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THE GENERAL FORM OF LOCAL BILINEAR FUNCTIONS

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Summary. The scalar product of the FEM basis functions with non-intersecting supports vanishes. This property is generalized and the concept of local bilinear functional in a Hilbert space is introduced. The general form of such functionals in the spaces $L_2(a,b)$ and $H^1(a,b)$ is given.

Keywords: bilinear functional, bilinear form, Sobolev spaces

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One of typical features that makes the finite element method useful for a still wider area of problems is the fact that the originating matrix of the discrete problem is sparse. The weak solution of a boundary-value problem is described by a bilinear functional and by a subspace of a suitably chosen Sobolev space. For standard boundary-value problems the bilinear functional has the property, we will call it the local property, that it vanishes for a pair of arguments with non intersecting supports. On the other hand, the functions of the chosen subspace may be approximated by linear combinations of functions with small supports and this implies sparsity of the stiffness matrix.

The purpose of this paper is to give a general form of a local bilinear functional on the space H^1 in one dimension. It appears that we obtain exactly those bilinear functions that correspond to standard boundary-value problems for second-order elliptic differential operators. The general form of a local bilinear functional on the space L_2 in one dimension is also given. Obviously, this latter simple case may be generalized to higher dimensions.

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1. Preliminaries

Definition 1. Let a bilinear functional $B(u,v)$ in $L_2(0,1)$ or $H^1(0,1)$ have the property that $B(u,v) = 0$ for every pair u, $v \in L_2(0,1)$ or $H^1(0,1)$ such that $\operatorname{meas}(\operatorname{supp} u \cap \operatorname{supp} v) = 0$. Such a bilinear functional will be called a local functional.

Let us denote by D_k a uniform partition of the interval $\langle 0,1\rangle$ into 2^k parts and let I_j^k be the j-th (open) interval of this partition, $j = 1, \ldots, 2^k$. Let us further denote by $x_j^k = j/2^k$, $j = 0, \ldots, 2^k$ the j-th node of the partition D_k.

Lemma 1. *Let $b^k(t)$ be a sequence of functions piecewise constant on $\langle 0,1\rangle$ such that $b^k(t) = b^k$ for $t \in I_j^k$. The value of b at the points x_j is arbitrary.*

Let the equality

$$2b_j^k = b_{2j-1}^{k+1} + b_{2j}^{k+1}$$

hold for all k and j. We define functions f_k by

$$f_k(t) = \int_0^t b^k(s)\,\mathrm{d}s.$$

Then we have

$$f_n\Big(\frac{j}{2^k}\Big) = f_k\Big(\frac{j}{2^k}\Big) \quad \text{for } n \geqslant k. \tag{1}$$

Proof. First we have

$$\int_{I_j^k} b^k(t)\,\mathrm{d}t = \frac{1}{2^k} b_j^k = \frac{1}{2^{k+1}}(b_{2j-1}^{k+1} + b_{2j}^{k+1})$$
$$= \int_{I_{2j-1}^{k}} b^{k+1}(t)\,\mathrm{d}t + \int_{I_{2j}^{k+1}} b^{k+1}(t)\,\mathrm{d}t,$$

which implies

$$f_k\Big(\frac{j}{2^k}\Big) - f_k\Big(\frac{j-1}{2^k}\Big) = f_{k+1}\Big(\frac{j}{2^k}\Big) - f_{k+1}\Big(\frac{j-1}{2^k}\Big)$$

and

$$f_{k+1}\Big(\frac{j}{2^k}\Big) = \int_0^{x_j^k} b^{k+1}(t)\,\mathrm{d}t = \sum_{s=1}^{j} \int_{I_s^k} b^{k+1}(t)\,\mathrm{d}t$$
$$= \sum_{s=1}^{j} \big(f_{k+1}(s/2^k) - f_{k+1}((s-1)/2^k)\big)$$
$$= \sum_{s=1}^{j} \big(f_k(s/2^k) - f_k((s-1)/2^k)\big)$$
$$= f_k(j/2^k)$$

since $f_k(0) = 0$. From here (1) is easily seen. □

Corollary. *Let k be arbitrary but fixed. Then the sequence of functions $f_n(t)$ converges for $n \to \infty$ at every point x_j^k, $j = 0, \ldots, 2^k$ towards $f_k(j/2^k)$.*

