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ON THE EFFICIENT USE OF THE GALERKIN-METHOD TO SOLVE FREDHOLM INTEGRAL EQUATIONS

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Summary. In the present paper we describe, how to use the Galerkin-method efficiently in solving boundary integral equations. In the first part we show how the elements of the system matrix can be computed in a reasonable time by using suitable coordinate transformations. These techniques can be applied to a wide class of integral equations (including hypersingular kernels) on piecewise smooth surfaces in 3-D, approximated by spline functions of arbitrary degree.

In the second part we show, how to use the panel-clustering technique for the Galerkin-method. This technique was developed by Hackbusch and Nowak in [6, 7] for the collocation method. In that paper it was shown, that a matrix-vector-multiplication can be computed with a number of $O(n \log^{\kappa+1} n)$ operations by storing $O(n \log^{\kappa} n)$ sizes. For the panel-clustering-technique applied to Galerkin-discretizations we get similar asymptotic estimates for the expense, while the reduction of the consumption for practical problems (1 000–15 000 unknowns) turns out to be stronger than for the collocation method.

Keywords: boundary element method, Galerkin method, numerical cubature, panel-clustering-algorithm

AMS classification: 65D30, 65D32, 65N38, 45B05, 45E05

1. INTRODUCTION

The boundary element method is an efficient tool to solve homogeneous linear PDE numerically. To sketch this technique let us consider boundary value problems of the form:

$$\begin{aligned} Lu &= 0 \quad \text{in } \Omega \\ Bu &= g \quad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{1.1}$$

where L denotes an elliptic differential operator of order $2m$:

$$L := \sum_{|\alpha| \leqslant 2 \cdot m} c_\alpha(x) D^\alpha,$$

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and Ω is a bounded domain in $\mathbf{R}^d$ or the complement of a bounded domain: $\Omega := \mathbf{R}^d \setminus \bar{\Omega}'$. B denotes a system of m differential operators acting on the boundary Γ of Ω. Throughout this paper we assume that Γ is a piecewise smooth surface in $\mathbf{R}^d$. For simplicity we assume that there is a set

$$\mathfrak{M} := \{i_1, i_2, \ldots, i_m\}, \qquad 0 \leqslant i_1 < i_2 < \ldots < i_m \leqslant 2m-1,$$

which defines the boundary operators:

$$B_j := \partial_n^{i_j} \quad \text{for } 1 \leqslant j \leqslant m.$$

Here ∂_n denotes the normal derivative. For more general boundary operators B_j see [1], [16].

The basic tool to transfer problem (1.1) to an integral equation on the boundary is the explicit knowledge of the fundamental solution S, which is defined via Dirac's functional δ_0:

$$LS = \delta_0.$$

The existence of a fundamental solutions is ensured by the theorem of Malgrange-Ehrenpreis (see [10]). For many common differential operators the corresponding fundamental solutions can be found in [10]. With the help of S it is always possible to transform the boundary value problem to a system of m integral equations for the unknown Cauchy data, that means for the computation of the components $u_j := \partial_n^{i_j} u$; $\{i_1, i_2, \ldots, i_m\} = \{0, 1, \ldots, 2m-1\} \setminus \mathfrak{M}$. Then the integral equations takes the form:

$$\lambda_i(x)u_i(x) := \sum_{j=1}^{m} \int_\Gamma k_{i,j}(x, y-x)u_j(x)\,\mathrm{d}o_x + r_i(x) \qquad \forall\, x \in \Gamma,\ 1 \leqslant i \leqslant m. \tag{1.2}$$

The functions $\lambda_i(x)$ are piecewise constant and contain removable discontinuous points. The kernel functions $k_{i,j}(x, y-x)$ are suitable Gâteau derivatives of the fundamental solution $S(y-x)$. The direction of the derivatives depend on the boundary operator B_j. If $B_j = \partial_n^{i_j}$ holds, then $k_{i,j}$ is defined by: $k_{i,j}(x, y-x) := \delta_{n_y}^{2m-i_j} S(y-x)$. For a detailed description of the so-called integral equation method we refer to [1], [2], [5], [16], [17].

In order to develop numerical methods for solving (1.2), it is important to investigate the form of the appearing kernel functions $k_{i,j}$. In the case that the differential operator L has constant coefficients c_α and contains only derivatives of order $2m$, the fundamental solution can always be written in the form:

$$S(x) := \|x\|^{2m-d}\big(K_0(x) + K_1(x)\log\|x\|\big), \tag{1.3}$$

where K_0 is real analytic and $K_0(tx) = K_0(x)$ holds for $x \neq 0, t > 0$. K_1 is vanishing identically, if the spatial dimension d is odd or $d > 2m$. Otherwise K_1 is a homogeneous polynomial of degree $2m - d$. If the coefficients $c_\alpha(x)$ depend analytically on x, (1.3) holds locally for small $\|x\|$. The concept of the integral equation method is not restricted to scalar equations but can also be applied to systems of differential equations. The coefficients of the corresponding kernel matrices can always be written in the form (1.3). These facts and more details can be found in [8, §3] and [3, theorem 4.1]. We restrict our analysis to the three spatial dimensions ($d = 3$), thus no logarithmic term appears in the representation of S. The kernels k of the boundary integral equation are Gâteau derivatives of the fundamental solution, so k (locally) takes always the form:

$$k(x, y - x) := \|y - x\|^{2\cdot(m-j)-3} K_{0,j}(x, y, y - x),$$

where j denotes the order of the Gâteau differentation. $K_{0,j}(x, y, z)$ is real analytic in the third variable and piecewise smooth in x and y, dependent on the smoothness of the surface.

For our further investigation of boundary integral equations we make the following

Assumption 1.1. Let Γ be a two-dimensional manifold in $\mathbf{R}^3$, which we assume to be piecewise smooth. We consider the integral equation:

$$\lambda(x)u(x) + Ku(x) = r(x) \qquad \forall x \in \Gamma \tag{1.4}$$

where

$$Ku(x) := \int_\Gamma k(x, y, y - x)u(y)\,\mathrm{d}o_y. \tag{1.5}$$

We assume that K_0 in (1.3) can be expanded into a power series containing only finitely many non-vanishing terms:

$$K_0(x) := \sum_{0 \leqslant |\nu| \leqslant t} c_\nu \left(\frac{x}{\|x\|}\right)^\nu$$

and for all indices with $c_\nu \neq 0$ the values of $|\nu|$ is even. This assumption is fulfilled in many applications. As a consequence the kernel function $k(x, y, y - x)$ can be written in the form:

$$k(x, y, y - x) = \frac{1}{\|y - x\|^{s+t}} \sum_{|\nu| \geqslant t} c_\nu(x, y)(y - x)^\nu, \tag{1.6}$$

where $s + t$ is odd and $t \in \mathbf{N}_0$. c_α are surface-dependent, piecewise smooth functions in both variables.

To simplify the notation, we restrict ourself to the case of a scalar integral equation. The application of the numerical methods described below to systems of equations is straightforward.

To discretize the boundary integral equation with Galerkin's method we have to transform (1.4) in a variational formulation. Let $\lambda I + K$ be an operator, which maps the Sobolev space $H^s(\Gamma)$ into $H^{-s}(\Gamma)$ for a suitable $s \in \mathbf{R}$. Then the weak formulation of (1.4) reads:

find $u \in H^s(\Gamma)$ such that

$$(\lambda u, v)_0 + (Ku, v)_0 = (r, v)_0 \qquad \forall v \in H^s(\Gamma)$$

holds, where $(.,.)_0$ denotes the scalar product in $L^2(\Gamma)$.

