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A NOTE ON DIFFERENTIAL EQUATIONS WITH PERIODIC SOLUTIONS

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I. Problem. It is known [1, p. 136] for two differential equations $y'' = q_1(t)\, y$ and $y'' = q_2(t)\, y$, q_1 and q_2 continuous on $(-\infty, \infty)$, with the same basic central dispersion of the 1st kind $\varphi(t)$ (for the definition, see [1, p. 105]), that the difference $q_1(t) - q_2(t)$ vanishes at least four times on any interval $a \leqq t < \varphi(a)$, a being an arbitrary number.

Especially, for $\varphi(t) = t + \pi$ and $q_2(t) = -1$, the expression $q_1(t) + 1$ vanishes at least four times on the interval $0 \leqq t < \pi$ for every differential equation $y'' = q_1(t)\, y$, q_1 continuous on $(-\infty, \infty)$, every solution of which is half-periodic[1]) with period π and has exactly one zero on $0 \leqq t < \pi$, or in the other words, for every differential equation $y'' = [\lambda + q_1(t)]\, y$, q_1 periodic with period π and continuous on $(-\infty, \infty)$ for which the first interval of instability (λ_1', λ_2') disappears, as $\lambda_1' = \lambda_2' = 0$ (see [1, p. 11]).

Another simple criterion of disappearance of the 1st interval of instability can be found in [2].

In this paper we shall investigate the behaviour of $q(t)$ for the differential equation $y'' = [\lambda + q(t)]\, y$, $q(t)$ continuous on $(-\infty, \infty)$, for which an interval of instability disappears.

For construction of all differential equations $y'' = [\lambda + q(t)]\, y$, for which one or two intervals of instability disappear, see [3].

II. Let C_I^n ($n \geqq 0$, integer) denote the set of all continuous functions on I having here continuous derivatives up to and including the order n; $C^n(-\infty, \infty) \equiv C^n$.

Consider differential equation

$$y'' = q(t)\, y, \qquad q \in C^0. \tag{1}$$

Let $\alpha(t)$ be a 1st phase of the equation (for the definition, see [1, p. 33]) and $\varphi(t)$ its basic central dispersion of the 1st kind; further on, only phase and dispersion.

In [3] there was proved (Theorem 1) that "Every solution of (1) is periodic or half-periodic of period π with exactly n zeros on $0 \leqq t < \pi$ iff

$$\alpha(t + \pi) = \alpha(t) + n\pi \,.\, \operatorname{sgn} \alpha', \tag{2}$$

[1]) $y(t)$ is half-periodic with period d if $y(t + d) = -y(t)$ for all t.

for all $t \in (-\infty, \infty)$, or equivalently

(3) $$\varphi^{[n]}(t) = t + \pi,$$

for all $t \in (-\infty, \infty)$,
where
$$\varphi^{[1]}(t) = \varphi(t), \qquad \varphi^{[n+1]}(t) = \varphi(\varphi^{[n]}(t)).\text{''}$$

Let us choose a periodic function $P(t) \in C^4$ with period π such that $P(t) \not\equiv 0$, $|P'(t)| < 1$ and $P'(t)$ has no more than two (then exactly two) zeros t_1, t_2 on $0 \leqq t < \pi$. Moreover, let $P''(t_1) \cdot P''(t_2) \neq 0$. Then there exist neighbourhoods $O(t_1)$ and $O(t_2)$ of t_1 and t_2, resp., such that $|P''(t)| > c_1 > 0$ for all $t \in O(t_1) \cup O(t_2)$ and $|P'(t)| > c_0 > 0$ for all $t \in \langle 0, \pi\rangle - O(t_1) \cup O(t_2)$. E. g., $P(t) = a \sin 2t$, $|a| < 1/2$, $t_1 = \pi/4$, $t_2 = 3\pi/4$, $O(t_1) = (\pi/8, 3\pi/8)$, $O(t_2) = (5\pi/8, 7\pi/8)$, $c_0 = c_1 = a$.