Lemma 2. *Let M be a dense set in $\langle 0, 1\rangle$. Let f_k be a sequence of equicontinuous functions on $\langle 0, 1\rangle$ that converges on M to a function φ defined on M. Then there exists function f continuous on $\langle 0, 1\rangle$ and such that $f(t) = \varphi(t)$ for $t \in M$.*

Proof. We will verify that the function φ is uniformly continuous on M. For an arbitrary $\varepsilon > 0$ let us choose a $\delta > 0$ such that $|f_k(t) - f_k(z)| < \varepsilon/3$ for $|t - z| < \delta$ and for all k. Now we choose t and $z \in M$ arbitrary but fixed so that $|t - z| < \delta$ and choose k such that $|f_k(t) - \varphi(t)| < \varepsilon/3$ and $|f_k(z) - \varphi(z)| < \varepsilon/3$. Then we have $|\varphi(t) - \varphi(z)| < \varepsilon$. Consequently the assumption of Theorem 177 of [1] is satisfied and applying this theorem we prove the existence of f. □

Lemma 3. *Let b^k be the sequence of functions from Lemma 1 satisfying, in addition,*

$$|b_j^k| < K \quad \text{for all } k \text{ and } j.$$

Then there exists a bounded function η such that

$$\int_{I_j^k} \eta(x)\,\mathrm{d}x = 2^{-k} b_j^k \quad \text{for all } k \text{ and } j.$$

Proof. According to Lemma 1 the corresponding sequence f_k converges to a function f at all nodes. The functions f_k are equicontinuous because they fulfill the Lipschitz condition with a constant K independent of k. Therefore according to Lemma 2 there exists a continuous prolongation of the function f onto the whole interval $\langle 0, 1\rangle$. This function satisfies, as is easily seen, the Lipschitz condition with the constant K and is therefore absolutely continuous. Let η be its derivative. By virtue of Lemma 1 we have further

$$\begin{aligned}\int_{I_j^k} \eta(x)\,\mathrm{d}x &= f(j/2^k) - f((j-1)/2^k)\\ &= f_k(j/2^k) - f_k((j-1)/2^k)\\ &= \int_{I_j^k} b^k(t)\,\mathrm{d}t = 2^{-k} b_j^k.\end{aligned}$$

We show that η is bounded. For all Lebesgue points of η,

$$\lim_{h\to 0_+} \frac{1}{h}\int_x^{x+h} \eta(t)\,\mathrm{d}t = \eta(x)$$

holds. On the other hand, we have $|\frac{1}{h}\int_x^{x+h}\eta(x)\,\mathrm{d}t| = \frac{1}{h}|f(x+h)-f(x)| \leqslant K$ and therefore $|\eta(x)| \leqslant K$, and the lemma is proved. □

Lemma 4. *Let b^k be the sequence of functions from Lemma 1 satisfying, in addition,*

$$\|b^k\|_{L_2}^2 = 2^{-k}\sum_{j=1}^{2^k}(b_j^k)^2 < K^2 \quad \textit{for all } k.$$

Then there exists a function $\eta \in L_2$ such that

$$\int_{I_j^k} \eta(x)\,\mathrm{d}x = 2^{-k}b_j^k \quad \textit{for all } k \textit{ and } j.$$

Proof. According to Lemma 1 the corresponding sequence of functions f_k converges at all nodes. We denote its limit by f. The functions f_k are equicontinuous. Indeed, choose an arbitrary $\varepsilon > 0$. Put $\delta = \varepsilon^2/K^2$ and choose x and y so that $|x-y| < \delta$. Then we have

$$|f_k(x) - f_k(y)| = \left|\int_x^y b^k(t)\,\mathrm{d}t\right| \leqslant |x-y|^{\frac{1}{2}}\|b^k\|_{L_2} \leqslant \frac{\varepsilon}{K}K.$$