With a sequence of finite-dimensional subspaces $\{H_n\}_{n\in\mathbf{N}}$ of $H^s(\Gamma)$, satisfying:

$$H_n \subset H_{n+1}\ \forall n \in \mathbf{N}, \qquad \lim_{n\to\infty} \inf_{v_n \in H_n} \|x - v_n\|_s = 0\ \forall x \in H^s(\Gamma),$$

the Galerkin formulation of (1.7) reads:

find $u_n \in H_n$ such that

$$(\lambda u_n, v_n)_0 + (Ku_n, v_n)_0 = (r_n, v_n)_0 \qquad \forall v_n \in H_n \tag{1.8}$$

holds. In order to construct the subspaces H_n we assume that Γ is partitioned into finitely many pieces π_i, called panels, forming the penalization $P := \{\pi_1, \pi_2, \ldots, \pi_{NP}\}$. Each π_i can be considered to be the image of a C^r-application $x = \chi_i(u)$, where u is defined on a polygonal parameter domain $\tilde{\pi}_i \subset \mathbf{R}^2$. We restrict our presentation to the case that the parameter domains $\tilde{\pi}_i$ are triangles and the mappings $\chi_i(u)$ are polynomials of degree r in each component:

$$\chi_{i,j}(u) := \sum_{|\alpha|\leqslant r} c_\alpha^j u^\alpha \qquad j \in \{1,2,3\}.$$

The usual regularity assumptions for the penalization P are:

— $\mathring{\pi}_i \cap \mathring{\pi}_j = \emptyset \qquad \forall \pi_i, \pi_j \in P \mid \pi_i \neq \pi_j$,

— each side of π_i is also the side of exactly one panel π_j, $i \neq j$

— the unisolvent set of each π_i w.r.t. the mapping χ_i lie on the surface Γ.

With an unisolvent set $\Xi := \{x_1, \ldots, x_N\} \subset \Gamma_n$ we define a Lagrangian basis $\{\varphi_i\}_{1\leqslant i\leqslant N}^{p,d}$, which are polynomials of degree p on each panel, globally continuous differentiable of order d, and satisfy $\varphi_i(x_j) = \delta_{i,j}$. Then the finite-dimensional subspaces H_n are defined by $H_n := \operatorname{span}\{\varphi_i : 1 \leqslant i \leqslant N\}$. Now the finite-dimensional problem (1.7) can be written in matrix form:

$$(\mathbf{M} + \mathbf{K})\mathbf{u} = \mathbf{f} \tag{1.9}$$

with the $N \times N$ matrices:

$$\mathbf{M}_{i,j} := (\lambda\varphi_j, \varphi_i)_0 \qquad \mathbf{K}_{i,j} := (K\varphi_j, \varphi_i)_0$$

and the right hand side:

$$\mathbf{f}_i := (f, \varphi_i)_0.$$

The solution vector $\mathbf{u}$ corresponds to the solution of (1.8) via:

$$u(x) = I_\Delta \mathbf{u} := \sum_{i=1}^{N} \mathbf{u}_i \varphi_i(x).$$

In many cases the condition number of $\mathbf{K}$ is bounded by a constant independent of the dimension of $\mathbf{K}$, thus iterative solvers need only few iteration steps to give a sufficiently accurate approximation of the solution. The main difficulty in solving the linear system (1.9) lies in the fact that $\mathbf{K}$ is a full matrix. The computation of the matrix elements is rather involved because of the (nearly) singular behavior of the kernel function $k(x, y-x)$ in the neighborhood of $y = x$. In the following section we explain how to compute these elements in a reasonable time to any desired accuracy by using suitable coordinate transformations. It remains the problem that the generation of $\mathbf{K}$ and each matrix-vector multiplication requires $O(N^2)$ operations. The storage requirements are of the same order. Hackbusch and Nowak introduced in [9], [6], [7] the panel-clustering algorithm for the collocation method. In the latter paper it was shown that by storing $O(N \log^\kappa N)$ quantities a matrix-vector multiplication can be approximated by $O(N \log^{\kappa+1} N)$ operations without influencing the asymptotic discretization error. In §3 we develop the panel-clustering algorithm for Galerkin discretizations and present the corresponding error analysis. In §4 we report about numerical test calculations, where it turns out that the reduction of the consumption to solve equation (1.9) by the panel-clustering method is considerable for practical problems (1,000–15,000 unknowns) and not only in the asymptotic behavior. It turned out that the reduction of the panel-clustering method is stronger for the Galerkin procedure than for the collocation method.

Computation of singular and nearly singular surface integrals, arising by discretizing boundary integral equations via the Galerkin method

2.1. Representation of the elements of the Galerkin matrix. Before we analyse the Galerkin integrals we have to introduce the concept of regularized integrals. Let f be a continuous function, defined on a $(d-1)$-dimensional, piecewise

smooth manifold $S \subset \mathbf{R}^d$. Let S be a split into finitely many patches S_i, where we assume that for every S_i there exists a diffeomorphism χ_i mapping a parameter domain $\tilde{S}_i$ bijectively onto S_i. The integral of f over S is defined by:

$$\int_S f(x)\,\mathrm{d}o_x := \sum_{i=1}^{q} \int_{\tilde{S}_i} \sqrt{g_i(\tilde{x})} f(\chi_i(\tilde{x}))\,\mathrm{d}\tilde{x}, \tag{2.1}$$

where g_i denotes Gram's determinant:

$$g_i(\tilde{x}) := \det\{\langle \partial\chi(x)/\partial\tilde{x}_i, \partial\chi(x)/\partial\tilde{x}_j\rangle_{1\leqslant i,j\leqslant q}\}.$$

It is straightforward to extend definition (2.1) to weakly singular functions. To be more concrete we define the order of a singularity for a function f:

Definition 2.1. A functional $f\colon S \to \mathbf{R}$ is called singular of order s in $x_0 \in S$, if s is the infimum over all $t \in \mathbf{R}_0^+$, which satisfy:

$$|f(x) - f(x_0)| \leqslant C\|x - x_0\|^{-t} \quad \forall\, x \in S \cap K_\varepsilon(x_0), \quad \varepsilon \in \mathbf{R}^+ \text{ sufficiently small,} \tag{2.2}$$

where $K_\varepsilon(x_0)$ denotes the ball with centre x_0 and radius ε. The constant C may not depend on t.

We state that f is weakly singular in x_0 if $s < d-1$. If $s = d-1$ the integral (2.1) diverges, thus we have to introduce the Cauchy-principal value:

Definition 2.2. Let $f\colon S \to \mathbf{R}$ be singular in $x_0 \in S$ of order $s = d-1$. Then the *Cauchy-principal value* is defined by:

$$\text{p. v.} \int_S f(x)\,\mathrm{d}o_x := \lim_{\varepsilon\to 0} \int_{S\setminus\{K_\varepsilon(x_0)\cap S\}} f(x)\,\mathrm{d}o_x, \tag{2.3}$$

provided that the right hand side of (2.3) exists.

In the integral equation method the resulting kernels may have a singularity of order d. In this case one has to define the part-fini integral:

Definition 2.3. Let $f\colon S \subset \mathbf{R}^d \to \mathbf{R}$ be a function of the form $f = f_1 \cdot f_2$, where f_1 is singular in x_0 of order d, and f_2 is Hölder continuous in x_0 with exponent $1 + \lambda > 0$. Then the *part-fini integral* of f is defined by:

$$\text{p. f.} \int_S f(x)\,\mathrm{d}o_x := \text{p. v.} \int_S f_1(x)\big(f_2(x) - f_2(x_0)\big)\,\mathrm{d}o_x. \tag{2.4}$$

If the right hand side of (2.4) is finite, f is called *hypersingular*.

