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AN ORDERING OF THE SET OF NATURAL NUMBERS BASED ON PEANO AXIOMS

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1. The set N of natural numbers is defined by Peano by requiring that there exists a mapping $'$ of N into itself such that

1° $1 \in N$
2° $(\forall\, x \in N)\; x' \neq 1$
3° $(\forall\, x, y \in N)\; x' = y' \Rightarrow x = y$
4° $(\forall\, P \subset N)\,\{[1 \in P \;\&\; (\forall\, x \in N)\; (x \in P \Rightarrow x' \in P)] \Rightarrow P = N\}$.[1])

The order-relation $<$ in N is usually introduced only after the binary operation of addition is defined (and investigated to some extent) by the definition

$$(\forall\, x, y \in N) \qquad [x < y \underset{Df}{\Leftrightarrow} (\exists\, z \in N)\; x + z = y].$$

In this paper we give a definition of order $\leqq$ which does not presuppose addition.[2])

2. First we derive some properties of $(N,')$ as defined by 1. 1°—4°.

2.1 $(\forall y \in N)\,\{[(\forall x \in N)\; x' \neq y] \Rightarrow y = 1\}$ i.e. 1 is the only element of N with the property 2°.

Suppose the contrary and let $N \ni a \neq 1\; \&(\forall x \in N)\; x' \neq a$. Then for $M = N \setminus \{a\}$ it would hold $1 \in M$ and $(\forall x \in N)\; (x \in M \Rightarrow x' \in M)$, hence by 1.4° $M = N$, a contradiction.

2.2. $(\forall P \subset N)\,\{[(\forall x \in N)\; (x \in P \Rightarrow x \in P') \Rightarrow P = \emptyset\}$[3]) i.e. no non-void subset of N is contained in its $'$-image.

Let $P \subset P'$ and denote $N \setminus P = M$. $1 \in M$ since, because of 2° and the supposition $P \subset P'$, $1 \notin P$. Furthermore, if $x \in M$ then $x \notin P$, hence by 1.3° $x' \notin P'$, hence $x' \notin P$, i.e. $x' \in M$. By 1.4° $M = N$ and $P = \emptyset$.

3. A binary relation R in N (i.e. a subset R of N^2) will be called regular if

(i) $(\forall x \in N) \quad R(x, x)$

(ii) $(\forall x, y \in N)\; (R(x, y) \Rightarrow R(x, y'))$.

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[1]) Throughout this paper we use logical symbols informally.

[2]) An introduction of order (related to this one) into the set of natural numbers based on another axiom-system was given in [1].

[3]) $P' = \{y \mid (\exists x \in P)\; x' = y\}$.

E.g. N^2 is regular. Let ϱ be the intersection of all regular R, i.e. $\varrho(x, y)$ if and only if for all regular R, $R(x, y)$. ϱ itself is regular.

We shall show that $\varrho(x, y)$ is a relation of (total) order (and even of well order) of N.

4. We prove first some properties of ϱ. Let

$$(\forall x \in N)\ (\varrho x \underset{Df}{=} \{y \mid \varrho(x, y)\}).$$

4.1. $(\forall x \in N) \quad \varrho x = \{x\} \cup (\varrho x)'$.

Proof. By (i) $x \in \varrho x$. By (ii) $y \in \varrho x \Rightarrow y' \in \varrho x$ i.e. $(\varrho x)' \subset \varrho x$. Hence

$$\varrho x \supset \{x\} \cup (\varrho x)'. \tag{1}$$

Let the binary relation R_0 be defined by

$$(\forall x, y \in N)\ (R_0(x, y) \underset{Df}{\Leftrightarrow} y \in \{x\} \cup (\varrho x)').$$

Obviously, R_0 satisfies (i). R_0 satisfies (ii) too, for, if $R_0(x, y)$ then $y \in \{x\} \cup (\varrho x)'$ hence by (1) $y \in \varrho x$ and $y' \in (\varrho x)'$ hence $R_0(x, y')$. R_0 is regular and by (1) and the definition of ϱ, $R_0 \equiv \varrho$.