According to Lemma 2, there exists continuous prolongation of the function f onto the whole interval $\langle 0,1\rangle$. On the other hand, we have

$$\begin{aligned}
\|f_k\|_{H^1}^2 &= \int_0^1 f_k^2(t)\,\mathrm{d}t + \int_0^1 \left[f_k'(t)\right]^2\mathrm{d}t \\
&= \int_0^1 \left(\int_0^t b^k(s)\,\mathrm{d}s\right)^2 \mathrm{d}t + \int_0^1 \left[b^k(t)\right]^2\mathrm{d}t \\
&\leqslant \int_0^1 \left\{\int_0^t \mathrm{d}s \cdot \int_0^t \left[b^k(s)\right]^2\mathrm{d}s\right\}\mathrm{d}t + \|b^k\|_{L_2}^2 \\
&\leqslant 2\|b^k\|_{L_2}^2 < 2K^2.
\end{aligned}$$

The set of the functions f_k is therefore bounded in H^1. We can choose a subsequence f_{k_r} that is weakly convergent. Denote its limit $\bar{f}$. The imbedding operator from H^1 to C is compact and, consequently, the sequence f_{k_r} converges to $\bar{f}$ in C. Thus we have $f = \bar{f}$, this function has the derivative $f' = \eta \in L_2$ a.e. and in the same way as above we show

$$\int_{I_j^k} \eta(x)\,\mathrm{d}x = 2^{-k} b_j^k.$$

The lemma is proved. □

Lemma 5. *Let $u \in L_2$ and let u^k be piecewise constant functions with*

$$u^k(x) = 2^k \int_{I_j^k} u(t)\,\mathrm{d}t = u_j^k \quad \text{for } x \in I_j^k.$$

Then u^k converge to u in L_2.

Proof. First, let the function u be continuous in $\langle 0,1\rangle$. Then for $x \in I_j^k$ we have

$$|u(x) - u_j^k| \leqslant 2^k \int_{I_j^k} |u(x) - u(t)|\,\mathrm{d}t < \varepsilon$$

for an arbitrary $\varepsilon > 0$ and sufficiently large k. Moreover,

$$\|u - u^k\|_{L_2}^2 = \sum_{j=1}^{2^k} \int_{I_j^k} \left[u(t) - u^k(t)\right]^2 \,\mathrm{d}t < 2^k \cdot 2^{-k}\varepsilon^2.$$

It can be easily proved that u^k is the orthogonal projection of u into the subspace of piecewise constant functions. Applying the properties of the projector we obtain the convergence for all functions $u \in L_2$. □

2. The L_2-case

Theorem 1. *Let $B(u,v)$ be a local, continuous, bilinear functional in $L_2(0,1)$. Then there exists a bounded function η such that*

$$B(u,v) = \int_0^1 \eta(t)u(t)v(t)\,\mathrm{d}t.$$

Proof. Approximate the functions u and v by sequences of piecewise constant functions according to Lemma 5. We use the local property of B obtaining

$$B(u^k, v^k) = \sum_{j=1}^{2^k} u_j^k v_j^k B(\chi_j^k, \chi_j^k),$$

where χ_j^k is the characteristic function of I_j^k.

Let us set $2^k B(\chi_j^k, \chi_j^k) = b_j^k$. We have $\chi_j^k = \chi_{2j-1}^{k+1} + \chi_{2j}^{k+1}$ and after a substitution

$$\begin{aligned} b_j^k &= 2^k B(\chi_{2j-1}^{k+1} + \chi_{2j}^{k+1}, \chi_{2j-1}^{k+1} + \chi_{2j}^{k+1}) \\ &= 2^k B(\chi_{2j-1}^{k+1}, \chi_{2j-1}^{k+1}) + 2^k B(\chi_{2j}^{k+1}, \chi_{2j}^{k+1}) \\ &= \frac{1}{2}(b_{2j-1}^{k+1} + b_{2j}^{k+1}). \end{aligned}$$

The quantities b_j^k are uniformly bounded. This is the consequence of the continuity of the functional. Therefore, according to Lemma 3, there exists a bounded function η such that

$$\int_{I_j^k} \eta(t)\,\mathrm{d}t = 2^{-k} b_j^k \quad \text{for all } k \text{ and } j.$$