With these definitions we can define the elements of the Galerkin matrix $\mathbf{K}$ by:

$$\mathbf{K}_{i,j} := \int_{\mathrm{supp}(\varphi_i)} \varphi_i(x)\ \text{p. v.} \int_{\mathrm{supp}(\varphi_j)} k(x, y-x) b_j(y,x)\, \mathrm{d}o_y \mathrm{d}o_x, \tag{2.5}$$

where $\{\varphi_i\}_{1 \leqslant i \leqslant N}$ denotes the Lagrangian basis, introduced in §1. In (2.5) b_j is defined via:

$$b_j(y,x) = \begin{cases} \varphi_j(y) - \varphi_j(x), & \text{if } k \text{ is hypersingular} \\ \varphi_j(y), & \text{if } k \text{ is Cauchy-singular.} \end{cases}$$

The essential advantage of the Galerkin method in comparison with the collocation procedure is that isolated corners and edges of the surface S do not cause any difficulties, and the integral (2.5) may be split into a sum over the panels of $\mathrm{supp}(\varphi_i)$, $\mathrm{supp}(\varphi_j)$. The details can be found in the following

Theorem 2.4. *Let the surface S be a piecewise smooth, $(d-1)$-dimensional manifold of $\mathbf{R}^d$, penalized by smooth panels (e.g. polynomial surface pieces). Further we assume that the basis functions $\{\varphi_i\}_{1 \leqslant i \leqslant N}$ are Lipschitz continuous on S and smooth on each panel. Let the integrand of (2.5): $k(x, y-x) b_j(y,x)$ be Cauchy-singular.*

Then $\mathbf{K}_{i,j}$ can be split into:

$$\mathbf{K}_{i,j} = \sum_{\pi_x \subset \mathrm{supp}(\varphi_i)} \sum_{\pi_y \subset \mathrm{supp}(\varphi_j)} \mathbf{K}_{i,j}^{\pi_x, \pi_y},$$

where for $\pi_x \neq \pi_y$ the sizes $\mathbf{K}_{i,j}^{\pi_x, \pi_y}$ exist as weakly singular integrals:

$$\mathbf{K}_{i,j}^{\pi_x, \pi_y} = \int_{\pi_x \times \pi_y} \varphi_i(x) k(x, y-x) b_j(y,x)\, \mathrm{d}o_y \mathrm{d}o_x, \tag{2.6}$$

and for $\pi_x = \pi_y$ the limit process can be interchanged with the outer integration:

$$\mathbf{K}_{i,j}^{\pi,\pi} = \lim_{\varepsilon \to 0} \int_{\substack{\pi \times \pi \\ \|x-y\| > \varepsilon}} \varphi_i(x) k(x, y-x) b_j(y,x)\, \mathrm{d}o_y \mathrm{d}o_x. \tag{2.7}$$

Proof. Lemma 3.1.2 in [12] shows that the function $H_\pi(x)$ defined by:

$$H_\pi(x) := \text{p. v.} \int_\pi k(x, y-x) b_j(y,x)\, \mathrm{d}o_y \mathrm{d}o_x$$

is weakly singular. If the case of $\pi_x \neq \pi_y$, (2.6) follows by Fubini's theorem. If $\pi_x = \pi_y$ holds, the result follows by Lebesgue's Lemma. □

We state that the condition of globally Lipschitz continuous basis functions is necessary only for hypersingular kernels.

In the following we shall develop numerical quadrature methods for approximating the integrals $\mathbf{K}_{i,j}^{\pi_x,\pi_y}$. We restrict our presentation to the three-dimensional case. Additional to assumption 1.1 we require that assumption 2.5 holds, which is natural if the approximation of the surface and the ansatz functions are piecewise polynomial.

Assumption 2.5. Let the integral (2.6/7) be represented over the parameter domain $\tilde{\pi}_x \times \tilde{\pi}_y$ via $\chi_{\pi_x}(u)$, $\chi_{\pi_y}(v)$. Usually the panels $\tilde{\pi}$ are polygons. We require all $\tilde{\pi}$ to be triangular, which can always be attained by a suitable partitioning of $\tilde{\pi}$. Let $\widetilde{\Sigma_k}$ be a unisolved set of points of $\tilde{\pi}$, and $\Sigma_k \subset \Gamma$ the corresponding set of points on the surface panel π. The mapping $\chi_\pi^k \colon \tilde{\pi} \to \pi$ is defined by:

$$\chi_{\pi,i}^k(u) := \sum_{|\nu| \leqslant k} c_{\nu,i} u^\nu,$$
$$\chi_{\pi,i}^k(u_j) = x_{j,i}$$

for all $u_j \in \widetilde{\Sigma_k}$ and corresponding $x_j \in \Sigma_k$ and all components $1 \leqslant i \leqslant 3$. In this notation $\chi_{\pi,i}^1$ denotes the affine transformation of $\tilde{\pi}$ onto π. We assume, that the integrands of (2.7) admit an expansion of the form:

$$\begin{aligned}
&\tilde{\varphi}_i(u) k\big(\chi_{\pi_x}^k(u), \chi_{\pi_y}^k(v), \chi_{\pi_y}^k(v) - \chi_{\pi_x}^k(u)\big) \tilde{b}_j(v,u) \sqrt{g_{\pi_x}^k(u) g_{\pi_y}^k(v)} \\
&\quad = \sqrt{g_{\pi_x}^1(u) g_{\pi_y}^1(v)}\, k\big(\chi_{\pi_x}^1(u), \chi_{\pi_y}^1(v) - \chi_{\pi_x}^1(u)\big) \\
&\qquad \times p_j\big(\chi_{\pi_y}^1(v) - \chi_{\pi_x}^1(u)\big) \sum_{|\nu| \geqslant 0} \sigma_\nu^{i,j}(u,v) \left(\frac{\chi_{\pi_y}^1(v) - \chi_{\pi_x}^1(u)}{\|\chi_{\pi_y}^1(v) - \chi_{\pi_x}^1(u)\|} \right)^{\kappa_1 \cdot \nu}
\end{aligned}$$

Here p_j denotes a polynomial of homogeneous degree κ_2. The numbers κ_1, κ_2 are determined by:

$$\kappa_1 = \begin{cases} 2, & \text{if } \pi_x = \pi_y \\ 1, & \text{otherwise} \end{cases}, \qquad \kappa_2 = \begin{cases} 1, & \text{if } \pi_x = \pi_y \text{ and } k \text{ is hypersingular} \\ 0, & \text{otherwise} \end{cases}$$

In [12] it is shown, that under weak assumptions expansion (2.8) holds. Beyond that, in the paper mentioned above, error estimates are presented, if the sum in (2.8) is replaced by the corresponding partial sums.

If assumption (1.6) holds, the integrals (2.6/7) consequently can be written in the form:

$$\mathbf{K}_{i,j}^{\pi_x,\pi_y} = \sum_{|\nu| \geqslant 0} \int_{\pi_x^1 \times \pi_y^1} \frac{\sigma_\nu^{i,j}(x,y)}{\|y-x\|^s} \left(\frac{y-x}{\|y-x\|} \right)^\nu \mathrm{d}o_y \mathrm{d}o_x, \tag{2.6'}$$

$$(2.7') \qquad \mathbf{K}_{i,j}^{\pi,\pi} = \sum_{|\nu|\geqslant 0} \lim_{\varepsilon\to 0} \int\limits_{\substack{\pi^1\times\pi^1\\ \|y-x\|>\varepsilon}} \frac{\bar\sigma_\nu^{i,j}(x,y,y-x)}{\|y-x\|^s}\Big(\frac{y-x}{\|y-x\|}\Big)^{2\nu} \mathrm{d}o_y \mathrm{d}o_x,$$

with plane triangles: $\pi_x^1 = \chi_{\pi_x}^1(\tilde\pi_x)$ and $\pi_y^1 = \chi_{\pi_y}^1(\tilde\pi_y)$. The coefficients $\bar\sigma_\nu^{i,j}$, $\bar\sigma_\nu^{i,j}$ are piecewise polynomials in the first and second variable. $\bar\sigma_\nu^{i,j}$ is homogeneous of degree one in the third variable, provided k is hypersingular.

