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NOTE ON THE SECOND PHASE OF THE DIFFERENTIAL EQUATION $y'' = q(t)\,y$

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1. We shall deal with the differential equation

$$y'' = q(t)\,y, \tag{1}$$

where the function $q(t)$ is defined in a certain interval j and continuous on the latter (more concisely $q(t) \in C^{\circ}(j)$). The class $C^n(i)$, where i is some interval, stands for a set of all functions defined on the interval i and having, on the latter, continuous derivatives up to and including the order n. By a solution of the differential equation (1) on the interval i, $i \subset j$, we understand every function $y(t) \in C^2(i)$ satisfying the equation (1). When we speak only about solutions of the differential equation (1), we mean solutions defined on the entire interval j. The identically zero solution will be excluded from our considerations.

With regard to the two linearly independent solutions u, v of the differential equation (1), Prof. O. Borůvka [1], [2] has defined the first and the second phase, respectively, as a continuous function on the interval j and complying with the relation

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \quad \text{resp.} \quad \operatorname{tg} \beta(t) = \frac{u'(t)}{v'(t)}. \tag{2}$$

For the existence of the 1^{st} phase, the postulates concerning the function $q(t)$ are obviously sufficient, since

$$\alpha'(t) = \frac{-W}{u^2 + v^2}, \tag{3}$$

where W is the Wronskian of the pair u, v. For the existence of the 2^{nd} phase, $\beta(t)$, the continuity of $q(t)$ only would not be sufficient, since the zeros of the function $v'(t)$ would not necessarily be isolated. Therefore one generally requires $q(t) \neq 0$ for $t \in j$. A number of properties of the 2^{nd} phase have been derived under the additional assumption that there exists a continuous 1^{st} derivative of the function $q(t)$. Let us therefore introduce a definition of the 2^{nd} phase so that the latter exist for every continuous function $q(t)$ and every two independent solutions of the equation (1) and that, at the same time, the 2^{nd} phase, thus defined, coincide with the above definition of the case $q(t) \neq 0$. We shall also show that a number of properties derived by O. Borůvka for the 2^{nd} phase remain valid even if only $q(t) \in C^{\circ}(j)$ is required. Let us, moreover, note the basic central dispersions of the differential equation (1) for the definition of which one assumes, with the exception of the basic central dispersion of the 1^{st} kind, that $q(t) \neq 0$. We shall see that there it will not be possible to introduce definitions of basic central dispersions of the 2^{nd}, 3^{rd} and 4^{sh} kind so that the properties of the basic central dispersions be, on the whole, maintained. But analogous functions may, at the same time, be defined so that the so called Abelian relations remain valid and that the mentioned functions coincide with the basic central dispersions for the case $q(t) \neq 0$.

2. Let u, v be two independent solutions of the differential equation (1), $q \in C^{\circ}(j)$. The second phase of the differential equation (1) with regard to the pair (u, v) is every function $\beta(t)$ satisfying the relation

$$\beta(t) = \int_{t_0}^{t} Wq(\sigma)/(u'^2(\sigma) + v'^2(\sigma))\, d\sigma + M, \tag{4}$$

where $t_0 \in j$ and

$$M = \begin{cases} \operatorname{arctg} \dfrac{u'(t_0)}{v'(t_0)} & \text{for } v'(t_0) \neq 0, \\ \pi/2 & \text{for } v'(t_0) = 0. \end{cases}$$

Obviously, for $q(t) \neq 0$, this definition coincides with the original one, particularly, as there need not be only one 2^{nd} phase of that kind. There holds $\beta(t) \in C^1(j)$. Since $\beta'(t) = Wq(t)/(u'^2(t) + v'^2(t))$, the 2^{nd} phase of the equation (1) need not be a monotone function.

