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VARIATIONAL PROBLEMS IN DOMAINS WITH CUSP POINTS

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Summary. The finite element analysis of linear elliptic problems in two-dimensional domains with cusp points (turning points) is presented. This analysis needs on one side a generalization of results concerning the existence and uniqueness of the solution of a continuous elliptic variational problem in a domain the boundary of which is Lipschitz contionuous and on the other side a presentation of a new finite element interpolation theorem and other new devices.

Keywords: finite element method, nonlipschitz boundary, cusp points (turning points), maximum angle condition, minimum angle condition

AMS classification: 65N30

1. Continuous problem

In this paper we shall study elliptic variational problems in two-dimensional bounded domains with cusp points. A typical domain Ω considered is indicated in Fig. 1. Points B, C, D are cusp points, i.e. the points of the boundary $\partial\Omega$ in which two smooth parts of $\partial\Omega$ have the same tangent. For simplicity we restrict ourselves to the case where these tangents are othogonal to the baseline OF which lies on the x-axis. The point O is the origin of the Cartesian coordinate system (x, y) whose axes x and y are oriented in the directions of OF and OA, respectively.

We assume that Ω is a bounded domain (not necessarily simply connected) and its boundary piecewise smooth; more precisely, $\partial\Omega$ is piecewise of class C^∞. (The domain Ω in Fig. 1 can be considered as an outline of an exotic building. In this case we prescribe the homogeneous Dirichlet boundary condition on the baseline OF. However, for a greater simplicity we shall not consider the two-dimensional elasticity in Problem 1.1 and the homogeneous Dirichlet boundary condition will not be restricted to the baseline OF.)

We distinquish two kinds of cusp points: internal ones and external ones. The point C is an internal cusp point, the points B, D are external cusp points.

The variational problem in which we are interested reads as follows (some notions are introduced in Problem 1.1 intuitively and will be explained and precised in subsections 1.3–1.7).

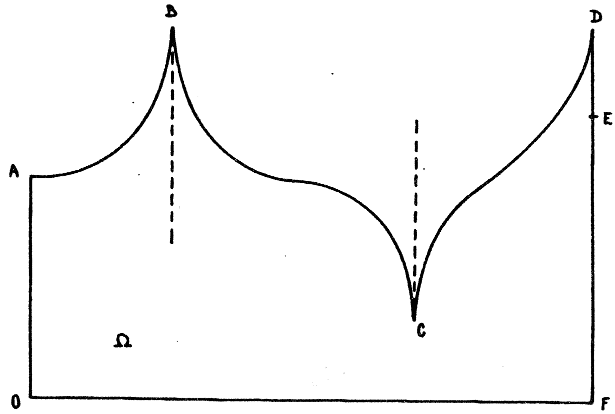

Figure 1.

1.1. Problem. Let Ω be a bounded two-dimensional domain of the type indicated in Fig. 1. Let Γ_1 be a part of $\partial\Omega$ with $\operatorname{mes}_1\Gamma_1 > 0$. Let

$$V = \left\{v \in H^1(\Omega) \colon \operatorname{tr} v = 0 \ \text{ a.e. on } \Gamma_1\right\}. \tag{1.1}$$

Find $u \in V$ such that

$$a(u,v) = L^{\Omega}(v) + L^{\Gamma}(v) \qquad \forall v \in V \tag{1.2}$$

where

$$a(u,v) = \int_{\Omega} \left(k_1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + k_2 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

$$(k_i = k_i(x,y) \geqslant \mu_0 > 0, k_i \in H^{1,\infty}(\Omega),\ i = 1,2), \tag{1.3}$$

$$L^{\Omega}(v) = \int_{\Omega} v\, f \mathrm{d}x\mathrm{d}y \quad (f \in L_2(\Omega)), \tag{1.4}$$

$$L^{\Gamma}(v) = \int_{\Gamma_2} (\operatorname{tr} v)\, q \,\mathrm{d}s \quad (q \in L_2(\Gamma_2),\ \Gamma_2 \equiv \partial\Omega - \overline{\Gamma}_1). \tag{1.5}$$

1.2. Remark. The symbol $H^{k,p}(\Omega)$ denotes the Sobolev space in the sense of [3, Section 5.4]. As usual, we denote $H^k(\Omega) := H^{k,2}(\Omega)$.

1.3. Line integral. We divide $\partial\Omega$ into a finite number of arcs $s_1, \ldots, s_m$ with nonoverlapping interiors (i.e., two arbitrary arcs can have common only the end points). Each arc is either a segment parallel to one of the coordinate axes or a part of a smooth curve. In the first case it has one of the expressions

$$(1.6) \qquad x = a, \quad y \in [y_i, y_{i+1}],$$

$$(1.7) \qquad y = b, \quad x \in [x_j, x_{j+1}];$$

in the second case it has both the expressions

$$(1.8) \qquad y = f_k(x), \quad x \in [x_k, x_{k+1}]$$

and

$$(1.9) \qquad x = g_k(y), \quad y \in [y_k, y_{k+1}].$$

We have

$$(1.10) \qquad x = g_k(f_k(x)), \quad x \in [x_k, x_{k+1}],$$

$$(1.11) \qquad y = f_k(g_k(y)), \quad y \in [y_k, y_{k+1}].$$

Moreover, we assume that the partition $s_1, \dots, s_m$ of $\partial\Omega$ is constructed in such a way that in the case of (1.8), (1.9) at least one of the following inequalities holds (this is always possible):

$$(1.12) \qquad |f_k'(x)| \leqslant K, \quad x \in [x_k, x_{k+1}],$$

$$(1.13) \qquad |g_k'(y)| \leqslant K, \quad y \in [y_k, y_{k+1}].$$

Let, for example, inequality (1.13) hold (this is true in a neighbourhood of the cusp-point B in Fig. 1). We say that the function $r\colon s_k \to \mathbf{R}^1$ belongs to $L_1(s_k)$ if the function $\psi\colon (y_k, y_{k+1}) \to \mathbf{R}^1$ belongs to $L_1(y_k, y_{k+1})$ where

$$\psi(y) = r(g_k(y), y).$$

In this case we define

$$(1.14) \qquad \int_{s_k} r(x,y)\,\mathrm{d}s := \int_{y_k}^{y_{k+1}} r(g_k(y), y)\sqrt{1 + (g_k'(y))^2}\,\mathrm{d}s$$

provided that $y_k < y_{k+1}$. Definition (1.14) is not contradictory because

$$\left|\int_{s_k} r(x,y)\,\mathrm{d}s\right| \leqslant \sqrt{1+K^2} \int_{y_k}^{y_{k+1}} |\psi(y)|\,\mathrm{d}y < \infty$$

and because we have, according to the theorem on the change of variables in the Lebesgue integral:

a) in the case $x_k = g_k(y_k) < x_{k+1} = g_k(y_{k+1})$:

$$\int_{y_k}^{y_{k+1}} r(g_k(y), y)\sqrt{1 + (g_k'(y))^2}\, \mathrm{d}y = \int_{x_k}^{x_{k+1}} r(x, f_k(x))\sqrt{1 + (f_k'(x))^2}\, \mathrm{d}x\,;$$

b) in the case $x_k = g_k(y_k) > x_{k+1} = g_k(y_{k+1})$:

$$\int_{y_k}^{y_{k+1}} r(g_k(y), y)\sqrt{1 + (g_k'(y))^2}\, \mathrm{d}y = \int_{x_{k+1}}^{x_k} r(x, f_k(x))\sqrt{1 + (f_k'(x))^2}\, \mathrm{d}x.$$

It should be noted that the boundary $\partial\Omega$ from Fig. 1 is not continuous in the sense of [5] in any neighbourhood of the cusp point D. This is the reason that we prove the following theorem.

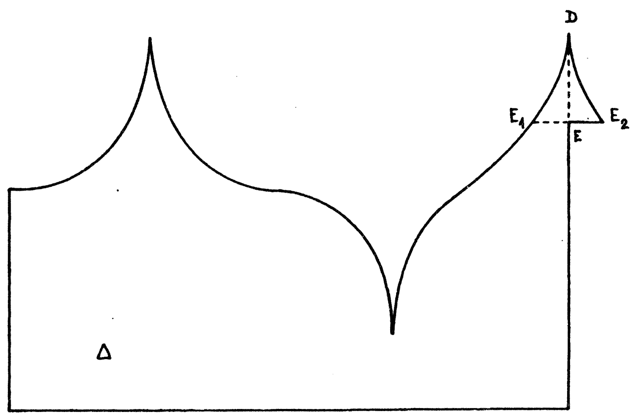

Figure 2.