We present now quadrature formulas, to approximate the integrals (2.6'/7'). Obviously it is sufficient to develop formulas to approximate:

$$(2.6^*) \qquad \mathbf{K}_{i,j}^{\pi_x,\pi_y} = \int\limits_{\pi_x\times\pi_y} \frac{q(x,y)}{\|y-x\|^{s+t}}(y-x)^\nu \mathrm{d}o_y \mathrm{d}o_x$$

$$(2.7^*) \qquad \mathbf{K}_{i,j}^{\pi,\pi} = \sum_{|\nu|\geqslant 0} \lim_{\varepsilon\to 0} \int\limits_{\substack{\pi\times\pi\\ \|y-x\|>\varepsilon}} \frac{q(x,y)}{\|y-x\|^{s+t}}(y-x)^\nu \mathrm{d}o_y \mathrm{d}o_x,$$

where π_x, π_y, π are plane triangles, $s+t$ is odd, and $|\nu| = t + \kappa_2$ holds.

2.2. Computation of the singular surface integrals in the case $\pi_x = \pi_y$. Consider first the case that $\pi_x = \pi_y =: \pi$ holds. We choose the coordinate system in such a way that π lies in the $(1,2)$ plane and the corner points A, B, C of π have the coordinates $A = (0,0,0)^T$, $B = (B_1, B_2, 0)^T$, $C = (B_1, C_2, 0)^T$ with $B_2 \leqslant 0$ and $C_2 \geqslant 0$. The difference $y - x$ now is two-dimensional: $y - x = (y^{1,2} - x^{1,2}, 0)^T$, where $v^{1,2}$ denotes the $(1,2)$ composed of a vector $v \in \mathbf{R}^3$. We introduce new variables $(\hat y, u)^T \in \mathbf{R}^4$ by:

$$\hat y := y^{1,2}, \qquad u := y^{1,2} - x^{1,2}.$$

Then the integral (2.7*) takes the form:

$$(2.9) \qquad \mathbf{K}_{i,j}^{\pi,\pi} = \sum_{|\nu|\geqslant 0} \lim_{\varepsilon\to 0} \int\limits_{\substack{(\hat y,u)^T\in\mathscr{P}\\ \|u\|>\varepsilon}} \frac{q(\hat y - u, \hat y)}{\|u\|^{s+t}} u^\nu \mathrm{d}u \,\mathrm{d}\hat y.$$

The parameter domain $\mathscr{P}$ is a four-dimensional polyhedron and can be described via:

$$\mathscr{P} = \{(\hat y, u)^T \in \mathbf{R}^4 \mid \hat y \in \pi, \exists x \in \pi \mid u = y - x\}.$$

This parameterization implies that the inner integration corresponds to the variable u. Let $H(u, \hat y)$ denote the integrand of (2.9). H is a polynomial w.r.t. $\hat y$ and singular

w.r.t. u. Our aim is to interchange the integration variables, to integrate analytically with respect to $\hat{y}$ and develop formulas for the u-integration, adepted to the order and location of the singularity. To be more concrete, one has to transform the inequalities below into an equivalent system, where u depends on $\hat{y}$.

$$\begin{aligned} 0 &\leqslant \hat{y}_1 \leqslant B_1 \\ m_1\hat{y}_1 &\leqslant \hat{y}_2 \leqslant m_2\hat{y}_1 \\ -\hat{y}_1 &\leqslant u_1 \leqslant B_1 - \hat{y}_1 \\ m_1(u_1+\hat{y}_1) - \hat{y}_2 &\leqslant m_2(u_1+\hat{y}_1) - \hat{y}_2. \end{aligned}$$

Here m_1, m_2 denote the ratios B_2/B_1 and C_2/B_1. We skip the detailed transformations, since they are rather involved and can be found in [12, §3]. The result is that the integral (2.9) can be written in the form:

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int\limits_{\substack{\pi \\ \|u\|>\varepsilon}} \Bigg\{ \int\limits_0^{B_1-u_1} \int\limits_{m_1\hat{y}_1}^{m_2\hat{y}_1} \big(H(u,\hat{y}) + H(-u,\hat{y}+u)\big)\,\mathrm{d}\hat{y} \\ &+ \int\limits_{\frac{m_2u_1-u_2}{m_2-m_1}}^{u_1} \int\limits_{m_1\hat{y}_1}^{m_2(\hat{y}_1-u_1)+u_2} \big(H(C-u,\hat{y}) + H(u-C,\hat{y}+C-u)\big)\,\mathrm{d}\hat{y} \\ &+ \int\limits_{\frac{m_1u_1-u_2}{m_1-m_2}}^{u_1} \int\limits_{m_2(\hat{y}_1-u_1)+u_2}^{m_2\hat{y}_1} \big(H(B-u,\hat{y}) + H(u-B,\hat{y}+B-u)\big)\,\mathrm{d}\hat{y} \Bigg\}\,\mathrm{d}u. \end{aligned} \tag{2.10}$$

Now it is trivial to perform the integration w.r.t. $\hat{y}$ analytically. It remains to consider integrals of the form:

$$\lim_{\varepsilon\to 0} \int\limits_{\substack{\pi \\ \|u\|>\varepsilon}} \big(\tilde{H}(P-u,u) + \tilde{H}(u-P,u)\big)\,\mathrm{d}u, \tag{2.11}$$

for $P \in \{A, B, C\}$. $\tilde{H}$ is a polynomial in the second variable and singular in the first one. A careful analysis [12, p. 54] shows that, with regard to our assumptions on s, t and κ_2, the strongest singularity cancels and the integrand is weakly singular, namely in a P corner of π. These integrals can be solved very cheaply by the use of Duffy's triangle coordinates or polar coordinates. The details can be found in [13], [14], [11, §3].

2.3. The computation of nearly singular surface integrals: $\operatorname{dist}(\pi_x, \pi_y) = O(h)$. It turns out, that the singular case ($\pi_x = \pi_y$) is the easiest one. We consider

now the case that $\operatorname{dist}(\pi_x, \pi_y) = O(h)$ holds, where $h := \operatorname{diam}(\pi_x)$ is assumed to be of the same magnitude as $\operatorname{diam}(\pi_y)$. Further we require that $\pi_x \cap \pi_y$ contains at most one point. The cartesian coordinate system has to be chosen in such a way that $\pi_x = \Delta(X_1, X_2, X_3)$ lies in the $(1,2)$ plane with $X_1 = (0,0,0)^T$, $X_2 = (X_{2,1}, 0, 0)^T$, $X_3 = (X_{3,1}, X_{3,2}, 0)^T$. First we consider the case that π_x and π_y are not parallel. We assume that the orthogonal projection $P_y^{1,3}$ of π_y onto the $(1,3)$ plane results in a non-degenerate triangle $\pi_y^{1,3}$. If $\pi_y^{1,3}$ is a straight line the projection onto the $(2,3)$ plane is regular and the indices in the following formulas have to be modified. $\pi_y^{1,3}$ can be mapped onto $\pi_y = \Delta(Y_1, Y_2, Y_3)$ via the transformation χ_y:

$$y(v) := \chi_y(v) := \begin{bmatrix} 1 & 0 \\ \alpha_1 & \alpha_3 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \alpha_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.12}$$

with suitable constants α_0, α_1, $\alpha_3 \in \mathbf{R}$. Thus integral (2.6*) may be parametrized over $(u,v)^T \in \pi_x \times \pi_y^{1,3}$.

$$\mathbf{K}_{i,j}^{\pi_x,\pi_y} = \frac{|\pi_y|}{|\pi_y^{1,3}|} \int\limits_{\pi_x \times \pi_y^{1,3}} \frac{q(u, y(v))}{\|y(v) - u\|^{s+t}} (y(v) - x)^\nu \, \mathrm{d}v \, \mathrm{d}u, \tag{2.13}$$

where $|.|$ denotes the area of π. We introduce the coordinate transform:

$$w = y(v) - (u, 0)^T, \qquad z = v_1.$$

Then (2.13) takes the following form:

$$\mathbf{K}_{i,j}^{\pi_x,\pi_y} = \frac{|\pi_y|}{|\pi_y^{1,3}|} \int\limits_{(w,z)^T \in \mathfrak{D}} \frac{\tilde{q}(w,z)}{\|w\|^{s+t}} w^\nu \mathrm{d}w \mathrm{d}z. \tag{2.14}$$