4.2. $(\forall x \in N)\ \varrho(x') = (\varrho x)'$.

Proof. By 4.1.

$$\varrho(x') = \{x'\} \cup (\varrho x')' = (\{x\} \cup (\varrho x'))',$$
$$(\varrho x)' = (\{x\} \cup (\varrho x)')'.$$

Since $\varrho(x') \setminus (\varrho x)' \supset (\{x\} \cup \varrho(x')) \setminus (\{x\} \cup (\varrho x)')$, so $[\varrho(x') \setminus (\varrho x)']' \supset [(\{x\} \cup \varrho(x')) \setminus (\{x\} \cup (\varrho x)')]' = (\{x\} \cup \varrho(x'))' \setminus (\{x\} \cup (\varrho x)')' = \varrho(x') \setminus (\varrho x)'$ hence by 2.2. $\varrho(x') \setminus (\varrho x)' = \emptyset$ i.e. $(\varrho x)' \supset \varrho(x')$. Similarly $[(\varrho x)' \setminus \varrho(x')]' \supset [(\{x\} \cup (\varrho x)') \setminus (\{x\} \cup \varrho(x'))]' =$ $(\{x\} \cup (\varrho x)')' \setminus (\{x\} \cup \varrho(x'))' = (\varrho x)' \setminus \varrho(x')$ hence by 2.2. $(\varrho x)' \setminus \varrho(x') = \emptyset$ i.e. $\varrho(x') \supset (\varrho x)'$.

4.3. By 4.1. and 4.2.

$(\forall x \in N)\ \varrho x = \{x\} \cup \varrho(x')$.

4.4. $(\forall x, y \in N) \quad x \in \varrho y \vee y \in \varrho x$.

Proof by induction on x. Let the predicate P be defined by $P(x) \underset{Df}{\Leftrightarrow}$ $\underset{Df}{\Leftrightarrow} (\forall y \in N)\ [x \in \varrho y \vee y \in \varrho x]$.

Induction basis. $1 \in \varrho\, 1$ by 3 (i) and $k \in \varrho\, 1 \Rightarrow k' \in \varrho 1$ by 3 (ii), hence by 1.4° $\varrho 1 = N$, hence $(\forall y \in N)\ y \in \varrho 1$ and a fortiori $P(1)$.

Induction step. Suppose for fixed $x = k \in N$: $P(k)$, i.e. $(\forall y \in N)$ $k \in \varrho y \ \vee\ y \in \varrho k$. In case $k \in \varrho y$ by 3 (ii) $k' \in \varrho y$; in case $y \in \varrho k$ by 4.1. $y \in \{k\} \cup (\varrho k)'$, hence either a) $y = k$ or b) $y \in (\varrho k)'$. If $y = k$ then $y' =$ $= k' \in \varrho(k')$ hence by 4.3 $k' \in \varrho k = \varrho y$, and if $y \in (\varrho k)'$ then by 4.2. $y \in \varrho(k')$. So in either case $k' \in \varrho y \ \vee\ y \in \varrho(k')$, i.e. $P(k')$.

5. Now we can prove that ϱ is a (total) ordering (and even a well-ordering) of N.

5.1. $(\forall x, y \in N)\ [\varrho(x, y) \lor \varrho(y, x)]$ by 4.4.

5.2. ϱ is reflexive, since by 3(i) $(\forall x \in N)\ \varrho(x, x)$.

5.3. ϱ is antisymmetric, i.e. $(\forall x, y \in N)\ \varrho(x, y)\ \&\ \varrho(y, x) \Rightarrow x = y$.

Proof by induction. Let the predicate P be defined by $P(x) \underset{Df}{\Leftrightarrow} (\forall y \in N)\ [\varrho(x, y)\ \&\ \varrho(y, x) \Rightarrow x = y]$.

Induction basis. If $\varrho(y, 1)$ then by 4.1. $1 \in \{y\} \cup (\varrho y)'$ hence by 1.2° $y = 1$, i.e. $P(1)$.