We thus have $B(u^k, v^k) = \int_0^1 \eta(t) u^k(t) v^k(t)\,\mathrm{d}t$ and hence also

$$B(u,v) = \int_0^1 \eta(t) u(t) v(t)\,\mathrm{d}t.$$

Indeed, we have

$$\begin{aligned} &\int_0^1 \eta(t)\big[u(t)v(t) - u^k(t)v^k(t)\big]\,\mathrm{d}t \\ &\qquad \leqslant K \int_0^1 |u(t)|\,|v(t) - v^k(t)|\,\mathrm{d}t + K \int_0^1 |v^k(t)|\,|u(t) - u^k(t)|\,\mathrm{d}t \\ &\qquad \leqslant K\big(\|u\|_{L_2} \cdot \|v - v^k\|_{L_2} + \|v^k\|_{L_2} \cdot \|u - u^k\|_{L_2}\big) \end{aligned}$$

and the right-hand side converges to zero for $k \longrightarrow \infty$. The theorem is proved. □

3. H^1-CASE

Let us denote by s_j^k a continuous function linear in every interval of the partition D_k and such that $s_j^k(x_\ell^k) = \delta_{j\ell}$ (a hat-function). The support of s_j^k is the interval $\langle x_{j-1}^k, x_{j+1}^k \rangle = I_j^k \cup I_{j+1}^k$. Let $B(u,v)$ be a symmetric continuous bilinear functional in H^1. Let us write $B(s_j^k, s_i^k) = B_{j,i}^k$. Of course, these values are equal to zero for $|i-j| \geqslant 2$. Let us further put

$$T_i^k = B_{i,i-1}^k + B_{i,i}^k + B_{i,i+1}^k, \quad i = 0, 1, \ldots, 2^k.$$

The values of $B_{j,i}^k$ with subscripts equal to -1 or $2^k + 1$ are set to zero. The values of T_i^k for the same subscripts are also zero.

We find the relation between s_j^k, T_i^k and $B_{i,j}^k$ for two neighboring partitions. We obviously have

$$s_i^k = \frac{1}{2}\, s_{2i-1}^{k+1} + s_{2i}^{k+1} + \frac{1}{2}\, s_{2i+1}^{k+1}$$

with the corresponding exceptions for $i = 0$ or 2^k. Further, we have

$$\begin{aligned}(2)\qquad B_{i,i+1}^k &= B\Big(\frac{1}{2}\, s_{2i-1}^{k+1} + s_{2i}^{k+1} + \frac{1}{2}\, s_{2i+1}^{k+1}, \frac{1}{2}\, s_{2i+1}^{k+1} + s_{2i+2}^{k+1} + s_{2i+3}^{k+1}\Big)\\ &= \frac{1}{2}\, B_{2i,2i+1}^{k+1} + \frac{1}{4}\, B_{2i+1,2i+1}^{k+1} + \frac{1}{2}\, B_{2i+1,2i+2}^{k+1}\\ &= \frac{1}{4} T_{2i+1}^{k+1} + \frac{1}{4}(B_{2i,2i+1}^{k+1} + B_{2i+1,2i+2}^{k+1})\end{aligned}$$

and

$$\begin{aligned}(3)\qquad B_{i,i}^k &= \frac{1}{4} B_{2i-1,2i-1}^{k+1} + B_{2i-1,2i}^{k+1} + B_{2i,2i}^{k+1} + B_{2i,2i+1}^{k+1} + \frac{1}{4} B_{2i+1,2i+1}^{k+1}\\ &= T_{2i}^{k+1} + \frac{1}{4}(B_{2i-1,2i-1}^{k+1} + B_{2i+1,2i+1}^{k+1}).\end{aligned}$$

From (2) and (3) we have

$$T_i^k = B_{i-1,i}^k + B_{i,i}^k + B_{i,i+1}^k = \frac{1}{2}\, T_{2i-1}^{k+1} + T_{2i}^{k+1} + \frac{1}{2}\, T_{2i+1}^{k+1}.$$

