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A FIXED POINT THEOREM IN LOCALLY CONVEX SPACES

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In this paper there is extended a fixed point theorem obtained in [1] to locally convex spaces. Therefore, let E be *a* Hausdorff locally convex space and $\mathscr{A}$ a sufficently and directed familly of seminorms that gives the topology of E.

Let φ be a mapping of the family $\mathscr{A}$ satisfying the condition

$$(1) \qquad \varphi[\varphi(\alpha)] = \varphi(\alpha) \qquad (\alpha \in \mathscr{A})$$

and H a closed, convex and bounded subset of E.

Theorem. *Let f be a mapping from H into H such that:*

(i) *for all* $\alpha \in \mathscr{A}$

$$2) \qquad |f(x_1) - f(x_2)|_\alpha \leqq |x_1 - x_2|_{\varphi(\alpha)} \qquad (x_1, x_2 \in H)$$

(ii) *there is a compact set $M \subset H$ such that for every $x \in H$ the sequence $\{f^n(x)\}$ has an accumulation point in M* $(f(^n = f \cdot f^{n-1}, f^1 = f.$

Then f has a fixed point in H.

Proof. We can suppose that H contains the null element of E such that if we put $f_q = q \cdot f$ with $0 < q < 1, f_q(x) = (1 - q) \cdot 0 + qf(x) \in H$, that is f_q is a contraction of H.

For every $\alpha \in \mathscr{A}$

$$|f_q(x_1) - f_q(x_2)|_\alpha \leqq q\, |x_1 - x_2|_{\varphi(\alpha)} \qquad (x_1, x_2 \in H)$$

such that, for $n = 1, 2, \dots$ and $x \in H$

$$|f_q^{n+1}(x) - f_q^n(x)|_\alpha \leqq q \cdot |f_q^n(x) - f_q^{n-1}(x)|_{\varphi(\alpha)} \leqq$$
$$\leqq q^2\, |f_q^{n-1}(x) - f_q^{n-2}(x)|_{\varphi(\alpha)} \leqq \dots \leqq q^n\, |f_q(x) - x|_{\varphi(\alpha)}$$

It follows that

$$|f_q^{n+k}(x) - f_q^n(x)|_\alpha \leqq (q^{n+k-1} + \dots + q^n)\, |f_q(x) - x|_{\varphi(\alpha)} \leqq$$
$$\leqq \frac{q^n}{1-q} \left| f_q(x) - x \right|_{\varphi(\alpha)}.$$

Consequently, for each $x \in H$ the sequence $\{f_q^n(x)\}$ is a Cauchy one. It results that there exists $x_q \in H$ such that:

$$|f\,(x_q) - x_q|_{\varphi(\alpha)} \leqq 1 - q,$$

Indeed we can take $x_q = f_2^n(x)$ with a sufficient large n.
On the other hand

$$(3) \qquad |f(x_q) - x_q|_{\varphi(\alpha)} \leqq |f(x_q) - qf(x_q)|_{\varphi(\alpha)} + \\ + |f_q(x_q) - x_q|_{\varphi(\alpha)} \leqq (1-q)\,[|f(x_q)|_{\varphi(\alpha)} + 1] = \\ = (1-q)\,r$$

where r is a positive number independent of x_q since the set H is bounded.

Hence, if $n = 1, 2, \ldots$

$$(4) \quad |f^{n+1}(x_q) - f^n(x_q)|_\alpha \leqq |f^n(x_q) - f^{n-1}(x_q)|_{\varphi(\alpha)} \leqq \cdots \leqq (1-q)\,r.$$

Since the sequence $\{f^n(x_q)\}$ has an accumulation point

$$y_q \in M,$$

for every $\varepsilon > o$, there exists n such that:

$$(5) \qquad |f^n(x_q) - y_q|_\beta \leqq \varepsilon \qquad [\beta \geqq \alpha,\ \varphi(\alpha)]$$

From (4) and (5) it follows that:

$$|f(y_q) - y_q|_\alpha \leqq |f(y_q) - f^{n+1}(x_q)|_\alpha + |f^{n+1}(x_q) - f^n(x_q)|_\alpha + \\ + |f^n(x_q) - y_q|_\alpha \leqq |y_q - f^n(x_q)|_{\varphi(\alpha)} + |f^{n+1}(x_q) - f^n(x_q)|_\alpha + \\ + |f^n(x_q) - y_q|_\alpha \leqq \varepsilon + (1-q)\,.\,r + \varepsilon$$

that is

$$|f(y_q) - y_q|_\alpha \leqq (1-q)\,.\,r.$$

Let $\{q_i\}$ be a sequence of real numbers such that

$$\operatorname*{Lim}_{i\to\infty} q_i = 1 \qquad (0 < q_i < 1,\ i = 1, 2, \ldots).$$

We consider a convergent subsequence $\{y_{q'_i}\}$ of the corresponding sequence $\{y_{q_i}\} \subset M$.

If $\operatorname*{Lim}_{i\to\infty} y_{q'_i} = y \in M$ it follows that

$$\operatorname*{Lim}_{i\to\infty} |f(y_{q'_i}) - y_{q'_i}|_\alpha \leqq \lim\,(1 - q'_i)\,.\,r = 0.$$

Since f is a continuous mapping, it holds

$$|f(y) - y|_\alpha = 0$$

for all $\alpha \in \mathscr{A}$. Hence $f(y) = y$, $q.e.d.$

When the space E is normed there is obtained theorem 5 given by D. Göhde in [1].

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