1.4. Theorem. *Let Ω be a bounded two-dimensional domain of the type indicated in Fig. 1. Then the set $\mathcal{E}_{\overline{\Omega}}(\mathbf{R}^2)$ is dense in $H^1(\Omega)$.*

Proof. A) Let us consider the domain Ω appearing in Fig. 1. Let ω be the interior of the curved triangle whose one side is the segment DE and the remaining two sides lie on the straight line $y = y_E$ and the curve CD. Let Δ be the domain defined by

$$\Delta = \Omega \cup S \cup \tilde{\omega}$$

where S is the interior of the segment DE and

$$\tilde{\omega} = \{[\xi, \eta] : \xi = 2x_D - x,\ \eta = y,\ [x, y] \in \omega\}.$$

The domain Δ has a continuous boundary $\partial\Delta$ in the sense of [5]; thus the set $\mathscr{E}_{\overline{\Delta}}(\mathbf{R}^2)$ is dense in $H^1(\Delta)$. In part B we shall prove the existence of an extension operator $E\colon H^1(\Omega) \to H^1(\Delta)$. These results imply the assertion of Theorem 1.3.

B) First we prove that the values of $u \in H^1(\Omega)$ are almost everywhere uniquely determined on DE. Let us choose $\varepsilon > 0$ arbitrarily small. Let $D_\varepsilon = [x_D, y_D - \varepsilon]$ and let

$$\omega_\varepsilon = \{[x,y]\colon [x,y] \in \omega,\ y < y_D - \varepsilon\}.$$

Then the domain ω_ε has a Lipschitz-continuous boundary $\partial\omega_\varepsilon$ and there exists the Calderon extension operator $E_C^\varepsilon\colon H^1(\omega_\varepsilon) \to H^1(\mathbf{R}^2)$. This result and the fact that every Sobolev space H^1 is the Beppo Levi space (see [5]) imply that the value of $E_C^\varepsilon(u)$, where $u \in H^1(\omega_\varepsilon)$, are uniquely determined at almost all points lying on the segment $D_\varepsilon E$ (because for every $u \in H^1(\omega_\varepsilon)$ the extension $E_C^\varepsilon(u)$ is absolutely continuous almost on all lines which are orthogonal to $D_\varepsilon E$ and intersect $D_\varepsilon E$). We can set $u = E_C^\varepsilon(u)$ on $D_\varepsilon E$. As $\varepsilon > 0$ is arbitrary we see that u is almost everywhere uniquely determined on DE.

Now we set

$$(Ev)(2x_D - x, y) = v(x,y), \quad [x,y] \in \omega,$$
$$(Ev)(x,y) = v(x,y), \quad [x,y] \in \Omega$$

These relations and the fact that u is on DE almost everywhere uniquely determined imply that that Eu has on Δ the structure of a function from the Beppo Levi space. Hence $Eu \in H^1(\Delta)$. □

Theorem 1.4 is used in the proof of the following trace theorem.

1.5. Theorem. *Let s_k be a curved arc with the cusp-point B or D as one of its end points. Then there exists a uniquely determined bounded linear mapping tr: $H^1(\Omega) \to L_2(\varrho_k; s_k)$ where $L_2(\varrho_k; s_k)$ denotes the weighted L_2 space defined on s_k with the weight function*

$$\varrho_k(x,y) = \operatorname{dist}([x,y], t_Q) \tag{1.15}$$

where t_Q is the straight line tangent to s_k at the cusp point Q ($Q = B$ or D). Thus the trace inequality has the form

$$\int_{s_k} \varrho_k(x,y)[(\mathrm{tr}u)(x,y)]^2\,\mathrm{d}s \leqslant C\|u\|^2_{1,\Omega} \quad \forall u \in H^1(\Omega). \tag{1.16}$$

The mapping $\mathrm{tr}: H^1(\Omega) \to L_2(\varrho_k; s_k)$ is such that $(\mathrm{tr}\,u)(x,y) = u(x,y)$ for all $u \in \mathscr{E}_{\overline{\Omega}}(\mathbf{R}^2)$ and $[x,y] \in s_k$.

Proof. The proof is a simple generalization of [5, p. 15]. We shall consider the case $x_k < x_{k+1}$, $y_k < y_{k+1}$ with $Q = [x_Q, y_Q] \equiv [x_{k+1}, y_{k+1}]$. Let $v \in \mathscr{E}_{\overline{\Omega}}(\mathbf{R}^2)$ be an arbitrary function. For every $y \in [y_k, y_{k+1}]$ we have

$$v(g_k(y), y) = -\int_{g_k(y)}^{\tau} \frac{\partial v}{\partial x}(\xi, y)\,\mathrm{d}\xi + v(\tau, y)$$

where $g_k(y) \leqslant \tau \leqslant x_Q$. Applying the Schwarz inequality we obtain

$$[v(g_k(y), y)]^2 \leqslant 2\left\{[x_Q - g_k(y)] \int_{g_k(y)}^{\tau_Q} \left[\frac{\partial v}{\partial x}(\xi, y)\right]^2 \mathrm{d}\xi + [v(\tau, y)]^2\right\}.$$

Let us integrate this inequality with respect to τ in the interval $[g_k(y), x_Q]$:

$$(x_Q - g_k(y))[v(g_k(y), y)]^2 \leqslant 2[x_Q - g_k(y)]^2 \int_{g_k(y)}^{x_Q} \left[\frac{\partial v}{\partial x}(\xi, y)\right]^2 \mathrm{d}\xi + 2\int_{g_k(y)}^{x_Q} [v(\tau, y)]^2 \mathrm{d}\tau.$$

The integration with respect to y in the interval $[y_k, y_{k+1}]$ gives

(1.17)
$$\int_{y_k}^{y_{k+1}} (x_Q - g_k(y))[v(g_k(y), y)]^2 \mathrm{d}y \leqslant 2\max\left(\max_{[y_k, y_{k+1}]} [x_Q - g_k(y)]^2, 1\right) \|v\|_{1,\Omega}^2.$$

Let $u \in H^1(\Omega)$ be an arbitrary function. According to Theorem 1.4, we can find a sequence $\{v_n\} \subset \mathscr{E}_{\overline{\Omega}}(\mathbf{R}^2)$ such that $\|v_n - u\|_{1,\Omega} \to 0$. By (1.17) the sequence $\{v_n(g_k(y), y)\}$ is a Cauchy sequence in $L_2(\varrho_k; (y_k, y_{k+1}))$. As the space $L_2(\varrho_k; (y_k, y_{k+1}))$ is complete (see, e.g., [3, p. 37]) the sequence $\{v_n(g_k(y), y)\}$ converges in this space to a function which will be denoted $(\operatorname{tr} u)(g_k(y), y)$. Hence relations (1.17) and (1.13) imply

(1.18)
$$\int_{y_k}^{y_{k+1}} (x_Q - g_k(y))[(\operatorname{tr} u)(g_k(y), y)]^2 \sqrt{1 + [g_k'(y)]^2}\,\mathrm{d}y \leqslant C\|u\|_{1,\Omega}^2 \quad \forall u \in H^1(\Omega).$$

Transforming the integral on the left-hand side of (1.18) by means of (1.8) we obtain

$$\int_{y_k}^{y_{k+1}} (x_Q - g_k(y))[(\operatorname{tr} u)(g_k(y), y]^2 \sqrt{1 + [g_k'(y)]^2}\,\mathrm{d}y$$

$$= \int_{x_k}^{x_{k+1}} (x_Q - x)[(\operatorname{tr} u)(x, f_k(x))]^2 \sqrt{1 + [f_k'(x)]^2}\,\mathrm{d}x \tag{1.19}$$

As

$$\operatorname{dist}([x, y], t_Q) = x_Q - g_k(y) = x_Q - x \quad [x, y] \in s_k$$

inequality (1.16) follows from (1.18), (1.19) and the definition of the line integral.