Finally, we put $Z_{j+1}^k = \sum_{r=0}^{j} T_r^k$, $j = 0, \ldots, 2^k$. We obtain

$$\begin{aligned}(4)\qquad Z_{j+1}^k &= \sum_{r=0}^{j} \Big(\frac{1}{2}\, T_{2r-1}^{k+1} + T_{2r}^{k+1} + \frac{1}{2}\, T_{2r+1}^{k+1}\Big)\\ &= \frac{1}{2} \sum_{r=0}^{j} (T_{2r-1}^{k+1} + T_{2r}^{k+1}) + \frac{1}{2} \sum_{r=0}^{j} (T_{2r}^{k+1} + T_{2r+1}^{k+1})\\ &= \frac{1}{2} \sum_{r=0}^{2j} T_r^{k+1} + \frac{1}{2} \sum_{r=0}^{2j+1} T_r^{k+1} = \frac{1}{2}(Z_{2j+1}^{k+1} + Z_{2j+2}^{k+1}).\end{aligned}$$

In particular,

$$(5)\qquad Z_{2^k+1}^k = Z_{2^{k+1}+1}^{k+1}$$

Let us notice that

$$(6)\qquad T_j^k = B(s_j^k, s_{j-1}^k + s_j^k + s_{j+1}^k) = B(s_j^k, 1)$$

because the support of the function $1 - s^k_{j-1} - s^k_j - s^k_{j+k}$ lies in the complement of the interior of the support of s^k_j. The symbol 1 here denotes the constant function equal to unity in the whole $\langle 0, 1\rangle$. Therefore, we have

$$(7) \qquad |T^k_j| = |B(s^k_j, 1)| \leqslant K\|s^k_j\|_{H^1} \cdot \|1\|_{H^1} \leqslant K\Big(\frac{2}{3}2^{-k} + 2^{k+1}\Big)^{\frac{1}{2}} \leqslant K_1 \cdot 2^{k/2}.$$

Lemma 6. *Let $u \in H^1$ and set $u^k = \sum_{j=0}^{2^k} u(x^k_j)s^k_j$. Then the sequence u^k converges to u in H^1.*

Proof. First, we notice that the derivatives $(u^k)'$ approximate u' in the sense of Lemma 5. Indeed, for $x \in I^k_j$ we have

$$\big[u^k(x)\big]' = 2^k\big[u(x^k_j) - u(x^k_{j-1})\big] = 2^k \int_{I^k} u'(t)\,\mathrm{d}t.$$

Thus, $\lim_{k\to\infty} \|(u^k)' - u'\|_{L_2} = 0$. In view of $u(0) = u^k(0)$ we have further

$$\begin{aligned}\|u^k - u\|^2_{L_2} &= \int_0^1 \big[u^k(t) - u(t)\big]^2\,\mathrm{d}t = \int_0^1 \mathrm{d}t\Big(\int_0^t \big[(u^k(s))' - u'(s)\big]\,\mathrm{d}s\Big)^2 \\ &\leqslant \int_0^1 \mathrm{d}t\Big(\int_0^t \mathrm{d}s \cdot \int_0^t \big[(u^k(s))' - u'(s)\big]^2\,\mathrm{d}s\Big) \leqslant \|(u^k)' - u'\|^2_{L_2}\end{aligned}$$

and the assertion follows. □

The function u will be approximated also by functions $\bar{u}^k$ and $\bar{\bar{u}}_k$ piecewise constant and such that $\bar{u}^k(t) = u(x^k_j)$ for $t \in I^k_j$ and $\bar{\bar{u}}^k(t) = u(x^k_j)$ for $t \in I^k_{j+1}$. Obviously, we have $\lim_{k\to\infty} \|\bar{u}^k - u\|_{L_2} = \lim_{k\to\infty} \|\bar{\bar{u}}^k - u\|_{L_2} = 0$.

In what follows, we denote the derivative of u^k by Du^k and its value in I^k_j by Du^k_j (Du^k is a piecewise constant function). Now, we give the general form of the local bilinear symmetric functional in H^1.