The parameter domain $\mathfrak{D}$ may be described by:

$$\mathfrak{D} := \left\{ (u,v)^T \in \mathbf{R}^2 \times \mathbf{R}^2 \mid u \in \pi_y^{1,3}, \exists \gamma \in \pi_x \mid v = \left(y_1(u), y_2(u)\right)^T - \gamma \right\}.$$

A careful examination of $\mathfrak{D}$ shows that $\mathfrak{D}$ is a polyhedron with nine corner points and the surface of $\mathfrak{D}$ consists of six distorted prisms $\mathfrak{P}_i$, $1 \leqslant i \leqslant 6$ (see [12, p. 66]). With $\mathfrak{P}_i^0$ we denote the convex envelope of $\{\mathfrak{P}_i, 0\}$. Consequently the integral (2.14) can be written in the form:

$$\mathbf{K}_{i,j}^{\pi_x,\pi_y} = \frac{|\pi_y|}{|\pi_y^{1,3}|} \sum_{i=1}^{6} s_i \int\limits_{(w,z)^T \in \mathfrak{P}_i^0} \frac{\tilde{q}(w,z)}{\|w\|^{s+t}} w^\nu \mathrm{d}w \, \mathrm{d}z, \tag{2.15}$$

where $s_i = \pm 1$ depend on the orientation of $\mathfrak{P}_i^0$. Let the reference element $\mathfrak{P}$ be defined by:

$$\mathfrak{P} := \{\eta \in \mathbf{R}^4 \mid 0 \leqslant \eta_1 \leqslant 1; 0 \leqslant \eta_2 \leqslant \eta_1; 0 \leqslant \eta_3 \leqslant \eta_1; 0 \leqslant \eta_4 \leqslant \eta_3\}.$$

A linear transformation $\mathfrak{T}_i$, which maps $\mathfrak{P}$ onto $\mathfrak{P}_i^0 = \operatorname{conv}\{E_1^i, E_2^i, E_3^i, E_1^i + v^i, E_2^i + v^i, E_3^i + v^i, 0\}$ ($v^i \in \mathbf{R}^4$ denotes a suitable vector) is defined by:

$$\mathfrak{T}_i(\eta) = \eta_1 E_1^i + \eta_2 v^i + \eta_3(E_2^i - E_1^i) + \eta_4(E_3^i - E_2^i).$$

Now integral (2.15) takes the form:

$$\mathbf{K}_{i,j}^{\pi_x,\pi_y} = \frac{|\pi_y|}{|\pi_y^{1,3}|} \sum_{i=1}^{6} s_i \frac{|\mathfrak{P}_i^0|}{|\mathfrak{P}|} \int\limits_{\eta \in \mathfrak{P}} \frac{\tilde{q}(\mathfrak{T}_i(\eta))}{\|\mathfrak{T}_i^3(\eta)\|^{s+t}} \mathfrak{T}_i^3(\eta)^\nu \, \mathrm{d}\eta, \tag{2.16}$$

where $\mathfrak{T}_i^3$ denotes the first three components of $\mathfrak{T}_i$. We introduce a four-dimensional analogous of Duffy's triangle coordinates by:

$$\eta_1 = \xi, \qquad \eta_i = \xi\theta_i, \quad 1 \leqslant i \leqslant 3,$$

which maps the parameter domain $\mathfrak{G} := \{0 \leqslant \theta_1 \leqslant 1; 0 \leqslant \theta_2 \leqslant 1; 0 \leqslant \theta_3 \leqslant \theta_2; 0 \leqslant \xi \leqslant 1\}$ onto $\mathfrak{P}$. With regard to the linearity of $\mathfrak{T}_i$, (2.16) takes the form:

$$\mathbf{K}_{i,j}^{\pi_x,\pi_y} = \frac{|\pi_y|}{|\pi_y^{1,3}|} \sum_{i=1}^{6} s_i \frac{|\mathfrak{P}_i^0|}{|\mathfrak{P}|} \int\limits_{(\theta,\xi)^T \in \mathfrak{G}} \xi^\kappa \frac{\tilde{q}(\mathfrak{T}_i(\xi,\theta))}{\|\mathfrak{T}_i^3(1,\theta)\|^{s+t}} \mathfrak{T}_i^3(1,\theta)^\nu \, \mathrm{d}\xi \, \mathrm{d}\theta, \tag{2.17}$$

where $\kappa = |\nu| + 3 - s - t$ is non-negative by assumption 2.5. $\tilde{q}(\mathfrak{T}_i(\xi,\theta))$ is polynomial in ξ, so the ξ-integration can be performed analytically. In [12, p. 67 ff] it is shown that the remaining integrand in (2.17) has no singular behavior as a function of θ, consequently the θ-integration can be performed via tensor-Gauß-formulas, which give the desired approximation of the integral (2.7).

It remains to consider the following cases:

(i) π_x, π_y are different triangles, but have one common edge. In [12, §3] this case is discussed. Similar transformation techniques yield integrands, where the singularity can be integrated analytically without any problems and the remaining terms are smooth.

(ii) $\operatorname{dist}(\pi_x, \pi_y) \geqslant C$. In this case the kernel function has no singular behavior , so the integrals can be approximated by a standard quadrature scheme.

(iii) $\operatorname{dist}(\pi_x, \pi_y) \leqslant Ch$; $\bar{\pi}_x \cap \bar{\pi}_y$ is at most one point and π_x and π_y are parallel. For sufficiently small h we can assume that π_x and π_y lie in a common plane, provided

the surface Γ is continuous. We can use the same technique as in the non-parallel case, if we replace the parameterization (2.12) by:

$$y(v) := \chi_y(v) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.12'}$$

and proceed in an analogous manner.

3 The panel-clustering algorithm for Galerkin descretizations of integral equations

3.1. The formulation of the panel-clustering algorithm. In the previous chapter we have developed techniques to approximate the system matrix, appearing by the Galerkin descretization. Nevertheless, the matrix is full, so the generation of this $n \times n$ system requires $O(n^2)$ operations and the storage of the matrix needs n^2 real numbers. If iterative schemes ar used to solve the linear system, a matrix-vector multiplication has to be performed in each iteration step requiring again $O(n^2)$ operations. In [6], [7], [9] Hackbusch and Nowak introduced the panel-clustering for the collocation method. With this techniques it is possible to approximate a matrix-vector multiplication with a number of $0(n \log^{\kappa+2} n)$ operations by storing $O(n \log^{\kappa} n)$ real numbers. The asymptotic reduction of the consumption is obvious and in [11] the consumption of the panel-clustering algorithm for collocation discretizations is estimated explicitly. Here we extend the concept of panel-clustering to Galerkin discretizations.

First we need the following definitions.

Definition 3.1. *cluster, tree of clusters, centre/radius of a cluster*

Let the surface Γ be partitioned into finitely many panels: $P := \{\pi_1, \pi_2, \ldots, \pi_{NP}\}$. A *cluster* is the union of some panels:

$$\tau := \bigcup_{i=1}^{k} \pi_{j_i} \qquad 1 \leqslant k \leqslant NP, \quad 1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant NP.$$

A *tree of clusters* T is a subset of all possible clusters relative to P which fulfills the following conditions:

— $\Gamma \subset T$

— Each $\tau \subset T$ is either a panel or the union of smaller clusters $\tau = \bigcup_{i=1}^{q} \tau_{j_i}$. τ is called *father* of the *sons* $\tau_{j_1}, \tau_{j_2}, \ldots, \tau_{j_q}$.

The midpoint $z(\tau)$ of the ball with smallest radius containing a cluster τ is called the *centre of* τ, the radius $\varrho(\tau)$ is called the *radius of* τ.