The linearity of the mapping $\operatorname{tr} : H^1(\Omega) \to L_2(\varrho_k, s_k)$ is obvious. □

1.6. Corollary. *Let us choose the point E in such a way that*

$$\operatorname{dist}(D, E) \leqslant |y_i - y_{i+1}|$$

where s_i is the curved arc which has the cusp point D as one of its end points. Then

$$\int_{y_D}^{y_E} [x_D - g_i(y)][(\operatorname{tr} u)(x_D, y)]^2 \mathrm{d}y \leqslant C\|u\|_{1,\Omega}^2 \quad \forall u \in H^1(\Omega). \tag{1.20}$$

1.7. Remark. Combining the preceding results with the standard trace theorem we can write

$$\int_\Gamma \varrho(x, y)[(\operatorname{tr} u)(x, y)]^2 \mathrm{d}s \leqslant C\|u\|_{1,\Omega}^2 \quad \forall u \in H^1(\Omega) \tag{1.21}$$

where Γ is an arbitrary measurable part of $\partial\Omega$ with $\operatorname{mes}_1\Gamma > 0$ and $\varrho(x, y) = 1$ outside certain neighbourhoods of external cusp points.

1.8. Friedrichs' inequality. *Let Ω be a domain of the type indicated in Fig. 1. Let Γ be a part of $\partial\Omega$ such that* $\operatorname{mes}_1 \Gamma > 0$. *Then we have*

$$\|u\|_{1,\Omega}^2 \leqslant C\left(\int_\Gamma \varrho \cdot (\operatorname{tr} u)^2 \mathrm{d}s + |u|_{1,\Omega}^2\right) \quad \forall u \in H^1(\Omega). \tag{1.22}$$

Proof. The proof is a simple modification of the proof of [5, Theorem 1.1.8]. We set ${}^1\|u\| = \|u\|_{1,\Omega}$ and

$${}^2\|u\| = \left(\int_\Gamma \varrho \cdot (\operatorname{tr} u)^2 \mathrm{d}s + |u|_{1,\Omega}\right)^{\frac{1}{2}}.$$

It is easy to se that ${}^2\|u\|$ is a norm in $H^1(\Omega)$. By the trace theorem we can write

$${}^2\|u\| \leqslant C({}^1\|u\|). \tag{1.23}$$

Let us denote $B_1 = H^1(\Omega)$ and let B_2 is a linear space with the same elements as B_1 and the norm ${}^2\|\cdot\|$. It is evident that the identical operator $I_{1,2}\colon B_1 \to B_2$ is additive. By (1.23) it is also bounded. We can show similarly as in the proof of [5, Theorem 1.1.8] that the space B_2 is complete. Then we use the Banach theorem, according to which the identical operator $I_{2,1}\colon B_2 \to B_1$, the inverse operator to $I_{1,2}$, is additive and continuous; hence it is bounded:

$${}^2\|u\| \leqslant C({}^1\|u\|).$$

This is inequality (1.22). □

1.9. Existence and uniqueness theorem. *Let the assumptions of Problem* 1.1 *be satisfied and let the function* $q \in L_2(\Gamma_2)$ *be on each curved arc* s_k, *which has an external cusp point as its end point, of the form*

$$q(x,y) = \sqrt{\varrho_k(x,y)}\, p(x,y), \quad p \in L_2(s_k) \tag{1.24}$$

and on the segment DE *of the form*

$$q(x,y) = \sqrt{x_D - g_i(y)}\, p(x,y), \quad p \in L_2(\overline{DE}). \tag{1.25}$$

Then Problem 1.1 *has a unique solution.*

Proof. We have, according to (1.3), $a(v,v) \geqslant \mu_0 |v|^2_{1,\Omega}$. Hence, by Friedrichs' inequality,

$$\|v\|^2_{1,\Omega} \leqslant C\, a(v,v) \quad \forall v \in V. \tag{1.26}$$

By Schwarz's inequality and (1.3), (1.4),

$$|a(v,w)| \leqslant M\, \|v\|_{1,\Omega} \|w\|_{1,\Omega} \quad \forall v, w \in H^1(\Omega), \tag{1.27}$$

$$|L^{\Omega}(v)| \leqslant \|f\|_{0,\Omega} \|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega). \tag{1.28}$$

Using trace inequality in the form mentioned in Remark 1.7 and Schwarz's inequality we obtain from assumptions (1.24), (1.25) and expression (1.5)

$$\begin{aligned} |L^{\Gamma}(v)| &\leqslant \sum_{k=1}^{m} \|p\|_{0,s_k} \sqrt{\int_{s_k} \varrho_k \cdot (\operatorname{tr} v)^2 \mathrm{d}s} \\ &\leqslant C \left(\sum_{k=1}^{m} \|p\|_{0,s_k} \right) \|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega). \end{aligned} \tag{1.29}$$

We see from (1.3)–(1.5) and (1.26)–(1.29) that the assumptions of Lax-Milgram lemma are satisfied. □

2. Discretization by the finite element method. Convergence theorems

To simplify our considerations in this section, we restrict ourselves to the case

$$k_i(x,y) \equiv 1 \quad (i = 1,2), \quad f(x,y) \equiv 1, \quad [x,y] \in \Omega. \tag{2.1}$$

Further, in order to be able to apply the finite element technique we shall assume that

$$q(x,y) = 0 \quad \forall [x,y] \in S_\delta(Q) \cap \Gamma_2 \quad (Q = B, D) \tag{2.2}$$

where $S_\delta(Q)\,(\delta > 0)$ is the square with the sides parallel to the axes x, y, with the center of gravity at the external cusp-point Q and with $\mathrm{mes}_2\, S_\delta(Q) = \delta^2$. This assumption is in some elasticity problems quite natural (see Fig. 3). In addition we assume (in accordance with assumptions used in the finite element theory) that the function q is piecewise of class C^2.

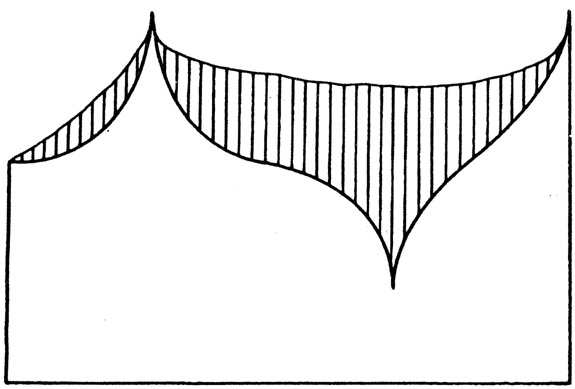

Figure 3.

Let us denote

$$\Omega_\delta^Q := \Omega \cap S_\delta(Q) \quad (Q = B, D), \tag{2.3}$$

$$\Omega_\delta := \Omega - (S_\delta(B) \cup S_\delta(D)). \tag{2.4}$$

We have $\overline{\Omega} = \overline{\Omega}_\delta \cup \overline{\Omega}_\delta^B \cup \overline{\Omega}_\delta^D$. Every triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ is constructed in such a way that it is a union of triangulations of $\overline{\Omega}_\delta$, $\overline{\Omega}_\delta^B$ and $\overline{\Omega}_\delta^D$. Triangulations of $\overline{\Omega}_\delta^Q$ $(Q = B, D)$ obey naturally the maximum angle condition (see Fig. 4 a,b,c)

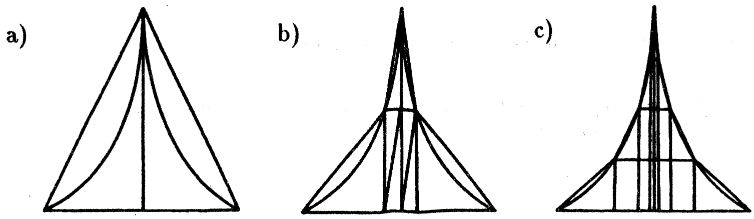

Figure 4.

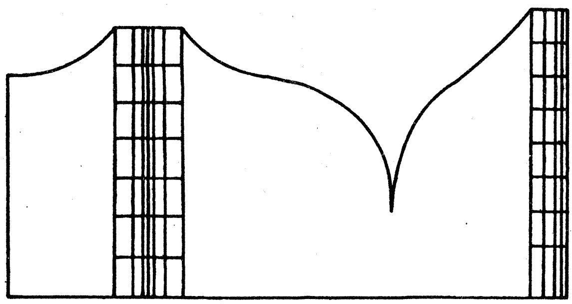

Figure 5.

and triangulations of $\overline{\Omega}_\delta$ can obey (and in this paper will obey) the minimum angle condition along the curved parts of the boundary $\partial\Omega_\delta$ (see Fig. 5).