Theorem 2. *Let $B(u, v)$ be a local continuous bilinear symmetric functional in $H^1(0, 1)$. Then there exist functions $\eta_1 \in L_\infty$, $\eta_0 \in L_2$ and a number α such that*

$$B(u, v) = \int_0^1 \big[\eta_1 u'v' + \eta_0(uv)'\big]\,\mathrm{d}t + \alpha u(1).$$

Proof. We approximate the functions u and v by sequences of piecewise linear functions u^k and v^k according to Lemma 6. We have

$$\begin{aligned} B(u^k, v^k) &= B\Big(\sum_{j=0}^{2^k} u(x_j^k)s_j^k, \sum_{j=0}^{2^k} v(x_j^k)s_j^k\Big) \\ &= \sum_{j=0}^{2^k} u(x_j^k)v(x_j^k)\cdot B_{j,j}^k + \sum_{j=0}^{2^k-1} u(x_j^k)v(x_{j+1}^k)B_{j,j+1}^k \\ &\quad + \sum_{j=1}^{2^k} u(x_{j+1}^k)v(x_j^k)B_{j,j+1}^k. \end{aligned}$$

We have utilized the local property and the symmetry of the functional. A simple manipulation leads to

$$\begin{aligned} B(u^k, v^k) &= \sum_{j=0}^{2^k} u(x_j^k)v(x_j^k)(B_{j-1,j}^k + B_{j,j}^k + B_{j,j+1}^k) \\ &\quad - \sum_{j=0}^{2^k-1} \big[u(x_{j+1}^k) - u(x_j^k)\big]\big[v(x_{j+1}^k) - v(x_j^k)\big] B_{j,j+1}^k. \end{aligned} \tag{8}$$

To the first sum on the right-hand side we apply the partial summation

$$\begin{aligned} \sum_{j=0}^{2^k} u(x_j^k)v(x_j^k)T_j^k &= u(1)v(1)Z_{2^k+1}^k - \sum_{j=1}^{2^k} \big[u(x_j^k)v(x_j^k) - u(x_{j-1}^k)v(x_{j-1}^k)\big] Z_j^k \\ &= u^k(1)v^k(1)Z_{2^k+1}^k - \sum_{j=1}^{2^k} \Big\{u(x_j^k)\big[v(x_j^k) - v(x_{j-1}^k)\big] \\ &\quad + v(x_{j-1}^k)\big[u(x_j^k) - u(x_{j-1}^k)\big]\Big\} Z_j^k \\ &= u^k(1)v^k(1)Z_{2^k+1}^k - \sum_{j=1}^{2^k} (\bar{u}_j^k Dv_j^k + \bar{\bar{v}}_j^k Du_j^k) Z_j^k 2^{-k}. \end{aligned}$$

We show that the numbers Z_j^k satisfy the assumptions of Lemma 4. They satisfy (4) and because of $T_r^k = B(s_r^k, 1)$, cf. (6), we have

$$Z_{j+1}^k = \sum_{r=0}^{j} T_r^k = B\Big(\sum_{r=0}^{j} s_r^k, 1\Big).$$

We put $w_j^k = \sum_{r=0}^{j} s_r^k$.

The bilinear functional may be represented by the scalar product

$$B(u,v) = (u, \mathcal{L} v)_{H^1}$$

were $\mathcal{L}$ is a linear operator from H^1 into H^1. Therefore,

$$B(w_j^k, 1) = (w_j^k, \mathcal{L}1)_{H^1} = -\int_{I_{j+1}^k} 2^k (\mathcal{L}1)' \,\mathrm{d}t + \int_0^{x_j^k} \mathcal{L}1 \,\mathrm{d}t + \int_{I_{j+1}^k} 2^k (x_{j+1}^k - t)\mathcal{L}1 \,\mathrm{d}t$$

and

$$\begin{aligned}(9) \quad |(w_j^k, \mathcal{L}1)_{H^1}|^2 &\leqslant 3\Big[2^k \int_{I_{j+1}^k} (\mathcal{L}1)'^2 \,\mathrm{d}t + \|\mathcal{L}1\|_{L_2}^2 \\ &\quad + \int_{I_{j+1}^k} 2^{2k}(x_{j+1}^k - t)^2 \,\mathrm{d}t \cdot \int_{I_{j+1}^k} (\mathcal{L}1)^2 \,\mathrm{d}t\Big] \\ &\leqslant 3 \cdot 2^k \Big[\int_{I_{j+1}^k} (\mathcal{L}1)'^2 \,\mathrm{d}t + 2^{-k}\|\mathcal{L}1\|_{L_2}^2 + K \int_{I_{j+1}^k} (\mathcal{L}1)^2 \,\mathrm{d}t,\end{aligned}$$