Let in the following $\mathbf{u}_n$ always denotes an n-vector and u_n the corresponding trial function: $u_n := I_\Delta \mathbf{u}_n = \sum_{i=1}^n \mathbf{u}_{n,i}\varphi_i$. We assume in the following that the kernel is hypersingular. A matrix-vector multiplication with the Galerkin matrix $\mathbf{K}$ reads:

$$(3.1)\quad \sum_{i=1}^n \mathbf{K}_{i,j}\mathbf{u}_{n,j} = \sum_{\pi_x \subset \mathrm{supp}(\varphi_i)} \int_{\pi_x} \varphi_i(x)\ \text{p. f.} \int_\Gamma k(x, y, y-x) u_n(y)\, do_y do_x$$
$$=: \sum_{\pi_x \subset \mathrm{supp}(\varphi_i)} Q_{i,\pi_x}(u_n).$$

By the panel-clustering method the sizes $Q_{i,\pi_x}(u_n)$ will be approximated. To this purpose, let for a panel π and a cluster τ the domain $D_{\pi,\tau}$ be defined via:

$$D_{\pi,\tau} := \{z \in \mathbf{R}^3 \mid \exists y \in \tau, x \in \pi \mid z = y - x\}.$$

We require that the kernel function can be written in the form:

$$(3.2)\quad k(x, y, z) = \sum_{i=1}^3 \gamma_i(y) k_i(x, z),$$

where k_i is singular for $z = 0$ and infinitely differentiable in $(x, z)^T \in D_{\pi,\tau}$, provided $z \neq 0$. The vector $\gamma(y)$ is assumed to be piecewise smooth, according to the smoothness of the surface Γ. The functions $k_i(x, z)$ can be expanded in a Taylor series around the centre $z_{\pi,\tau} := z(D_{\tau,\pi})$, which is convergent, provided 0 is not contained in the ball $B_{\varrho(D_{\pi,\tau})}(z_{\pi,\tau})$. The Taylor expansion of order m takes the following form:

$$k_i(x, z) = k_{i,m}(x, z) + R_{i,m}(x, z),$$

with

$$(3.3)\quad k_{i,m}(x, z) = \sum_{|\nu|<m} \frac{1}{\nu!}\Big(\frac{\partial}{\partial z}\Big)^\nu k_i(x, z)\Big|_{z=z_{\pi,\tau}} (z - z_{\pi,\tau})^\nu.$$

If in (3.3) the variable z is replaced by $y - x$ and the sum is ordered w.r.t. powers of y, $k_{i,m}$ can be written in the following form:

$$(3.4)\quad k_{i,m}(x, z) = \sum_{|\nu|<m} \kappa_\nu^i(x, z_{\pi,\tau}) y^\nu,$$

with suitable expansion coefficients $\kappa_\nu^i(x, z_{\pi,\tau})$. The kernel function k will be approximated for $(x, z) \in \pi \times D_{\tau,\pi}$ by:

$$k(x, z) \approx \sum_{i=1}^3 \gamma_i(y) k_{i,m}(x, z) =: k_m(x, z).$$

If the kernel function k is replaced by k_m, the resulting error has to be adapted by a suitable order of consistency. The exact requirements are stated in the following

Assumption 3.2. Let the kernel function $k(x, y, .)$ be singular of order s at the origin. We assume that for arbitrary $\varepsilon > 0$ and $m \in \mathbf{N}$ there exists an $\eta \in [0, 1[$, such that for all x, $z_0 \in \mathbf{R}^3$, $z_0 \neq 0$:

$$|k_m(x, z) - k(x, z)| \leqslant \varepsilon(\|z\|^{-s} + \|z_0\|^{-s}), \qquad \forall \frac{\|z - z_0\|}{\|z_0\|} \leqslant \eta \tag{3.5}$$

holds.

The idea of the panel-clustering algorithm is that on surface pieces (clusters), where the kernel is not singular, the kernel function k will be replaced by the expansion k_m. To control the size of the resulting error we need the following

Definition 3.3. Let $\eta \in [0, 1[$ be chosen according to assumption 3.2. A cluster $\tau \in T$ is called *admissible* w.r.t. a panel π if

$$\varrho(D_{\pi,\tau}) \leqslant \frac{\eta}{\sqrt{1 + \eta^2}} \operatorname{dist}(\pi, \tau)$$

holds. A set $\mathscr{C}(\pi) = \{\tau_1, \tau_2, \ldots, \tau_\sigma\}$ consisting of clusters $\tau_j \in T$ with pairwise disjoint interiors is called an *admissible covering* w.r.t. a panel π, if the following condition are fulfilled:

— $\Gamma = \bigcup_{j=1}^{\sigma} \tau_j$,

— either τ_j is a panel or τ_j is admissible w.r.t. π.

An admissible covering $\mathscr{C}(\pi)$, which contains a minimal number of clusters, is called a *minimal admissible covering*. $\mathscr{C}^{\text{near}}(\pi) := \{\tau \in \mathscr{C}(\pi) \mid \tau \text{ is not admissible}\}$ is called the nearfield part, $\mathscr{C}^{\text{far}}(\pi) := \mathscr{C}(\pi) \setminus \mathscr{C}^{\text{near}}(\pi)$ denotes the farfield part.

Now we can formulate the panel-clustering algorithm for the Galerkin method. For this let the so-called nearfield-matrix be defined by:

$$\mathbf{K}_{i,j}^{\text{near}} := \sum_{\pi_x \subset \operatorname{supp}(\varphi_i)} \sum_{\pi_y \subset \operatorname{supp}(\varphi_j) \cap \mathscr{C}^{\text{near}}(\pi_x)} \int_{\pi_x} \varphi_i(x) \times \text{ p. v. } \int_{\pi_y} k(x, y - x)\big(\varphi_j(y) - \varphi_j(x)\big)\, \mathrm{d}o_y \mathrm{d}o_x,$$

and the farfield-coefficients $\vec{J}_\tau^\nu(u)$ via:

$$\vec{J}_\tau^\nu(u) := \int_\tau y^\nu u(y)\gamma(y)\, \mathrm{d}o_y, \qquad \forall \tau \in T,\ \nu \in \mathbf{N}_0^3,\ |\nu| < m,$$

where γ denotes the surface-dependent vector in (3.2). Finally the expansion coefficients $\kappa_\nu^\tau(x, z(D_{\pi,\tau}))$ (see (3.4)) we have to be integrated together with the trial functions:

$$\vec{\kappa}_{\nu,\tau,r}^{\pi,i}(u) := \int_\pi \varphi_i(x)\kappa_\nu^\tau(x, z(D_{\pi,\tau}))u(x)\,do_x$$

for all $\pi \in P$, $\tau \in \mathcal{C}^{\text{far}}(\pi)$, $|\nu| < m$, $1 \leqslant r \leqslant 3$, i with $\text{supp}(\varphi_i) \cap \pi \neq \emptyset$.

With the help of $\mathbf{K}^{\text{near}}$, $\vec{J}_\tau^\nu(u)$ and $\vec{\kappa}_{\nu,\tau,r}^{\pi,i}(u)$ a matrix-vector multiplication can be approximated by:

$$\begin{aligned}(3.6)\qquad \sum_{j=1}^{N} \mathbf{K}_{i,j}\mathbf{u}_{n,j} &\approx \sum_{j;\mathbf{K}_{i,j}\neq 0} \mathbf{K}_{i,j}^{\text{near}}\mathbf{u}_{n,j} \\ &+ \sum_{\pi_x \subset \text{supp}(\varphi_i)} \sum_{\tau \in \mathcal{C}^{\text{far}}(\pi_x)} \sum_{|\nu|<m} \left(\left\langle \vec{\kappa}_{\nu,\tau}^{\pi,i}(1), \vec{J}_\tau^\nu(u_n) \right\rangle - \left\langle \vec{\kappa}_{\nu,\tau}^{\pi,i}(u_n), \vec{J}_\tau^\nu(1) \right\rangle \right).\end{aligned}$$

This representation can easily be derived by partitioning the surface Γ, appearing in formula (3.1), into the nearfield panels and the farfield clusters. Replacing the kernel function k by k_m in the farfield part results in (3.6) (cf. [12, remark 2.1.10]).