2.1. Notation. We shall denote by Ω_h ($h \in (0, h_0)$) polygonal domains approximating the domain Ω such that $\overline{\Omega}_h$ is a union of all $\overline{T} \in \mathcal{T}_h$ (for more detail see Assumption 2.2). By τ_h, ω_h we denote the sets

$$\tau_h = \Omega_h - \overline{\Omega}, \quad \omega_h = \Omega - \overline{\Omega}_h.$$

We further denote

$$\tau_{h,\delta} = \tau_h \cap \Omega_{\delta,h}, \quad \tau_{h,Q} = \tau_h \cap \Omega^Q_{\delta,h},$$
$$\omega_{h,\delta} = \omega_h \cap \Omega_\delta, \quad \omega_{h,Q} = \omega_h \cap \Omega^Q_\delta$$

where $\overline{\Omega}_{\delta,h}$ and $\overline{\Omega}^Q_{\delta,h}$ are the parts of $\overline{\Omega}_h$ approximating $\overline{\Omega}_\delta$ and $\overline{\Omega}^Q_\delta$, respectively.

2.2. Assumption. Let $\sigma_h = \{P_1, \ldots, P_n\}$ be the set of all vertices in $\mathcal{T}_h$. We assume:

a) $\sigma_h \subset \overline{\Omega}$, $\sigma_h \cap \partial\Omega_h \subset \partial\Omega$,

b) $\overline{\Gamma}_1 \cap (\partial\Omega - \Gamma_1) \subset \sigma_h$,

c) the points of $\partial\Omega$, where the conditions of C^3-smoothness of $\partial\Omega$ and C^2-smoothness of q are not satisfied, are elements of σ_h.

2.3. Assumption. Triangulations of $\overline{\Omega}_\delta$ obey the minimum angle condition along the curved parts of $\partial\Omega_\delta$.

2.4. Assumption. We consider only such domains Ω for which

$$\begin{gathered}\operatorname{mes}_2(T - T^{\mathrm{id}}) \leqslant K \operatorname{mes}_2 T^{\mathrm{id}} \\ \forall T \in \mathscr{T}_h \cap S_\delta(Q) \quad \text{with} \quad \operatorname{mes}_2(T \cap T^{\mathrm{id}}) > 0;\ h \in (0, h_0)\end{gathered} \tag{2.5}$$

where K depends only on Ω (see Remark 2.5). The ideal triangle T^{id} corresponding to a boundary triangle T is the part of Ω which is approximated by T.

2.5. Remark. The set of domains satisfying (2.5) is not empty. For example, condition (2.5) is satisfied if $\partial\Omega$ is described in $S_\delta(B)$ by the relations

$$x - x_B = \pm(y_B - y)^n, \quad y \in [y_B - \delta, y_B] \quad (n \geqslant 2)$$

and in $S_\delta(D)$ by the relations

$$\begin{gathered}x - x_D = -(y_D - y)^m, \quad y \in [y_D - \delta, y_D] \quad (m \geqslant 2) \\ x = x_D, \quad y \in [y_D - \delta, y_D].\end{gathered}$$

Now we generalize the discrete Friedrichs inequality introduced in [8, Section 29].

2.6. Notation. We denote

$$X_h = \{w \in C(\overline{\Omega}_h)\colon w \text{ is a linear function on } T \text{ for all } T \in \mathscr{T}_h\}. \tag{2.6}$$

Let $w \in X_h$. The function $\overline{w}\colon \overline{\Omega}_h \cup \overline{\Omega} \to \mathbf{R}^1$ such that $\overline{w} = w$ on $\overline{\Omega}_h$ and

$$\overline{w}\big|_{T^{\mathrm{id}} - T} = p\big|_{T^{\mathrm{id}} - T} \text{ if } \operatorname{mes}_2(T^{\mathrm{id}} - T) > 0,$$

where $p(x, y)$ is the polynomial of first degree satisfying $p\big|_T = w\big|_T$, is called the natural extension of the function w. (T^{id} denotes the ideal triangle which is approximated by T.)

2.7. Assumption. For a greater simplicity of exposition, which does not concern the cusp points, we assume that Γ_1 is the segment OF. In this case the finite element approximation of the space V has the form

$$V_h = \{w \in X_h : w = 0 \text{ on the baseline } OF\}. \tag{2.7}$$

Let us note that Assumption 2.7 enables us both to avoid the use of functions $\widehat{w} \in V$ associated with $w \in V_h$ (see [7, Sections 37–50]) and to satisfy assumptions of Theorem 2.14.

2.8. Theorem. (Generalized discrete Friedrichs' inequalities). *We have*

$$\|v\|_{1,\Omega_h} \leqslant C_0 |v|_{1,\Omega_h} \quad \forall v \in V_h \;, \forall h \in (0, h_0). \tag{2.8}$$

Proof. A) We have

$$\frac{\|v\|^2_{1,\Omega_h}}{|v|^2_{1,\Omega_h}} = \frac{\|\overline{v}\|^2_{1,\Omega}}{|\overline{v}|^2_{1,\Omega}} \cdot \frac{1 + \|v\|^2_{1,\tau_h} \|\overline{v}\|^{-2}_{1,\Omega} - \|\overline{v}\|^2_{1,\omega_h} \|\overline{v}\|^{-2}_{1,\Omega}}{1 + |v|^2_{1,\tau_h} |\overline{v}|^{-2}_{1,\Omega} - |\overline{v}|^2_{1,\omega_h} |\overline{v}|^{-2}_{1,\Omega}}$$

By Theorem 1.8,

$$\|\overline{v}\|^2_{1,\Omega} |\overline{v}|^{-2}_{1,\Omega} \leqslant C \quad \forall \overline{v} \in V.$$

Considering the relations

$$|v|^2_{i,\tau_h} = |v|^2_{i,\tau_{h,\delta}} + |v|^2_{i,\tau_{h,B}} + |v|^2_{i,\tau_{h,D}} \quad (i = 0, 1),$$

$$|\overline{v}|^2_{i,\omega_h} = |\overline{v}|^2_{i,\omega_{h,\delta}} \quad (i = 0, 1)$$

and inspecting the proof of [8, (29.1)] we see that it is sufficient to prove

$$|v|^2_{1,\tau_{h,Q}} |\overline{v}|^{-2}_{1,\Omega} \leqslant C, \tag{2.9}$$

$$\|v\|^2_{1,\tau_{h,Q}} \|\overline{v}\|^{-2}_{1,\Omega} \leqslant C. \tag{2.10}$$

In part B we shall prove

$$|v|^2_{1,T-T^{\mathrm{id}}} \leqslant K |v|^2_{1,T^{\mathrm{id}}} \quad \forall T \quad \text{with } \operatorname{mes}_2(T \cap \tau_{h,Q}) > 0. \tag{2.11}$$

Summing (2.11) over all T appearing in (2.11) we easily obtain (2.9).

In part C we shall prove that

$$\|v\|^2_{0,T-T^{\mathrm{id}}} \leqslant C \|v\|^2_{1,T^{\mathrm{id}}} \quad \forall T \text{ with } \operatorname{mes}_2(T \cap \tau_{h,Q}) > 0. \tag{2.12}$$

Summing (2.12) over all T appearing in (2.12) and using also (2.9) we easily obtain (2.10).