and finally

$$\begin{aligned}\sum_{j=0}^{2^k-1} (Z_{j+1}^k)^2 &= \sum_{j=0}^{2^k-1} (w_j^k, \mathcal{L}1)_{H^1}^2 \\ &\leqslant 3 \cdot 2^k \Big[\int_0^1 (\mathcal{L}1)'^2 \,\mathrm{d}t + \|\mathcal{L}1\|_{L_2}^2 + K \int_0^1 (\mathcal{L}1)^2 \,\mathrm{d}t\Big] \\ &\leqslant K 2^k \|\mathcal{L}1\|_{H^1}^2\end{aligned}$$

or

$$2^{-k} \sum_{j=0}^{2^k-1} (Z_{j+1}^k)^2 \leqslant K \|\mathcal{L}_1\|_{H^1}^2.$$

Thus, the assumptions of Lemma 4 are verified and there exists a function $\eta_0 \in L_2$ such that $Z_{j+1}^k = -2^k \int_{I_{k+1}^k} \eta_0 \,\mathrm{d}t$. We denote by α the value $Z_{2^k+1}^k$ that is independent of k, cf. (5). Analogously as in the proof of Theorem 1 we prove that the first sum in (8) converges to

$$\int_0^1 \eta_0 (uv)' \,\mathrm{d}t + \alpha u(1)v(1).$$

The second sum in (8) will be transformed to

$$\sum_{j=0}^{2^k-1} Du_{j+1}^k Dv_{j+1}^k B_{j,j+1}^k 2^{-2k}.$$

We use (2) obtaining

$$2^{-k}B^k_{j,j+1} = 2^{-(k+2)}T^{k+1}_{2j+1} + 2^{-(k+2)}(B^{k+1}_{2j,2j+1} + B^{k+1}_{2j+1,2j+2}).$$

We write $b^k_{j+1} = 2^{-k}B^k_{j,j+1}$ and $\varepsilon^k_{j+1} = 2^{-(k+2)}T^{k+1}_{2j+1}$. Consequently

$$b^k_{j+1} = \varepsilon^k_{j+1} + \frac{1}{2}(b^{k+1}_{2j+1} + b^{k+1}_{2j+2}) \tag{10}$$

and the estimates

$$|b^k_j| \leqslant K \cdot 2^{-k}\|s^k_j\|_{H^1}\|s^k_{j+1}\|_{H^1}, \quad |\varepsilon^k_{j+1}| \leqslant K2^{-k/2} \tag{11}$$

hold, cf. (7).

Let b^k and d^k be piecewise constant functions with $b^k(t) = b^k_j$, $d^k(t) = \varepsilon^k_j$ for $t \in I^k_j$.

Further we put $c^k = b^k - \sum_{s=k}^{\infty} d^s$. The sum is convergent in view of (11) and

$$\left|\sum_{s=k}^{\infty} d^s(t)\right| \leqslant K2^{-k/2}.$$

With the notation $c^k_j = 2^k \int_{I^k_j} c^k(t)\,\mathrm{d}t$ we have

$$\begin{aligned} c^k_j &= 2^k \int_{I^k_j}\left[b^k(t) - \sum_{s=k}^{\infty} d^s(t)\right]\mathrm{d}t = b^k_j - \varepsilon^k_j - 2^k\int_{I^k_j}\sum_{s=k+1}^{\infty} d^s(t)\,\mathrm{d}t \\ &= \frac{1}{2}(b^{k+1}_{2j-1} + b^{k+1}_{2j}) - \frac{1}{2}2^{k+1}\left[\int_{I^{k+1}_{2j-1}}\sum_{s=k+1}^{\infty} d^s(t)\,\mathrm{d}t + \int_{I^{k+1}_{2j}}\sum_{s=k+1}^{\infty} d^s(t)\,\mathrm{d}t\right] \\ &= \frac{1}{2}(c^{k+1}_{2j-1} + c^{k+1}_{2j}). \end{aligned}$$