Let $\alpha(t)$ and $\beta(t)$ be the 1^{st} and the 2^{nd} phase of the same pair of independent solutions (u, v) of the differential equation (1), respectively. There always holds $\alpha(t) - \beta(t) \neq k\pi$, k integer. If $v(t) \mathrel{\overset{=}{\neq}} 0$ and $v'(t) \mathrel{\overset{\neq}{=}} 0$, then $\alpha(t) \mathrel{\overset{=}{\neq}} \frac{1}{2}\pi + k_1\pi$ and $\beta(t) \mathrel{\overset{\neq}{=}} \frac{1}{2}\pi + k_2\pi$. If $v(t) \neq 0$, $v'(t) \neq 0$ and $\alpha(t) = \beta(t) + k\pi$, then $u(t)/v(t) = u'(t)/v'(t)$, so that $W = u(t)\,v'(t) - u'(t)\,v(t) = 0$, which contradicts the linear independence of the solutions $u(t)$ and $v(t)$. In other words:

Let $\alpha(t)$ and $\beta(t)$ be the 1^{st} and the 2^{nd} phase of the same pair of independent solutions $u(t)$ and $v(t)$ of the differential equation (1), *respectively. Then there exists an integer k such that*

$$\alpha(t) + k\pi < \beta(t) < \alpha(t) + \overline{k+1}\pi. \tag{5}$$

Let us now define, with O. Borůvka, the 1st and the 2nd amplitude of the independent solutions u, v of the differential equation (1) by the relation

$$r(t) = \sqrt{u^2(t) + v^2(t)} \text{ and } s(t) = \sqrt{u'^2(t) + v'^2(t)}.$$

According to [2], every solution $y(t)$ of the differential equation (1) may be written in the form $y(t) = k_1 \dfrac{\sin [\alpha(t) - k_2]}{\sqrt{|\alpha'(t)|}}$, where k_1 and k_2 are suitable constants. If $v'(t) \neq 0$, then $\operatorname{tg} \beta(t) = \dfrac{u'(t)}{v'(t)}$, or $u'(t) = \varepsilon \,.\, s(t) \,.\, \sin \beta(t)$, $v'(t) = \varepsilon \,.\, s(t) \,.\, \cos \beta(t)$, where $\varepsilon = +1$ or -1. If $v'(t) = 0$, then $\beta(t) = \pi/2 + k\pi$, or

$$u'(t) = |u'(t)| \,.\, \operatorname{sign} u'(t) = \varepsilon_1 \sqrt{u'^2(t) + v'^2(t)} \,.\, \sin \beta(t),$$

$0 = v'(t) = \varepsilon \,.\, s(t) \,.\, \cos \alpha(t)$, where, with regard to the continuity of $u'(t)$, one has $\varepsilon = \varepsilon_1$. Hence we may write:

$$u'(t) = \varepsilon \,.\, s(t) \,.\, \sin \beta(t), \quad v'(t) = \varepsilon \,.\, s(t) \,.\, \cos \beta(t). \tag{6}$$

Therefore:

Let $\alpha(t)$ and $\beta(t)$ be the 1st and the 2nd phase of the same pair of independent solutions of the differential equation (1) with the Wronskian $W(\neq 0)$, respectively. Then the derivative of the solution

$$y(t) = k_1 \frac{\sin [\alpha(t) - k_2]}{\sqrt{|\alpha'(t)|}} \tag{7}$$

of the differential equation (1) *is*

$$y'(t) = \pm k_1 \frac{s(t) \,.\, \sin [\beta(t) - k_2]}{\sqrt{|W|}} \,. \tag{8}$$

Proof. Let us, first, write the solution in the form

$$y(t) = k_1 \frac{\sin [\alpha(t) - k_2]}{\sqrt{|\alpha'(t)|}} = k_1 \cos k_2 \,.\, \sin \alpha(t)/\sqrt{|\alpha'(t)|} -$$

$$- k_1 \sin k_2 \,.\, \cos \alpha(t)/\sqrt{|\alpha'(t)|} = \frac{k_1}{\sqrt{|W|}} \left[\cos k_2 \,.\, \sqrt{\frac{-W}{\alpha'(t)}} \,.\, \sin \alpha(t) - \right.$$

$$\left. - \sin k_2 \,.\, \sqrt{\frac{-W}{\alpha'(t)}} \,.\, \cos \alpha(t) \right] = \frac{\varepsilon' k_1}{\sqrt{|W|}} [\cos k_2 \,.\, u(t) - \sin k_2 \,.\, v(t)],$$

where

$$\varepsilon' = +1 \text{ or } -1.$$

Therefore

$$y'(t) = \varepsilon' \,.\, \frac{k_1}{\sqrt{|W|}} [\cos k_2 \,.\, u'(t) - \sin k_2 \,.\, v'(t)] =$$

$$= \varepsilon\varepsilon' \frac{k_1}{\sqrt{|W|}} \,.\, s(t) \,.\, \sin [\beta(t) - k_2],$$

which was to be proved.