B) As $|(\nabla v|_T)| = \text{const.}$ for $v \in X_h$ we have, according to Assumption 2.4,

$$|v|^2_{1,T-T^{\mathrm{id}}} = |(\nabla v|_T)|^2 \operatorname{mes}_2(T - T^{\mathrm{id}}) \leqslant K |(\nabla v|_T)|^2 \operatorname{mes}_2 T^{\mathrm{id}} \leqslant K |v|^2_{1,T^{\mathrm{id}}}.$$

C) Let the boundary triangle $T \in \mathcal{T}_h \cap S_\delta(Q)$ have the vertices $P_1(g(b), b)$, $P_2(g(b), a)$, $P_3(g(a), a)$, where $y_Q - \delta \leqslant a < b \leqslant y_Q$ (we write for simplicity $g(y)$ instead of $g_k(y)$). As $T \in \mathcal{T}_h$ we have

$$b - a < h, \quad g(a) - g(b) < h. \tag{2.13}$$

Let $v(x, y)$ be the linear polynomial satisfying

$$v(P_i) = z_i \quad (i = 1, 2, 3).$$

Then

$$v(x, y) = A(z_1, z_2, z_3) + \frac{z_3 - z_2}{g(a) - g(b)}\, x + \frac{z_2 - z_1}{a - b}\, y \tag{2.14}$$

where

$$A(z_1, z_2, z_3) = \begin{vmatrix} g(b) & b & 1 \\ g(b) & a & 1 \\ g(a) & a & 1 \end{vmatrix}^{-1} \begin{vmatrix} g(b) & b & z_1 \\ g(b) & a & z_2 \\ g(a) & a & z_3 \end{vmatrix}. \tag{2.15}$$

We shall prove (2.12).

Let $z_M = \max(|z_1|, |z_2|, |z_3|)$. If $z_M = 0$ then (2.12) is satisfied. Let now $z_M \neq 0$. First we consider the case when the condition

$$z_i > 0 \quad (i = 1, 2, 3) \quad \text{or} \quad z_i < 0 \quad (i = 1, 2, 3) \tag{2.16}$$

is not satisfied. We have

$$\|v\|^2_{0, T - T^{\mathrm{id}}} < z_M^2 \ \mathrm{mes}_2(T - T^{\mathrm{id}}). \tag{2.17}$$

If $z_M = |z_1|$ then either $z_M \leqslant |z_1 - z_2|$ or

$$z_M \leqslant |z_1 - z_2| + |z_2| \leqslant |z_1 - z_2| + |z_2 - z_3|.$$

Using (2.13) with $h < 1$ we see that in both cases we have

$$z_M^2 < \frac{2(z_1 - z_2)^2}{(b - a)^2} + \frac{2(z_2 - z_3)^2}{(g(a) - g(b))^2} = 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2. \tag{2.18}$$

If $z_M = |z_2|$ or $z_M = |z_3|$ then we again obtain (2.18).

Combining (2.5), (2.17) and (2.18) we find that

$$\|v\|_{0,T-T^{\mathrm{id}}}^2 \leqslant 2K|v|_{1,T^{\mathrm{id}}}^2$$

which implies (2.12).

It remains to analyze the case (2.16). If $z_1 = z_2 = z_3$ then (2.12) is evident. In the opposite case the mean value theorem gives

$$\|v\|_{0,T^{\mathrm{id}}}^2 = z_0^2 \operatorname{mes}_2 T^{\mathrm{id}} \tag{2.19}$$

where $\min(z_1, z_2, z_3) < z_0 < \max(z_1, z_2, z_3)$. It follows from (2.15) that

$$A(z_1, z_2, z_3) - z_0 = A(z_1 - z_0, z_2 - z_0, z_3 - z_0).$$

Hence

$$\begin{aligned} v(x,y) &= z_0 + v(x,y) - z_0 = z_0 + \Big\{A(z_1 - z_0, z_2 - z_0, z_3 - z_0) \\ &+ \frac{(z_3 - z_0) - (z_2 - z_0)}{g(a) - g(b)}\, x + \frac{(z_2 - z_0) - (z_1 - z_0)}{a - b}\, y\Big\}. \end{aligned} \tag{2.20}$$

Using the preceding results and relations (2.19), (2.20) we find that

$$\begin{aligned} \|v\|_{0,T-T^{\mathrm{id}}}^2 &\leqslant 2z_0^2 \operatorname{mes}_2(T - T^{\mathrm{id}}) + 2\|v - z_0\|_{0,T-T^{\mathrm{id}}}^2 \leqslant 2Kz_0^2 \operatorname{mes}_2 T^{\mathrm{id}} + \\ &+ 2K|v - z_0|_{1,T^{\mathrm{id}}}^2 = 2K\|v\|_{0,T^{\mathrm{id}}}^2 + 2K|v|_{1,T^{\mathrm{id}}}^2 = 2K\|v\|_{1,T^{\mathrm{id}}}^2 \end{aligned}$$

which completes the proof of relation (2.12). □

Now we formulate the discrete problem which approximates Problem 1.1 in case of (2.1)–(2.7) and prove a corresponding abstract error estimate.

2.9. Discrete problem. Find $u_h \in V_h$, where V_h is defined by (2.7), such that

$$\widetilde{a}_h(u_h, v) = \widetilde{L}_h^{\Omega}(v) + L_h^{\Gamma}(v) \quad \forall v \in V_h \tag{2.21}$$

with

$$\widetilde{a}_h(w, v) = \int_{\Omega_h} \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y \quad \forall w, v \in H^1(\Omega_h), \tag{2.22}$$

$$\widetilde{L}_h^{\Omega}(v) = \int_{\Omega_h} v \mathrm{d}x\mathrm{d}y \quad \forall v \in H^1(\Omega_h) \tag{2.23}$$

and with $L_h^\Gamma(v)$ as an approximation of the linear form

$$\widetilde{L}_h^\Gamma(v) = \int_{\Gamma_{h2}} q_h \, v \, \mathrm{d}s \quad \forall v \in H^1(\Omega_h) \tag{2.24}$$

by means of a quadrature formula. Here Γ_{h2} is the part of $\partial\Omega_h$ approximating Γ_2 and q_h is a function defined on Γ_{h2} and approximating q. For our purposes it is sufficient to know that

$$q_h(P_i) = q(P_i) \quad \forall P_i \in \Gamma_{h2} \cap \sigma_h \tag{2.25}$$

where σ_h is the set of all nodal points of $\mathscr{T}_h$. (As to the definition of q_h see [2].) Relations (2.25) enable us to use the trapezoidal formula.

2.10. Remark. We shall prove by an example that, in general, we cannot find any neighbourhood $U(Q)$ of an external cusp point Q such that there exist a linear bounded operator

$$E_k \colon H^k(\Omega \cap U(Q)) \to H^k(U(Q))$$

with the property $E_k u = u$ on $\Omega \cap U(Q)$ for all $u \in H^k(\Omega)$. This is the reason that we restrict ourselves to the spaces $H^1(\Omega \cap \Omega_h)$ in the following theorem.

Let us consider the domain Ω bounded by the curves $x = x_0$, $y = x^\alpha$ and $y = -x^\alpha$ where $\alpha > 2 + \varepsilon$ $(0 < \varepsilon < 1)$. The function

$$u(x, y) = x^{-(1+\varepsilon)/2}$$

belongs to $H^1(\Omega)$ because

$$\|u\|_{1,\Omega}^2 = \frac{x_0^{\alpha-\varepsilon}}{\alpha - \varepsilon} + \frac{(1+\varepsilon)^2}{4(\alpha - 2 - \varepsilon)} x_0^{\alpha-2-\varepsilon}.$$

Let us assume that there exist any domain $\widetilde{\Omega} \in \mathscr{C}^{0,1}$ and any function $\widetilde{u} \in H^1(\widetilde{\Omega})$ such that $\overline{\Omega} \subset \widetilde{\Omega}$ and $\widetilde{u} = u$ on Ω. Then $u \in L_p(\Omega)$ for arbitrary $p \in [1, \infty)$ (because $H^1(\widetilde{\Omega})$ is compactly imbedded in $L_p(\widetilde{\Omega})$ in the two-dimensional case). However, it is easy to see that $u \notin L_p(\Omega)$ for $p > 2(\alpha + 1)/(1 + \varepsilon)$ which is a contradiction.