For an arbitrary integer n, $n > 1$, put $\alpha_1(t) = nt + nP(t)$, $\alpha_2(t) = nt - nP(t)$. Let $\alpha_1(t)$ and $\alpha_2(t)$ be phases of the differential equations (it is possible according to [1, p. 36])

$(4_{1,2})$ $$y'' = q_1(t)\, y \quad \text{and} \quad y'' = q_2(t)\, y,$$

resp. Since α_1 and α_2 satisfy (2), every solution of any of the two differential equations $(4_{1,2})$ is periodic or half-periodic of period π with n zeros on $0 \leqq t < \pi$. Let us note that both q_1 and q_2 must be continuous and periodic with period π.

Let $\alpha_i(t)/n$ be a phase of the differential equation

(5_i) $$y'' = q_i^*(t)\, y, \qquad i = 1, 2.$$

Functions $q_i(t)$ and $q_i^*(t)$ are (cf. [1, p. 90]) in the relations

(6) $$q_i(t) = q_i^*(t) - (n^2 - 1)\left(\frac{\alpha_i'(t)}{n}\right)^2, \qquad i = 1,2.$$

Thus

$$q_1(t) - q_2(t) = [q_1^*(t) - q_2^*(t)] - (n^2 - 1)\,\frac{\alpha_1'^2(t) - \alpha_1'^2(t)}{n^2},$$

or

$$q_1(t) - q_2(t) = [q_1^*(t) - q_2^*(t)] - 4(n^2 - 1)\, P'(t).$$

The function $[q_1^*(t) - q_2^*(t)]$ is continuous and periodic with period π, and has continuous derivative (cf. $P(t) \in C^4$). Thus $[q_1^*(t) - q_2^*(t)]$ and $[q_1^*(t) - q_2^*(t)]'$ are bounded. Since $P(t)$ has the above properties, there exists a sufficently large n that $q_1(t) - q_2(t)$ vanishes only twice on $0 \leqq t < \pi$.

Thus, we have proved

Theorem 1: *There exist an integer n and two differential equations*

$$y'' = q_1(t)\, y, \quad y'' = q_2(t)\, y, \qquad q_1 \in C^\circ,\ q_2 \in C^\circ,$$

which admit only periodic or half-periodic solutions of period π with n zeros on $0 \leqq t < \pi$ (n even or odd, resp.) such that the difference $q_1(t) - q_2(t)$ vanishes only twice on $0 \leqq t < \pi$.

If $\alpha_1(t)$ is $nt + nP(t)$ again, but $\alpha_2(t) = nt$, we have

$$q_1(t) + n^2 = [q_1^*(t) + 1] - (n^2 - 1)\,(2P'(t) + P'^2(t)).$$

The function $[q_1^*(t) + 1]$ and its derivative are bounded again. The function $[2P'(t) + P'^2(t)] = P'(t)\,[2 + P'(t)]$ vanishes on $0 \leqq t < \pi$ only at t_1 and t_2, as $2 + P'(t) \geqq 1$. Further $|\,2P'(t) + P'^2(t)\,| \geqq |\,P'(t)\,|$ and $[2P'(t) + P'^2(t)]' = 2P''(t)\,[1 + P'(t)] \geqq P''(t)\,.\,\gamma$, where $0 < \gamma = \min\limits_{\langle 0,\pi\rangle} [1 + P'(t)]$. Thus, according to the selection of $P(t)$, for sufficiently large n, the value of $q_1(t) + n^2$ vanishes only twice on $0 \leqq t < \pi$. Then we can state

Theorem 1': *There exists a differential equation*

$$y'' = q(t)\, y, \qquad q \in C^\circ$$

every solution of which is periodic or half-periodic of period π with n zeros on $0 \leqq t < \pi$ (n even or odd, resp.) such that $q(t) + n^2$ vanishes only twice on $0 \leqq t < \pi$.

III. We have dealt with periodic solutions only. Now we are going to show how the above considerations can be extended to the general case.

Let a function $\varphi(t)$ be given on an interval (a, b)[1], $\varphi \in C^3_{(a,b)}$, $\varphi(t) > t$, $\varphi'(t) > 0$. Let $\beta(t)$ be chosen such that $\beta(t) \in C^3_{(a,\varphi(b))}$, $\beta'(t) \neq 0$, and

$$\beta(\varphi(t)) = \beta(t) + \pi \tag{7}$$

holds for all $t \in (a, b)$. According to [1, p. 124], it is always possible.