Induction step. Suppose for fixed $x = k \in N$: $P(k)$, i.e. (first induction hypothesis) $(\forall y \in N)\ \varrho(k, y)\ \&\ \varrho(y, k) \Rightarrow k = y$ and suppose $\varrho(k', y)\ \&\ \varrho(y, k')$.

Second induction basis. $\varrho(k', 1)\ \&\ \varrho(1, k') \Rightarrow k' = 1$ is trivially true: by the (first) induction basis $\varrho(1, k')\ \&\ \varrho(k', 1) \Rightarrow 1 = k'$ i.e. $\varrho(k', 1)\ \&\ \varrho(1, k') \Rightarrow k' = 1$, and since $k' = 1$ is impossible by 1.2°, so $\varrho(k', 1)\ \&\ \varrho(1, k')$ is also impossible.

Second induction step. Suppose for any fixed m $\varrho(k', m)\ \&\ \varrho(m, k') \Rightarrow k' = m$ and suppose $\varrho(k', m')\ \&\ \varrho(m', k')$. Then by 4.1. $k' \in \{m'\} \cup (\varrho(m'))'$ & $m' \in \{k'\} \cup (\varrho(k'))'$. If $k' = m'$ the second induction step (and therefore the first inductions step, too) is proved, so suppose $k' \in (\varrho(m'))'$ & $m' \in (\varrho(k'))'$. Then by 1.3° $k \in \varrho(m')$ & $m \in \varrho(k')$, hence by 4.3. $k \in (\varrho\, m)'$ & $m \in (\varrho\, k)'$ hence by 4.1. $k \in \varrho\, m$ & $m \in \varrho\, k$, hence by the first induction hypothesis $k = m$, hence $k' = m'$ again.

5.3.1. Another variant of the proof of 5.3.[1]) Let

$$(2) \qquad M \underset{Df}{=} \{x \mid (\exists y \in N)\ [\varrho(x, y)\ \&\ \varrho(y, x)\ \&\ x \neq y]\}.$$

If $x \in M$, then for some $y \in N$ it is $x \in \varrho y = \{y\} \cup (\varrho y)'$ and $x \neq y$, i.e. $x \in (\varrho y)'$. Hence there is a $u \in \varrho y$ such that $x = u'$. Similarly $y \in \varrho x = \{x\} \cup (\varrho x)'$, $y \in (\varrho x)'$. i.e. there is a $v \in \varrho x$ such that $y = v'$. But then $u \in \varrho(v')$ and by 4.3. $u \in \varrho v$ and similarly $v \in \varrho(u')$ and by 4.3 $v \in \varrho u$. Hence $\varrho(u, v)\ \&\ \varrho(v, u)$; but $u = v$ is impossible since $u = v \Rightarrow u' = v'$ i.e. $x = y$. In other words, if $x \in M$ then $x = u'$ with $u \in M$, i.e. $u' = x \in M'$. Hence $M' \supset M$ and by 2.2. $M = \emptyset$, i.e. 5.3. holds good.

5.4. ϱ is transitive, i.e.

$(\forall x, y, z \in N)\ \varrho(x, y)\ \&\ \varrho(y, z) \Rightarrow \varrho(x, z)$.

Proof. Let

$M \underset{Df}{=} \{x \mid (\exists y, z \in N)\ [\varrho(x, y)\ \&\ \varrho(y, z)\ \&\ \varrho(z, x)\ \&\ \text{non}\ (x = y = z)]\}$.

$\partial(x, y)\ \&\ \varrho(y, z)\ \&\ \varrho(z, x)\ \&\ \text{non}\ (x = y = z)$ yields $x \neq y\ \&\ y \neq z\ \&\ z \neq$

[1] For 5.3.—5. cf. [1].

$\neq x$, since e.g. $x = y$ and $\varrho(y, z)$ & $\varrho(z, x)$ would imply (by 5.3.) that $y = z$.