2.11. Theorem. *Let $u \in V$ and $u_h \in V_h$ be the solutions of Problems 1.1 and 2.4, respectively. Then*

$$\begin{aligned} \|u - u_h\|_{1,\Omega\cap\Omega_h} \leqslant C \Bigg\{ & \inf_{v \in V_h} \Bigg(\|u - v\|_{1,\Omega\cap\Omega_h} + \sup_{w \in V_h} \frac{|v|_{1,\tau_h} |w|_{1,\tau_h}}{\|w\|_{1,\Omega\cap\Omega_h}} \Bigg) \\ & + \sup_{w \in V_h} \Bigg(\frac{|u|_{1,\omega_h} |\overline{w}|_{1,\omega_h}}{\|w\|_{1,\Omega\cap\Omega_h}} + \frac{\sqrt{\operatorname{mes}_2 \tau_h}\, \|w\|_{0,\tau_h}}{\|w\|_{1,\Omega\cap\Omega_h}} + \frac{\sqrt{\operatorname{mes}_2 \omega_h}\, \|\overline{w}\|_{0,\omega_h}}{\|w\|_{1,\Omega\cap\Omega_h}} \\ & + \frac{|\widetilde{L}_h^\Gamma(w) - L^\Gamma(\overline{w})|}{\|w\|_{1,\Omega\cap\Omega_h}} + \frac{|L_h^\Gamma(w) - \widetilde{L}_h^\Gamma(w)|}{\|w\|_{1,\Omega\cap\Omega_h}} \Bigg) \Bigg\}. \end{aligned} \tag{2.26}$$

Proof. The existence and uniqueness of $u_h \in V_h$ is evident. We have

$$\|u-u_h\|_{1,\Omega\cap\Omega_h} \leqslant \|u-v\|_{1,\Omega\cap\Omega_h} + \|u_h - v\|_{1,\Omega\cap\Omega_h} \quad \forall v \in V_h. \tag{2.27}$$

We estimate the second term on the right-hand side of (2.27). By (2.22), $\widetilde{a}_h(w,w) = |w|^2_{1,\Omega_h}$. Using this relation and Theorem 2.8 we find

$$\begin{aligned} C_0^{-1}\|u_h - v\|^2_{1,\Omega\cap\Omega_h} &\leqslant C_0^{-1}\|u_h - v\|^2_{1,\Omega_h} \leqslant |u_h - v|^2_{1,\Omega_h} \\ &= \widetilde{a}_h(u_h - v, u_h - v) \quad \forall v \in V_h. \end{aligned} \tag{2.28}$$

Further, using (2.21) and the forms

$$\widetilde{L}^{\Omega}_G(\overline{v}) = \int_G \overline{v}\,\mathrm{d}x\mathrm{d}y \quad \forall v \in X_h \tag{2.29}$$

$$\widetilde{a}_G(\overline{w},\overline{v}) = \int_G \left(\frac{\partial \overline{w}}{\partial x}\frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{w}}{\partial y}\frac{\partial \overline{v}}{\partial y}\right)\mathrm{d}x\mathrm{d}y, \quad v, w \in X_h, \tag{2.30}$$

where $G \subset \overline{\Omega} \cup \overline{\Omega}_h$ is an arbitrary measurable set, we can write

$$\begin{aligned} \widetilde{a}_h(u_h - v, u_h - v) &= \widetilde{L}^{\Omega}_h(u_h - v) + L^{\Gamma}_h(u_h - v) - \widetilde{a}_h(v, u_h - v) \\ &= L^{\Omega}(\overline{u}_h - \overline{v}) - \widetilde{L}^{\Omega}_{\omega_h}(\overline{u}_h - \overline{v}) + \widetilde{L}^{\Omega}_{\tau_h}(u_h - v) \\ &\quad + L^{\Gamma}_h(u_h - v) + L^{\Gamma}(\overline{u}_h - \overline{v}) - L^{\Gamma}(\overline{u}_h - \overline{v}) \\ &\quad + \widetilde{L}^{\Gamma}_h(u_h - v) - \widetilde{L}^{\Gamma}_h(u_h - v) \\ &\quad - [a(\overline{v}, \overline{u}_h - \overline{v}) - \widetilde{a}_{\omega_h}(\overline{v}, \overline{u}_h - \overline{v}) + \widetilde{a}_{\tau_h}(v, u_h - v)] \\ &= a(u - \overline{v}, \overline{u}_h - \overline{v}) - \widetilde{L}^{\Omega}_{\omega_h}(\overline{u}_h - \overline{v}) + \widetilde{L}^{\Omega}_{\tau_h}(u_h - v) \\ &\quad + \{\widetilde{L}^{\Gamma}_h(u_h - v) - L^{\Gamma}(\overline{u}_h - \overline{v})\} \\ &\quad + \{L^{\Gamma}_h(u_h - v) - \widetilde{L}^{\Gamma}_h(\overline{u}_h - \overline{v})\} \\ &\quad + \widetilde{a}_{\omega_h}(\overline{v}, \overline{u}_h - \overline{v}) - \widetilde{a}_{\tau_h}(v, u_h - v). \end{aligned} \tag{2.31}$$

Using the relation

$$a(u - \overline{v}, \overline{w}) = a_{\Omega\cap\Omega_h}(u - v, w) + \widetilde{a}_{\omega_h}(u - \overline{v}, \overline{w})$$

and the estimates

$$|\widetilde{L}^{\Omega}_G(\overline{w})| \leqslant \sqrt{\operatorname{mes}_2 G}\|\overline{w}\|_{0,G} \quad (G = \tau_h, \omega_h),$$

$$|\widetilde{a}_G(\overline{v}, \overline{w})| \leqslant 2|\overline{v}|_{1,G}|\overline{w}|_{1,G} \quad (G = \tau_h, \omega_h, \Omega \cap \Omega_h)$$

we easily obtain (2.26) from (2.27), (2.28) and (2.31). □

The estimates of the terms on the right-hand side of (2.26) are introduced in the following five lemmas. The proof of the first lemma is based on a very simple (but new) finite element interpolation theorem and on a density theorem in V.

2.12. Interpolation theorem. *Let $\overline{T}^{\text{id}}$ be a closed concave triangle with two straight sides which are parallel to the axes x, y and have the vertex P_1^T as a common point. Let $w \in C^2(\overline{T}^{\text{id}})$ and let w_I be the linear polynomial satisfying*

$$w_I(P_i^T) = w(P_i^T) \quad (i = 1, 2, 3) \tag{2.32}$$

where P_1^T, P_2^T, P_3^T are the vertices of $\overline{T}^{\text{id}}$. Then

$$|(D^\alpha(w - w_I))(x, y)| \leqslant C(|\alpha|)\, M_2(w, \overline{T}^{\text{id}}) h_T^{2-|\alpha|}, \ |\alpha| \leqslant 1, \ [x, y] \in \overline{T}. \tag{2.33}$$

where $h_T = \max_{i,j} \operatorname{dist}(P_i^T, P_j^T)$, $C(0) = 4$, $C(1) = 3$ and

$$M_i(w, \overline{G}) = \max_{|\alpha| = i, [x,y] \in \overline{G}} |(D^\alpha w)(x, y)| \tag{2.34}$$

with $D^\alpha w$ the multiindex notation for derivatives.

Proof. The proof is a simple application of the Rothe and Taylor theorems.

Let x_i, y_i be the coordinates of the vertex P_i^T. Then either $y_1 = y_2$, $x_1 = x_3$ or $y_1 = y_3$, $x_1 = x_2$. Let, for example, the first possibility hold with $x_1 < x_2$. We have $w(x_i, y_1) = w_I(x_i, y_1)$ $(i = 1, 2)$. Hence, by the Rolle Theorem, there exists ξ such that $(\partial w/\partial x)(\xi, y_1) = \partial w_I/\partial x$, $x_1 < \xi < x_2$. Let us choose arbitrary $\overline{x} \in [x_1, x_2]$, $\overline{x} \neq \xi$ and let $K(\overline{x})$ be such a constant that

$$\frac{\partial w}{\partial x}(\overline{x}, y_1) - \frac{\partial w_I}{\partial x} = K(\overline{x})(\overline{x} - \xi). \tag{2.35}$$

To find the expression for $K(\overline{x})$ let us set

$$g(x) = \frac{\partial w}{\partial x}(x, y_1) - \frac{\partial w_I}{\partial x} - K(\overline{x})(x - \xi).$$

The function $g(x)$ satisfies $g(\overline{x}) = g(\xi) = 0$. By the Rolle theorem $g'(\eta) = 0$ where η is a point on the x-axis lying between $\overline{x}$ and ξ. Hence

$$K(\overline{x}) = \frac{\partial^2 w}{\partial x^2}(\eta, y_1).$$

As $\overline{x}$ is chosen in $[x_1, x_2]$ arbitrarily we obtain from (2.35) and (2.34)

$$\left| \frac{\partial w}{\partial x}(x, y_1) - \frac{\partial w_I}{\partial x} \right| \leqslant M_2(w, \overline{T}^{\text{id}}) |x_1 - x_2|, \quad x \in [x_1, x_2]. \tag{2.36}$$