With regard to the relations (3), (7) and (8), there holds:

(9) $$W = uv' - u'v = \varepsilon\varepsilon' \,.\, s(t) \,.\, r(t) \,.\, \sin [\alpha(t) - \beta(t)].$$

Employing $\beta'(t) = Wq(t)/s^2(t)$ and the relation (3), we obtain:

(10) $$q(t) = \frac{\beta'(t) s^2(t)}{W} = -\frac{\beta'(t)\, s^2(t)}{\alpha'(t)\, r^2(t)}.$$

Or

$$\alpha'(t) . \beta'(t) = \frac{-W^2 q(t)}{r^2(t) \,.\, s^2(t)} = -q(t) \,.\, \sin^2[\alpha(t) - \beta(t)];$$

hence

(11) $$q(t) = -\frac{\alpha'(t)\, \beta'(t)}{\sin^2 [\alpha(t) - \beta(t)]}.$$

Thus all the relations derived on the assumption that $q(t) \in C^0(j)$ and $q(t) \neq 0$ (see, e. g., [3]) and employing only relations derived in the present section remain valid even if only $q(t) \in C^0(j)$ is required.

3. In the previous section we have seen that a number of statements concerning the 2nd phase $\beta(t)$ of the differential equation (1) do not need the assumption that $q(t) \neq 0$ as long as they do not require the function inverse to $\beta(t)$. Still, this is not absolutely true for basic central dispersions of the 2nd, 3rd or 4th kind.

If only $q(t) \in C^0(j)$ is required, then the solution $u(t)$ of the differential equation (1) need not have a zero between the two zeros of its first derivative. Let us therefore consider, for a while, the differential equation (1) oscillatory in $(-\infty, \infty)$ and for which $q(t) \in C^0((-\infty, \infty))$. According to [1] we may always define the basic central dispersion of the 1st kind of this equation as the function $\varphi(t)$; $\varphi(t)$ is the first zero lying to the right of t of an arbitrary solution of the differential equation (1) which vanishes at t. There holds: $\varphi(t) \in C^3((-\infty, \infty))$, $\varphi'(t) > 0$. In case of $q(t) \neq 0$, the b. c. dispersion of the 2nd kind is defined as a function $\psi(t)$; $\psi(t)$ is the first zero lying to the right of t of the derivative $u'(t)$ of an arbitrary solution $u(t)$ of the differential equation (1) such that $u'(t) = 0$. Without the requirement $q(t) \neq 0$, however, this definition cannot be generally applied, since the function u' could be zero either on some entire interval, or the number t could be an accumulation point of the zeros of u'. Furthermore, the b. c. dispersion of the 3rd kind is defined as $\chi(t)$; $\chi(t)$ is the first zero lying to the right of t of the derivative u' of an arbitrary solution $u(t)$ such that $u(t) = 0$. In the oscillatory case, this definition may always be applied. It would seem that the b. c. dispersion of the 2nd kind could, in analogy with the properties of the b.c. dispersion of the 2nd kind in case of $q(t) \neq 0$, be defined as a function $\psi(t) = \varphi[\chi^{-1}(t)]$.

Let then $q(t) \in C^0(j)$. Let $t' \in j$ and $z(t)$ be a solution of the differential equation (1) such that $z(t') = 0$. Suppose there exists $t > t'$ such that $z'(t) = 0$. Define $\chi(t') = \min\{t : t > t' \text{ and } z'(t) = 0\}$. (In case of $q(t) \neq 0$, the function $\chi(t)$ obviously coincides with the b.c. dispersion of the 3rd kind.) The general solution of the differential equation (1) may be written in the form:

$$y(t;\ k_1, k_2) = k_1 \frac{\sin[\alpha(t) - k_2]}{\sqrt{|\alpha'(t)|}}. \tag{12}$$