Suppose $x \in M$. Then there are elements $y, z \in N$ such that $x \neq y$ & $y \neq z$ & $z \neq x$ and $y \in \varrho x$ & $z \in \varrho y$ & $x \in \varrho z$ i.e. by 4.1. $y \in \{x\} \cup (\varrho x)'$ & $z \in \{y\} \cup (\varrho y)'$ & $x \in \{z\} \cup (\varrho z)'$. Hence $y \in (\varrho x)'$ & $z \in (\varrho y)'$ & $x \in (\varrho z)'$, i.e. there are elements $u, v, w \in N$ such that $u \in \varrho x$ & $v \in \varrho y$ & $w \in \varrho z$ and $y = u'$ & $z = v'$ & $x = w'$ i.e. $u \in \varrho(w')$ & $v \in \varrho(u')$ & $w \in \varrho(v')$. $u = v \vee v = w \vee w = u$ is impossible since this would imply $u' = v' \vee v' = w' \vee w' = u'$ and hence by 5.3. $x = y = z$. By 4.3. $u \in \varrho w$ & $v \in \varrho u$ & $w \in \varrho v$. Thus for $u \in N$ with $u' = x$ there exist elements $v, w \in N$ such that $\varrho(u, v)$ & $\varrho(v, w)$ & $\varrho(w, u)$ & non $(u = v = w)$, i. e. $u \in M$. In other words, $x \in M$ implies $u \in M$ hence $x = u' \in M'$ i.e. $M' \supset M$. By 2.2. $M = \emptyset$ and therefore

(non $\exists\, x \in N$) $(\exists y, z \in N)$ $[\varrho(x, y)$ & $\varrho(y, z)$ & $\varrho(z, x)$ & non $(x = y = z)]$ hence

$(\forall x, y, z \in N)$ $[\varrho(x, y)$ & $\varrho(y, z) \Rightarrow$ (non $\varrho(z, x)) \vee x = y = z]$. Since by 5.1. non $\varrho(z, x) \Rightarrow \varrho(x, z)$ and by 5.2. $x = y = z \Rightarrow \varrho(x, z)$, 5.4 is proved.

5.1.—4. express that ϱ is a relation of (total) ordering of N.

5.5. Proof that ϱ is a relation of well-ordering of N.

Let M be a subset of N with the property

$(\forall y \in M)$ $(\exists\, x \in M)$ $[\varrho(x, y)$ & $x \neq y]$.

Let

$$M_1 = \bigcup_{z \in M} \varrho z.$$

By 4.1. $M_1 \supset M$. If $y_1 \in M_1$, there is an $y \in M$ such that $y_1 \in \varrho y$ i.e. $\varrho(y, y_1)$. By the supposition on M, there is an x, $x \neq y$, such that $\varrho(x, y)$. Because of 5.4. $\varrho(x, y_1)$, i.e. $y_1 \in \varrho x = \{x\} \cup (\varrho x)'$. But $y_1 = x$ is impossible, for then we would have $\varrho(y, x)$ and this, together with the supposition $\varrho(x, y)$ by 5.3. yields $x = y$, contrary to the supposition that $x \neq y$. Hence $x \neq y_1$ and therefore $y_1 \in (\varrho x)'$ or $y_1 = y_2'$ with $y_2 \in \varrho x \subset M_1$. In other words, if $y_1 \in M_1$ then $y_1 = y_2' \in M_1'$ i.e. $M_1' \supset M_1$. By 2.2. $M_1 = \emptyset$ and a fortiori $M = \emptyset$. Hence

$M \neq \emptyset \Rightarrow$ non $\{(\forall y \in M)$ $(\exists x \in M)$ $[\varrho(x, y)$ & $x \neq y]\}$, i.e.

$M \neq \emptyset \Rightarrow (\exists y \in M)$ $(\forall x \in M)$ [non $\varrho(x, y) \vee x = y]$.

Since by 5.3. non $\varrho(x, y)$ implies $\varrho(y, x)$ and by 5.2 $x = y$ implies $\varrho(y, x)$ it follows

$$M \neq \emptyset \Rightarrow (\exists y \in M)\ (\forall x \in M)\ \varrho(y, x)$$

i.e. N is well-ordered.

[1] Devidé Vladimir, An Axiom System for Natural Numbers and their Ordering, Period. mat.-phys. astr. 15 (1960), p. 153—159.

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