Similarly we find

$$(2.37)\qquad \left|\frac{\partial w}{\partial y}(x_1, y) - \frac{\partial w_I}{\partial y}\right| \leqslant M_2(w, \overline{T}^{\mathrm{id}})|y_1 - y_3|, \quad y \in [y_1, y_3].$$

Using relation (2.32) with $i = 1$, relation (2.36) with $x = x_1$, relation (2.37) with $y = y_1$ and the fact that $D^\alpha w_I \equiv 2$ for $|\alpha| = 0$ we derive (2.33) by means of Taylor's theorem written for functions $\varphi_\alpha \equiv D^\alpha(w - w_I)$ $(|\alpha| \leqslant 1)$ at the point (x_1, y_1). □

2.13. Remark. a) If $\gamma_T \neq \frac{1}{2}\pi$, where γ_T is the angle made by the segments $P_1^T P_2^T$ and $P_1^T P_3^T$, then we can derive estimate (2.33) with

$$C(0) = 2 + \frac{8}{\sin\gamma_T}, \quad C(1) = 2 + \frac{4}{\sin\gamma_T}.$$

This result is a generalization of Synge's interpolation theorem (see [7, p. 209–213)) to the case of concave triangles.

b) The result mentioned in a) holds, of course, for triangles with straight sides, γ_T being the maximum angle of the triangle T.

2.14. Density theorem. *Let Ω be a bounded two-dimensional domain of the type indicated in Fig. 1. Let for each cusp point Q_k there exist a neighbourhood B_k such that $B_k \cap \Gamma_1 = \emptyset$ where Γ_1 is the part of $\partial\Omega$ on which the homogeneous Dirichlet boundary condition is prescribed (see (1.1)). Then the set $V \cap C^\infty(\overline{\Omega})$, where V is given by (1.1), is dense in V.*

Proof. The proof is a combination of the proof of Doktor's similar result for $\Omega \in \mathscr{C}^{0,1}$ (see [1]) and the proof of density theorem in $H^k(\Omega)$, $\Omega \in \mathscr{C}^{0,0}$ (see [4, Theorem 5.5.9]) with the generalization introduced in Theorem 1.4.

For a greater simplicity we shall consider the case that $\partial\Omega$ has a continuous boundary in the sense of [5] with one cusp point Q_1 only. A generalization to a more general case is straightforward (see also the proof of Theorem 1.4). Let B_1 be the corresponding neighbourhood of Q_1. Let $\{G_0, G_1, \ldots, G_n\}$ be a family of open sets such that

$$\Omega \subset \bigcup_{r=0}^{n} G_r, \quad \partial\Omega \subset \bigcup_{r=1}^{n} G_r$$

with $\overline{G}_1 \subset B_1$, $Q_1 \in G_1$, $Q_1 \notin G_r$ $(r \neq 1)$ and let $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ be the corresponding partition of unity. Using the method of the proof of [4, Theorem 5.5.9] we can find a sequence $\{v_{n,1}\} \subset C^\infty(\overline{\Omega})$ $(v_{n,1} = 0$ outside of $\overline{G}_1)$ such that $v_{n,1} \to v\varphi_1$

in $H^1(\Omega)$, where $v \in H^1(\Omega)$ is a chosen function. Using the method of [1] we can find sequences $\{v_{n,r}\} \subset C^\infty(\overline{\Omega}) \cap V$ ($v_{n,r} = 0$ outside of $\overline{\Gamma}_r$, $r = 2, \ldots, n$) such that $v_{n,r} \to v\varphi_r$ in $H^1(\Omega)$. Finally, there exists a sequence $\{v_{n,0}\} \subset C_0^\infty(G_0)$ such that $v_{n,0} \to v\varphi_0$ in $H^1(\Omega)$ ($v_{n,0} = 0$ outside of $\overline{\Omega}_0$). Let us set $v_n = \sum_{r=0}^{n} v_{n,r}$. We have $v_n \in V \cap C^\infty(\overline{\Omega})$ and

$$v_n \to \sum_{r=0}^{n} v\varphi_r = v \text{ in } H^1(\Omega)$$

which proves the assertion. □

2.15. Lemma. *Let $\varepsilon > 0$. Let $u_\varepsilon \in V \cap C^\infty(\overline{\Omega})$ be such that $\|u - u_\varepsilon\|_{1,\Omega} < \frac{\varepsilon}{2}$. Then we have for h sufficiently small*

$$\|u - I_h u_\varepsilon\|_{1,\Omega\cap\Omega_h} < \varepsilon$$

where $I_h u_\varepsilon \in V_h$ is the interpolant of u_ε, i.e. the function satisfying $(I_h u_\varepsilon)(P_i) = u\varepsilon(P_i)$ $\forall P_i \in \sigma_h$. Moreover, if $u \in C^2(\overline{\Omega})$ then

$$\|u - I_h u\|_{1,\Omega\cap\Omega_h} \leqslant C h.$$

Proof. In both parts of the proof we shall consider for a greater simplicity the situation that every triangle $T \in \mathcal{T}_h$ lying along the boundary $\partial\Omega_h$ satisfies one of the following inclusions:

$$T \subset T^{\text{id}} \quad \text{or} \quad T^{\text{id}} \subset T. \tag{2.38}$$

In the general case there exists a neighbourhood of each internal cusp point where every boundary triangle T satisfies $(2.38)_1$. Similarly, there exists a neighbourhood of each external cusp point where $(2.38)_2$ is satisfied only.

This means that the situation where we have simultaneously for boundary triangles $T \in \mathcal{T}_h$

$$\text{mes}_2(T - T^{\text{id}}) > 0, \quad \text{mes}_2(T^{\text{id}} - T) > 0 \tag{2.39}$$

can occur only outside certain neighbourhoods of the cusp points. Outside these neighbourhoods we can locally extend every function $v \in H^k(\Omega)$. The extension Ev belongs to $H^k(\Omega \cup \Lambda)$ where Λ is a strip (or a union of strips) lying along a certain part $\partial_1\Omega$ of $\partial\Omega$. This part $\partial_1\Omega$ belongs to Λ and is a part of $\partial\Lambda$. (The proof follows from [6, pp. 22–24].) This fact enables us to use in the case of (2.39) the approaches introduced in [8]. We omit these technical details.

A) The existence of the function u_ε with properties introduced in Lemma 2.15 follows from Theorem 2.14. Thus the proof of the first part of Lemma 2.15 follows from the inequality

$$\|u - I_h u_\varepsilon\|_{1,\Omega\cap\Omega_h} \leqslant \|u - u_\varepsilon\|_{1,\Omega\cap\Omega_h} + \|u_\varepsilon - I_h u_\varepsilon\|_{1,\Omega\cap\Omega_h}$$

provided that

$$\|u_\varepsilon - I_h u_\varepsilon\|_{1,\Omega\cap\Omega_h} < \tfrac{1}{2}\varepsilon \quad \forall h < h_1(\varepsilon). \tag{2.40}$$

It remains to prove (2.40). We have

$$\|u_\varepsilon - I_h u_\varepsilon\|^2_{1,\Omega\cap\Omega_h} = \sum_{T^{\mathrm{id}}} \|u_\varepsilon - I_h u_\varepsilon\|^2_{1,T^{\mathrm{id}}} + \sum_{T} \|u_\varepsilon - I_h u_\varepsilon\|^2_{1,T} \tag{2.41}$$

where the sum in the first term on the right-hand side of (2.41) is taken over all T^{id} for which $T^{\mathrm{id}} \subset T$; in the second term we sum over all interior triangles of $\mathcal{T}_h$ and all boundary triangles satisfying $T \subset T^{\mathrm{id}}$. Owing to Remark 2.13b Theorem 2.12 can be used on all terms appearing on the right-hand side of (2.41) and we obtain

$$\|u_\varepsilon - I_h u_\varepsilon\|_{1,\Omega\cap\Omega_h} \leqslant C\, h \sqrt{\operatorname{mes}_2 \Omega}\, M_2(u_\varepsilon, \overline{\Omega}).$$

Thus estimate (2.40) is satisfied for $h < h_1(\varepsilon)$.