Further,

$$|c^k_j| \leqslant |b^k_j| + K2^{-k/2} \leqslant K,$$

and therefore according to Lemma 3 there exists a bounded function $-\eta_1$ such that $2^{-k}c^k_j = -\int_{I^k_j}\eta_1(t)\,\mathrm{d}t$. Thus, we have

$$\begin{aligned} \sum_{j=0}^{2^k-1} Du^k_{j+1}Dv^k_{j+1}b^k_{j+1}2^{-k} &= \sum_{j=0}^{2^k-1} Du^k_{j+1}Dv^k_{j+1}\left(2^{-k}c^k_{j+1} + 2^{-k}\frac{1}{2^{-k}}\int_{I^k_{j+1}}\sum_{s=k}^{\infty} d^s(t)\,\mathrm{d}t\right) \\ &= -\int_0^1 Du^k(t)Dv^k(t)\eta_1(t)\,\mathrm{d}t + \int_0^1 Du^k(t)Dv^k(t)\sum_{s=k}^{\infty} d^s(t)\,\mathrm{d}t. \end{aligned}$$

However, this expression converges for $k \to \infty$ to $-\int_0^1 u'v'\eta_1(t)\,dt$. The second integral is bounded by

$$K2^{-k/2}\int_0^1 |Du^k(t)Dv^k(t)|\,dt$$

and the integrals here form a convergent and, consequently, bounded sequence. The theorem is proved. □

Next we give the general form of a local antisymmetric bilinear functional.

Theorem 3. *Let $B(u,v)$ be a local continuous antisymmetric bilinear functional in $H^1(0,1)$. Then there exists a function $\eta_2 \in L_2$ such that*

$$B(u,v) = \int_0^1 \eta_1(uv' - u'v)\,dt.$$

P r o o f. We approximate again the functions u and v by sequences u^k and v^k according to Lemma 6. Then we have

$$\begin{aligned} B(u^k, v^k) &= B\Big(\sum_{j=0}^{2^k} u(x_j^k)s_j^k, \sum_{j=0}^{2^k} v(x_j^k)s_j^k\Big) \\ &= \sum_{j=0}^{2^k-1} \big[u(x_j^k)v(x_{j+1}^k) - u(x_{j+1}^k)v(x_j^k)\big] B_{j,j+1}^k \\ &= \sum_{j=0}^{2^k-1} \Big\{u(x_j^k)\big[v(x_{j+1}^k) - v(x_j^k)\big] - v(x_j^k)\big[u(x_{j+1}^k) - u(x_j^k)\big]\Big\} B_{j,j+1}^k \\ &= \sum_{j=0}^{2^k-1} \big[u(x_j^k)Dv_{j+1}^k - v(x_j^k)Du_{j+1}^k\big] B_{j,j+1}^k 2^{-k}. \end{aligned}$$

Now, in the antisymmetric case we have, instead of (2),

$$B_{j,j+1}^k = \frac{1}{2}(B_{2j,2j+1}^{k+1} + B_{2j+1,2j+1}^{k+2}). \tag{12}$$

Moreover,

$$B_{j,j+1}^k = B(s_j^k, s_{j+1}^k) = B(w_j^k, 1 - w_j^k) = B(w_j^k, 1) = (w_j^k, \mathscr{L}1)_{H^1}.$$

We have used the antisymmetry, writing again $w_j^k = \sum_{r=0}^{j} s_r^k$. According to (9) we have

$$\sum_{j=0}^{2^k-1} (w_j^k, \mathscr{L}1)^2 \leqslant K \cdot 2^k \|\mathscr{L}1\|_{H^1}^2.$$

The numbers $b_j^k = B_{j,j+1}^k$ satisfy the assumptions of Lemma 4. There exists a function $\eta_2 \in L_2$ such that

$$B(u^k, v^k) = \int_0^1 \eta_2(t)\big[\bar{\bar{u}}^k(t)Dv^k(t) - Du^k(t)\bar{\bar{v}}^k(t)\big]\,\mathrm{d}t,$$

and passing to the limit we finally obtain

$$B(u, v) = \int_0^1 \eta_2(t)\big[u(t)v'(t) - u'(t)v(t)\big]\,\mathrm{d}t,$$

qed. □

References

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