Remark 3.4. In the case that the kernel function is not hypersingular, the term $\left\langle \vec{\kappa}_{\nu,\tau}^{\pi,i}(u_n), \vec{J}_\tau^\nu(1) \right\rangle$ in (3.6) vanishes. Furthermore $\vec{\kappa}_{\nu,\tau}^{\pi,i}(.), \vec{J}_\tau^\nu(.)$ are scalar sizes in the case, that either the surface is globally smooth or the kernel function does not depend on the surface (e.g. the kernel is a fundamental solution).

A possible structuring of the panel-clustering algorithm consists in computing and storing the quantities K^{near}, $\vec{J}_\tau^\nu(\varphi_j)$, $\vec{J}_\tau^\nu(1)$, $\vec{\kappa}_{\nu,\tau,r}^{\pi,i}(\varphi_j)$, $\vec{\kappa}_{\nu,\tau,r}^{\pi,i}(1)$ in a first step, which corresponds to the usual generation phase of the system matrix. To perform a matrix-vector multiplication one has to compute at first the quantities $\vec{J}_\tau^\nu(u_n)$, $\vec{\kappa}_{\nu,\tau,r}^{\pi,i}(u_n)$ by summing up over the tree of clusters and then evaluating formula (3.6). An explicit formulation of the algorithm is presented in [12, algorithm 2.1.9].

3.2. Error analysis for the panel-clustering method. Here we proceed, by sketching the error analysis of the panel-clustering technique. In the first step we have to investigate the relative error $\varepsilon(\eta, m)$, which results if the kernel function k is replaced by the function k_m. In [12, §2.2] the following error estimate is proved:

Theorem 3.5. *Let the kernel function $k(x, y, y - x)$ be of the form:*

$$k(x, y, z) := \sum_{i=1}^{3} \gamma_i(y)k_i(x, z),$$

with

$$k_i(x,z) := \|z\|^{-s-t} \sum_{|\nu|\geqslant t} c_\nu(x) z^\nu$$

(viz. (1.6), (3.2)) approximated by k_m which is defined as the Taylor expansion of k_i around z_0 via (3.3/4). k is singular of order s at the origin:

$$k(x,y,z) \leqslant C_S \|z\|^{-s},$$

and $s+t$ is assumed to be odd.

Then for all x, y, $z_0 \in \mathbb{R}^3$ the following error estimate holds:

$$|k(x,y,z) - k_m(x,y,z)| \leqslant \varepsilon(\eta, m)(\|z\| + \|z_0\|), \qquad \forall \frac{\|z - z_0\|}{\|z\|} \leqslant \eta < 1. \tag{3.7}$$

In order to explain the function ε, let $C_k^r(x)$ denote the Gegenbauer polynomials and $x_1(\eta), \ldots, x_m(\eta)$ the roots of the polynomial $C_{m-1}^{s/2+1}(x) - \eta C_{m-2}^{s/2+1}(x)$. The function $\tilde{E}_m^{s+t}(\sigma, \eta)$ is defined by:

$$\tilde{E}_m^{s+t}(\eta, \theta) := 1 - (1 - 2\eta\cos\theta + \sigma^2)^{(s+t)/2} \sum_{k=0}^{m-1} C_k^{(s+t)/2}(\cos\theta)\eta^k$$

and $E_m^{s+t}(\eta)$ by:

$$E_m^{s+t}(\eta) := \begin{cases} \sigma^m & \text{if } s+t = 1 \\ \max\limits_{1\leqslant i\leqslant m} \{|E_m^{s+t}(\eta, 1)|, |E_m^{s+t}(\eta, -1)|, |E_m^{s+t}(\eta, x_i)|\} & \text{otherwise.} \end{cases}$$

With these definitions, $\varepsilon(\eta, m)$ in (3.7) is given by:

$$\varepsilon(\eta, m) \leqslant C(k) \max\left(E_m^{s+t}(\eta), \binom{m+s+t-1}{s+t} \eta^m \right),$$

where $C(k) := \max(C_S, C_P) \max \sum_{i=1}^{3} |\gamma_i(y)|$ and

$$C_P := \max_{|\lambda|\leqslant \max(t, m-l)} \left\{ |c_\lambda(x)|\ |z_0|^{|\lambda|-t} \sum_{\substack{\nu\leqslant\lambda \\ |\nu|\geqslant m-l}} \binom{\lambda}{\nu} \right\}$$

holds. We state that these error bounds are strict. Asymptotically ε can be estimated by $\varepsilon(\eta, m) \leqslant C_1(C_2\eta)^m$ with suitable constants C_1, C_2.

In the next step we have to investigate the effect of the relative error $\varepsilon(\eta, m)$ caused by clustering the panels for a matrix-vector multiplication. In [12, Satz 2.2.1] the following theorem is proved.

Theorem 3.6. *Let the kernel function $k(x, y, .)$ be singular of order s at the origin. For the approximation k_m according (3.3/4) k_m we require that assumption 3.1 is fulfilled with $\eta = \eta(\varepsilon, m)$. We assume the panelization P to be quasi-uniform: $\varrho(\pi) \geqslant h/C_u$. Let the trial function are bounded: $\|\varphi_i\|_\infty \leqslant C_b$. We assume that for all panels, there holds:*

$$\int_{\pi} \int_{\mathscr{C}^{\text{far}}(\pi)} \|z(D_{\pi,\tau})\|^{-s} \mathrm{d}o_y \mathrm{d}o_x \leqslant C_I \int_{\pi} \int_{\mathscr{C}^{\text{far}}(\pi)} \|x - y\|^{-s} \mathrm{d}o_y \mathrm{d}o_x,$$

where C_I is independent of π. The panel-clustering method defines an operator K_m, which approximates $\mathbf{K}\mathbf{u}_n \approx K_m u_n$. Then the following error estimate:

$$|(\mathbf{K}\mathbf{u}_n)_i - (K_m u_n)_i| \leqslant \varepsilon C_b (1 + C_I) C_k(h) \|\mathbf{u}_n\|_\infty \tag{3.8}$$

holds, where $C_k(h)$ is defined by:

$$\max_{1 \leqslant i \leqslant n} \sum_{\pi_x \subset \text{supp}(\varphi_i)} \int_{\pi} \int_{\mathscr{C}^{\text{far}}(\pi)} \|x - y\|^{-s} \mathrm{d}o_y \mathrm{d}o_x \leqslant C_k(h).$$

Estimate (3.8) corresponds to a consistency error, which has to be adapted to a suitable estimate of the required discretization error, e.g.: $\varepsilon \approx h^\kappa$, where h denotes a "stepsize" parameter and κ the order of consistency. In theorem 3.5 it was shown that under weak assumptions on the form of the kernel function the error-function $\varepsilon(\eta, m)$ asymptotically behaves like $\varepsilon \sim C_1(C_2, \eta)^m$, so any desired accuracy can be achieved by a suitable choice of η and m. If for the dimensions n of the trial space the condition $n^{-1/(d-1)} \sim h$ holds, $\eta \in [0, 1[$ should be chosen sufficiently small (independent of h) and $m = O\left(\frac{\kappa}{d} \log n\right)$.

