplane $x = 0$ is split in two regions by the isotropic cone:

$$y^2 + z^2 - t^2 = 0$$

Any geodesic of $\mathscr{F}_4$ which passes through origine is of temporal type (is found into this cone). The goedesics of $\mathscr{F}_5$ are of spatial type and the geodesics of $\mathscr{F}_8$ are of isotropic type. We designe by P one point of $V_4'(P \in V_4')$.

If the projection of $\overline{OP}$ on $x = 0$ is a temporal or isotropic vector, then through O and P passes always *only one* geodesic. If this projection is a spatial vector, then through O and P passes only one geodesic iff the pseudoeuclidian length of this projection is less than π.

Similar considerations may be also made for V_4''.

REFERENCES

1. F. Amato, *Rev. Roum. Math. pures et appl.* XIV, 5 (1969), p. 601
2. M. F. Atiyah, R. Bott and A. Shapiro, *Topology* 3 (1964), p. 3—38
3. R. Brauer and H. Weyl, *Amer. J. Math.* 57 (1935), 425—449.
4. E. Cartan, *Leçons sur la théorie des spineurs*, Exposés de Géométrie, Paris, 1938.
5. C. Chevalley, *The construction and study of certain important algebras.* The Math. Soc. of Japan, 1955.
6. R. Iordanescu and I. Popovici, *Graduations maximales associatives simple quaternioniennes Rev. Roum. Math. pures et appl.* XV, 3 (1970), p. 359—366.
7. R. Iordanescu, *Géométrie différentielle sur les formes réelles de Jordan de type* A_I. Boll. Un. Mat. It. 4 (1970), p. 585—594.
8. N. Jacobson, *Lie algebras*, New—York—London, 1962.
9. N. Jacobson, *Lectures on Jordan algebras*, Lectures notes, Univ. of Chicago, Math. 446, 1964.
10. A. O. Morris, *Quart..J. of Math.* (Oxford), 18 (1967), p. 7—12.
11. A. O. Morris, *Quart. J. of Math.* (Oxford) 19 (1968).
12. A. Popovici, *Rev. Roum. Math. pures et appl.* V, 2 (1960).
13. A. Popovici, A. Hristev and I. Popovici, *C. R., Acad. Sci. Paris*, 257 (1963).
14. I. Popovici and C. Cheorghe, *Rev. Roum. Math. pures et appl.* XI, 8 (1966), p. 989.
15. I. Popovici and G. Gheorghe, *C. R. Acad. Sci. Paris*, 262(1966), p. 682.
16. I. Popovici, *Studii si Cercet. Mat.* 3 (1969), p. 463—470.
17. I. Popovici, *Rev. Roum. Math. pures et appl.*, XIV, 8 (1969), p. 1187.
18. I. Popovici, *Applications d'une théorie généralisée des spineurs. Rendiconti del Circolo Ma tematico di Palermo.* XVIII (1969), p. 23—39.
19. I. Popovici and A. Turtoi, *Propriétés métriques des formes réelles de Jordan de type* A_{II}. *Rev. Roum. Math. pures et appl.* XV, 2 (1970), p. 303—314.
20. P. K. Rashevski, *Usp. Mat. Nauk*, X, 2 (64) (1965), p. 3.
21. C. Teleman, *Rev. Roum. Math. pures et appl.*, III, 1 (1958) p. 107.
22. G. Vranceanu, *St. cerc. mat.* XII, 2 (1961), p. 319.
23. G. Vranceanu, *St. cerc. mat.* XIII, 2 (1962), p. 209.
24. G. Vranceanu, *Lectii de geometrie diferentiala*, vol. II (1964) vol. IV (1968) Bucuresti.
25. H. Weyl, *The classical groups*, Princeton, 1946.
26. K. Yamazaki, *J. Fac. Sci. Univ. Tokyo*, Sect. I 10 (1964), p. 147—195.

Mathematical Institute
of the Romanian Academy
Calea Grivitei 21, Bucuresti 12
Romania