B) If $u \in C^2(\overline{\Omega})$ then considering similarly as in part A of this proof we find that

$$\|u - I_h u\|_{1,\Omega\cap\Omega_h} \leqslant C\, h \sqrt{\operatorname{mes}_2 \Omega}\, M_2(u, \overline{\Omega})$$

which proves the second part of Lemma 2.15. □

2.16. Lemma. *We have*

$$|I_h u_\varepsilon|_{1,\mathcal{T}_h} |w|_{1,\mathcal{T}_h} \leqslant C\, K(u_\varepsilon) h^{\frac{1}{2}} \|w\|_{1,\Omega\cap\Omega_h}, \quad w \in V_h, \quad h < h_1(\varepsilon),$$

$$|I_h u|_{1,\mathcal{T}_h}\, |w|_{1,\mathcal{T}_h} \leqslant C\, K(u)\, h^{\frac{1}{2}} \|w\|_{1,\Omega\cap\Omega_h}, \quad v \in V_h, \quad h < h_0.$$

where $K(v) = \max_{i=1,2} \{M_i(v, \overline{\Omega})\}$.

Proof. Similarly as in the proof of Lemma 2.15 we consider the simplified situation (2.38).

A) As $|(\nabla w|_T)| = \text{const.}$ we have, according to Assumption 2.4,

$$|w|^2_{1,\mathcal{T}_h\cap S_\delta(Q)} \leqslant K \|w\|^2_{1,\Omega\cap\Omega_h}. \tag{2.42}$$

Assumption 2.3 and [8, Lemma 28.8] give

$$|w|_{1,\tau_{h,\delta}} \leqslant C h^{\frac{1}{2}} |w|_{1,\Omega\cap\Omega_h}. \tag{2.43}$$

Similarly we obtain

$$|I_h u_\varepsilon|^2_{1,\tau_h\cap S_\delta(Q)} \leqslant K \sum_{T^{\mathrm{id}}\in S_\delta(Q)} |I_h u_\varepsilon|^2_{1,T^{\mathrm{id}}}, \tag{2.44}$$

$$|I_h u_\varepsilon|_{1,\tau_{h,\delta}} \leqslant C h^{\frac{1}{2}} \sum_{T^{\mathrm{id}}\notin S_\delta(Q)} |I_h u_\varepsilon|_{1,T^{\mathrm{id}}}. \tag{2.45}$$

Inequalities (2.42)–(2.45) imply

$$|I_h u_\varepsilon|_{1,\tau_h} |w|_{1,\tau_h} \leqslant C\|w\|_{1,\Omega\cap\Omega_h} \Big(\sum_{T^{\mathrm{id}}} |I_h u_\varepsilon|^2_{1,T^{\mathrm{id}}}\Big)^{\frac{1}{2}}$$

$$\leqslant 2C\|w\|_{1,\Omega\cap\Omega_h} \Big\{\Big(\sum_{T^{\mathrm{id}}} |u_\varepsilon|^2_{1,T^{\mathrm{id}}}\Big)^{\frac{1}{2}} + \Big(\sum_{T^{\mathrm{id}}} |u_\varepsilon - I_h u_\varepsilon|^2_{1,T^{\mathrm{id}}}\Big)^{\frac{1}{2}}\Big\}. \tag{2.46}$$

The i-th term in the brackets on the right-hand side of (2.46) is bounded by

$$C M_i(u_\varepsilon, \overline{\Omega}) h^{\frac{i}{2}} \ (i = 1, 2)$$

because the measure of the union of T^{id} is bounded by Ch. Hence the first assertion of Lemma 2.16 follows.

B) The second estimate of Lemma 2.16 can be obtained in a similar way. It suffices to replace u_ε by u. In the case of triangulations indicated in Fig. 4 we cannot obtain better estimate than in the case of u_ε. (It should be noted that the term $h^{\frac{1}{2}}$ appearing in (2.43) and (2.45) has not any influence on the resulting estimate in this case. For further developments see Remark 2.21.) □

2.17. Lemma. *We have for $u \in H^j(\Omega)$ $(j = 1, 2)$ and all $w \in V_h$*

$$|u|_{1,\omega_h} |\overline{w}|_{1,\omega_h} \leqslant C h^{j+\frac{1}{2}} \|w\|_{1,\Omega\cap\Omega_h}, \quad h < h_0.$$

Proof. The proof follows from Assumption 2.3 and [8, Lemmas 28.3 and 28.8]. □

2.18. Lemma. *We have for all $w \in V_h$*

$$\sqrt{\mathrm{mes}_2\, G_h}\, \|\overline{w}\|_{0,G_h} \leqslant C h \|w\|_{1,\Omega\cap\Omega_h} \quad (G_h = \tau_h, \omega_h).$$

Proof. As $\mathrm{mes}_2\, G_h \leqslant Ch^2$ the proof follows from considerations similar to the proof of Theorem 2.8 which give $\|\overline{w}\|_{0,G_h} \leqslant C\|w\|_{1,\Omega\cap\Omega_h}$. □

2.19. Lemma. *Let q be piecewise of class C^2. Then we have for all $w \in V_h$*

$$|\widetilde{L}_h^\Gamma(w) - L^\Gamma(\overline{w})| + |L_h^\Gamma(w) - \widetilde{L}_h^\Gamma(w)| \leqslant C\,h\,\|w\|_{1,\Omega\cap\Omega_h}.$$

Proof. The proof follows from [8, Lemmas 30.1 and 30.2] and the assumption that $q = 0$ on $\Gamma_2 \cap S_\delta(Q)$ $(Q = B,\, D)$. □

The following theorem is an immediate consequence of Theorem 2.11 and Lemmas 2.15–2.19.

2.20. Convergence theorem. *Under the preceding asuumptions we have*

$$\lim_{h\to 0+} \|u - u_h\|_{1,\Omega\cap\Omega_h} = 0.$$

In addition, if $u \in C^2(\overline{\Omega})$ then

$$\|u - u_h\|_{1,\Omega\cap\Omega_h} \leqslant C\,h^{\frac{1}{2}}.$$

2.21. Remark. If we refine the triangulation $\mathcal{T}_h$ along the boundary $\partial\Omega$ in the neighbourhoods of external cusp points by segments which are tangent to $\partial\Omega$ (see Fig. 6 where the corresponding part of $\partial\Omega$ is only dashed) then we improve the rate of convergence in Lemma 2.12 (and thus also in Theorem 2.16) from $O(\sqrt{h})$ to $O(h)$. This result follows from the fact that the measure of the new narrow triangles is $O(h^3)$, according to Taylor's theorem.

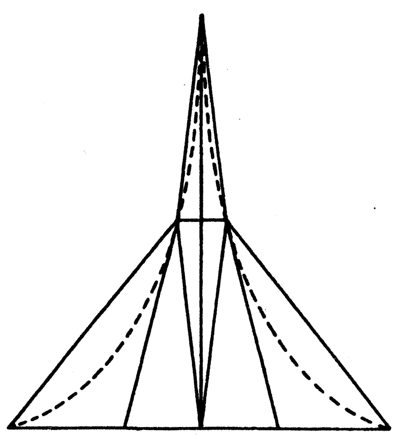

Figure 6.

2.22. Remark. The procedure just described and its analysis is not restricted to the case of linear triangular finite elements. We can use also quadratic and cubic triangular finite elements of the Lagrange type. In the neighbourhoods of external cusp points we use the following modifications of these finite elements:

a) In the case of *quadratic* finite elements we prescribe along the curved side the function values only at the vertices of the concave triangle. The remaining four parameters are as follows: function values at the third vertex and at the mid-points of two straight sides and the second mixed derivative at the vertex not lying on the boundary.

b) In the case of *cubic* finite elements we again prescribe along the curved side the function values only at the vertices of the concave triangle. The remaining eight parameters are as follows: function values at P_1 (the vertex not lying on the boundary) and at the four points which divide the straight sides into thirds and the mixed derivatives

$$\frac{\partial^2 p}{\partial x \partial y}(P_1),\ \frac{\partial^3 p}{\partial x^2 \partial y}(P_1),\ \frac{\partial^3 p}{\partial x \partial y^2}(P_1).$$

c) The accuracy of interpolation is $O(h_T^{n+1-|\alpha|})$ $(n = 2, 3;\ |\alpha| \leqslant 1)$. (The proof is similar to the proof of Theorem 2.12.)

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