The derivative of the solution $y(t;\ k_1, k_2)$ is, by (8), given by the relation

$$y'(t; k_1, k_2) = \pm k_1 \frac{s(t)}{W} \sin[\beta(t) - k_2],$$

where $\alpha(t)$ and $\beta(t)$ are the 1st and the 2nd phase of the differential equation (1), respectively, with regard to the solutions u, v, whose Wronskian is equal to W. The function $s(t) = \sqrt{u'^2 + v'^2} \neq 0$. Let, for example, $\alpha'(t) > 0$ and $0 < \beta(t) - \alpha(t) < \pi$. Choose t_0 in the interval of definition of the function $\chi(t)$. Set $k_2 = \alpha(t_0)$. According to (5), there is always $\beta(t) > \alpha(t) > \alpha(t_0)$ for $t > t_0$. Therefore $\chi(t_0)$ is the least of the numbers $t \in (t_0, \infty)$ for which $\beta(t) - \alpha(t_0) = \pi$. Hence there holds:

$$\beta(\chi(t)) = \alpha(t) + \pi \tag{13}$$

for those of the t, for which the corresponding functions are defined. From this relation it can also be seen that the function $\chi(t)$ need not be continuous. Neither can we, on the other hand, state that every function $\chi^*(t)$ complying with the Abelian relation (13) will be the function $\chi(t)$ we have defined.

Suppose, again, $q \in C^0(j)$, $t' \in j$. Let $z(t)$ be a solution of the differential equation (1) such that $z'(t') = 0$ and that there exists a $t > t'$ such that $z(t) = 0$. Set

$$\omega(t') = \min\{t : t > t' \text{ and } z(t) = 0\}.$$

(Obviously, in case of $q \neq 0$, the function $\omega(t)$ coincides with the b.c. dispersion of the 4th kind of the differential equation (1).) If $\alpha(t)$ and $\beta(t)$ stand for the 1st and the 2nd phase of the same pair of independent solutions of the differential equation (1), respectively, and there holds $\alpha'(t) > 0$ as well as $0 < \beta(t) - \alpha(t) < \pi$, then

$$\alpha(\omega(t)) = \beta(t). \tag{14}$$

The relation (14) applies for all the t for which the above functions are defined. For such t we can already write $\omega(t) = \alpha^{-1}(\beta(t))$. Hence the function $\omega(t)$ is continuous, its 1st derivative included, but it need not be generally increasing.

Analogously as in the case of $q \neq 0$, we may employ the functions $\chi(t)$ and $\omega(t)$ to define the function

$$\psi(t) = \chi(\omega(t))$$

for those of the t, for which the function $\chi(\omega(t))$ applies. (It is again obvious that the function $\psi(t)$ coincides, in the case $q \neq 0$, with the b.c. dispersion of the 2nd kind.) But, generally, the function $\psi(t)$ need be neither continuous nor increasing. For the 1st phase $\alpha(t)$ and the 2nd phase $\beta(t)$ of the same pair of independent solutions of the differential equation (1) for which $\alpha'(t) > 0$ as well as $0 < \beta(t) - \alpha(t) < \pi$, and for those of the t for which the functions are defined, there holds

$$\beta(\psi(t)) = \beta[\chi(\omega(t))] = \alpha(\omega(t)) + \pi = \beta(t) + \pi. \tag{15}$$

But it may not be stated, generally, that every function $\psi^*(t)$ satisfying the relation (15) is the above defined function $\psi(t)$.

To sum up:

The functions ψ, χ and ω may be defined, for every differential equation (1) *with $q(t) \in C^0(j)$, so that they:*

1. *coincide, in case of $q(t) \neq 0$, with the b.c. dispersions of the 2nd, 3rd and 4th kind of the differential equation* (1),

2. *satisfy the Abelian relations in those intervals in which the functions in* (13), (14) *and* (15) *are defined.*

Nevertheless:

1. the functions ψ and χ need not be uniquely determined by these Abelian relations,

2. the function χ is increasing, but it need not be continuous; the function ω is continuous, its 1st derivative included, but it need not be increasing; the function ψ need be neither increasing nor continuous.

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