Let $\alpha_1(t) = nt + nP(t)$ and $\alpha_2(t) = nt - nP(t)$, where a function $P(t)$ satisfies the conditions in Sec. II, and n is an integer, $n \geqq 2$. Put $\beta_1(t) \overset{\text{def.}}{=} \alpha_1\beta(t)$ and $\beta_2(t) \overset{\text{def.}}{=} \alpha_2\beta(t)$ for all t for which β is defined. Then

$$\begin{aligned} \beta_i(\varphi(t)) &= \alpha_i\beta(\varphi(t)) = \alpha_i(\beta(t) + \pi) = \alpha_i\beta(t) + n\pi \operatorname{sgn} \alpha_i', \\ &= \beta_i(t) + n\pi \operatorname{sgn} \beta_i' \quad \text{for all} \quad t \in (a, b). \end{aligned} \tag{8}$$

Let β_i be a phase of the differential equation $y'' = Q_i(t)\, y$, $i = 1,2$. If $\varphi_i(t)$ denotes the dispersion of the differential equation $y'' = Q_i(t)\, y$, $i = 1,2$, then, with respect to (8),

$$\varphi_1^{[n]}(t) = \varphi(t) = \varphi_2^{[n]}(t).$$

[1]) $a = -\infty$ and $b = \infty$ are not excluded.

That means that the distributions of the n-th zeros of the solutions of the both differential equations are the same.

In general, if α, β, $\alpha\beta = \alpha(\beta(t))$ are phases of differential equations $y'' = p_\alpha(t)\, y$, $y'' = p_\beta(t)\, y$, $y'' = p_{\alpha\beta}(t)\, y$, resp., then the relation

$$p_{\alpha\beta}(t) = p_\beta(t) + [p_\alpha(\beta(t)) + 1]\, \beta'^2(t)$$

holds for all t for which the functions are defined, see [1, p. 6 (17) or 188]. Thus we can write

$$Q_1(t) - Q_2(t) = [q_1(\beta(t)) - q_2(\beta(t))]\, \beta'^2(t). \tag{9}$$

Because the function $\beta(t)$ is defined on $(a, \varphi(b))$, the functions $Q_i(t)$ are defined on the same interval. Thus, according to (7), we can consider an interval $\langle c, \varphi(c)) \subset (a, \varphi(b))$ which is transformed by β onto $\langle\beta(c), \beta(c) + \pi)$. And, with respect to (9) and Theorem 1 of Sec. II., the difference $Q_1(t) - Q_2(t)$ vanishes only twice on every such interval $\langle c, \varphi(c))$.

Thus we can state

Thoerem 2: *There exist two differential equations*

$$y'' = Q_1(t)\, y \quad \text{and} \quad y'' = Q_2(t)\, y, \qquad Q_1 \in C^0_{\mathrm{I}},\ Q_2 \in C^0_{\mathrm{I}},$$

with the dispersions $\varphi_1(t)$ and $\varphi_2(t)$, resp., satisfying the relation

$$\varphi_1^{[n]}(t) = \varphi_2^{[n]}(t)$$

everywhere, where they are defined, such that the difference $Q_1(t) - Q_2(t)$ vanishes only twice on every interval $\langle c, \varphi_1^{[n]}(c)) \subset I$.

IV. The differences $q_1(t) - q_2(t)$ in Sec. II or $Q_1(t) - Q_2(t)$ in Sec. III. must vanish at least twice on considered intervals, because the relations $q_1(t) - q_2(t) > 0$, or $q_1(t) - q_2(t) < 0$, or $Q_1(t) - Q_2(t) > 0$, or $Q_1(t) - Q_2(t) < 0$ on these intervals together with the Separation Theorem lead to the contradiction with the supposition on distributions of zeros,

Thus, the above considerations do not give any criterion, but they show the nonexistence of such criterion. On the other side, we could demonstrate the ability of Prof. O. Borůvka's theory of phases and dipsersions to solve problems of this sort.

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