3.3. The complexity of the panel-clustering algorithm. In order to estimate the total cost of the panel-clustering algorithm, it is essential to estimate the number $\sigma^{\text{near}}(\pi)$ of the nearfield panels and the number $\sigma^{\text{far}}(\pi)$ of the farfield clusters per panel. In [7] it is shown that under the condition of a quasi-uniform penelset and some further technical requirements which are usually fulfilled, the estimates:

$$\sigma^{\text{near}}(\pi) \leqslant C_{\text{near}} \frac{1}{\eta^{d-1}} \tag{3.8}$$

$$\sigma^{\text{far}}(\pi) \leqslant C_{\text{far}} \frac{1}{\eta^{d-1}} \log(e + \eta^{d-1} n)$$

hold. To implement the panel-clustering method one has to develop fast algorithms for computing the tree of clusters, the minimal admissible coverings for all panels, the farfield-coefficients $\vec{J}_{\pi}^{\nu}(\varphi_i)$, the expansion coefficients $\vec{\kappa}_{\nu,\tau}^{\pi,i}(\varphi_i)$, the nearfield matrix $\mathbf{K}_{i,j}^{\text{near}}$, and finally for the evolution procedure. The description of these algorithms is rather technically and can be found in detail in [12, §4]. The cost of these procedures depends on the parameters m and η. By summing up the total storage and computational cost, we readily state that the computational cost takes $O(nm^d)$ operations for the generation part and $O(nm^{d+4})$ operations for the evaluation of a matrix-vector multiplication, where the storage amount comes to $O(nm^d)$ real numbers. Asymptotically the reduction of the total cost is obvious. In [12, §4] the cost of the algorithms is estimated strictly and computational tests are performed to estimated the surface-dependent constants C_{far}, C_{near} in (3.8). Further we use the exact error function $\varepsilon(\eta, m)$ to choose the expansion order m and the relative admissible size η of the clusters τ in an optimal way. Thus we get the explicit consumption of the panel-clustering algorithm, dependent on the number of unknowns, to compare it with the usual matrix-oriented procedure. The figures below show this comparison under different optimization strategies. One observes that the panel-clustering algorithm gives a considerable reduction of the storage and computational amount even for problems of medium size.

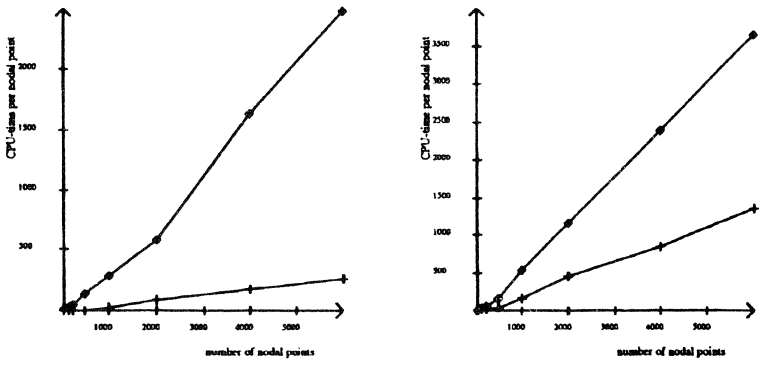

Figure 3.1. Optimization of the panel-clustering method w.r.t. the computational cost, applied to the single layer and double layer potential.

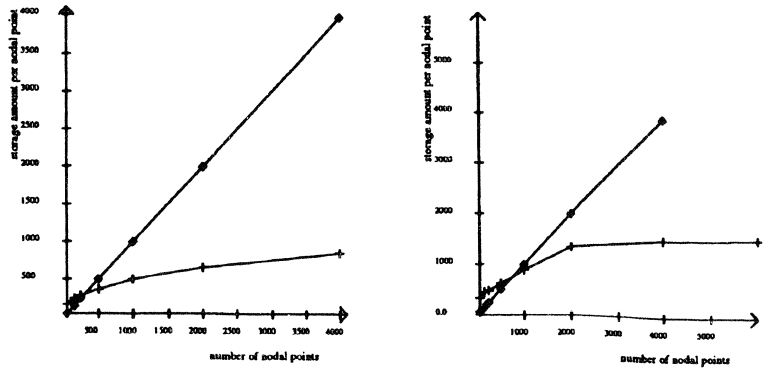

Figure 3.2. Optimization of the panel-clustering method w.r.t. the storage amount, applied to the single layer and double layer potential.

4. Computational examples

To compare the real amount of the panel-clustering method with the theoretical estimations of figures 3.1/2, the algorithm was implemented for Fredholm integral equations of the second kind with weakly singular kernel functions, specifically with the single layer potential for the Laplacian on the surface of a three-dimensional cube and the double layer potential on the surface of the unit sphere in 3 D. We used piecewise linear trial functions over plane triangular panels. We used the nested multigrid method as iterative solver, combined with the Picard-iteration as smoothing procedure (see [6]). The finest grid of the surface of the cube contained 3072 panels, the surface of the sphere 4096 panels. The tables presented below show the amount of the panel-clustering compared with the usual matrix-oriented technique.

ε	q	$\Delta(h)$	$\Delta(\varepsilon, h)$	RZ_0^2	RZ_0^3	RZ_q
0.2	1.31	$1.24e-3$	$1.63e-3$	338	1118	116
0.3	2.01	$1.24e-3$	$2.49e-3$	338	1118	78
0.4	2.44	$1.24e-3$	$3.03e-3$	338	1118	63

ε	q	$\delta(RZ^2)$	$\delta(RZ^3)$	MEM_0	MEM_q	$\delta(MEM)$
0.2	1.31	2.91	9.6	51.2	22.5	2.28
0.3	2.01	4.33	14.3	51.2	17.3	2.96
0.4	2.44	5.37	17.7	51.2	14.2	3.62

Table 4.1. Amount of the panel-clustering method, applied to the single layer potential. $\Delta(h)$ denotes the relative discretization error without panel-clustering, $\Delta(\varepsilon, h)$ the error due to the clustering tolerance ε (viz. 3.2). q is defined via

$q := \Delta(\varepsilon, h)/\Delta(h)$. RZ_0^k resp. RZ_q denote the CPU-time of the program without resp. with panel-clustering w.r.t. the tolerance q (in minutes). k denotes the degree of exactness for the quadratures scheme, applied to the regular integrals. $\delta(RZ) := RZ_0^k/RZ_q$. The storage amount MEM_0, MEM_q is defined analogously and is declared in mega-byte. $\delta(MEM) := MEM_0/MEM_q$.

ε	q	$\Delta(h)$	$\Delta(\varepsilon, h)$	RZ_0^2	RZ_0^3	RZ_q
0.3	1.64	$2.48e-4$	$4.07e-4$	1469	3104	974
0.4	2.64	$2.48e-4$	$6.44e-4$	1469	3104	803

ε	q	$\delta(RZ^2)$	$\delta(RZ^3)$	MEM_0	MEM_q	$\delta(MEM)$
0.3	1.64	1.5	3.2	90.7	43.1	2.10
0.4	2.64	1.8	3.9	90.7	36.2	2.51

Table 4.2. Amount of the panel-clustering method applied to the double layer potential where the algorithm was optimized w.r.t. the storage amount. The sizes q, $\Delta(h)$, $\Delta(\varepsilon, h)$, RZ_0, RZ_q etc. are defined in table 4.1.

More numerical tests are reported in [12, §5]. One observes that the reduction of the consumption is stronger for the single layer potential than for the double layer. The reason for this is that the single layer kernel is globally smooth, so the farfield coefficients and the expansion coefficients are scalars instead of vectors in the case of the double layer kernel. In each case the reduction of the consumptions is satisfactory. It remains to discuss the question, to what extend these results are valid for practical problems. Our examples were chosen so simple, since one knows explicitly the exact solution for these problems, so one can study systematically the influence of the additional error caused by the panel-clustering method. The exact solutions always had an extremely simple behavior, lying nearly in the trial space. This effects caused an unnaturally small discretization error (without panel-clustering). In practice this error will be essentially larger, so the parameters (η, m), which controls the panel-clustering-algorithm, could be chosen more favourable. In fact by our three-dimensional expansion of the kernel function the error, caused by panel-clustering, depend neither on the smoothness of the surface nor on the smoothness of the solution.

The effect of the panel-clustering method strongly depends on the singular behavior of the kernel function. That is if the kernel function is strongly singular, that means decreases fast with increasing distance to the singularity, the admissible covering could be chosen more favourable. A detailed study if the panel-clustering method for hypersingular kernels will be the topic of further research.

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