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FINITE ELEMENT SOLUTION OF A STATIONARY HEAT CONDUCTION EQUATION WITH THE RADIATION BOUNDARY CONDITION

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Summary. In this paper we present a weak formulation of a two-dimensional stationary heat conduction problem with the radiation boundary condition. The problem can be described by an operator which is monotone on the convex set of admissible functions. The relation between classical and weak solutions as well as the convergence of the finite element method to the weak solution in the norm of the Sobolev space $H^1(\Omega)$ are examined.

Keywords: heat conduction, heat radiation, finite elements

AMS classification: 65N30

1. FORMULATION OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary Γ. The stationary heat conduction equation is

$$-\operatorname{div}(\lambda \operatorname{grad} u) = f, \tag{1.1}$$

where u is the temperature, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u \geqslant 0$, since the temperature is measured in Kelvin's scale, λ is a symmetric positive definite matrix of heat conduction coefficients, $\lambda \in \left(C^1(\overline{\Omega})\right)^{2\times 2}$, $f \in L_2(\Omega)$ is the volume heat energy.

Let Γ_1, Γ_2 be relatively open sets in Γ, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma$. We prescribe boundary conditions

$$\begin{aligned} u &= g && \text{on } \Gamma_1, \\ -\nu^{\mathsf{T}} \lambda \operatorname{grad} u &= \alpha u + \beta u^4 && \text{on } \Gamma_2, \end{aligned} \tag{1.2}$$

where g is the temperature prescribed on Γ_1, $g \geqslant 0$, $g \in H^1(\Omega)$, ν is the outward unit normal to Γ, $\alpha \geqslant 0$ is the coefficient of convective heat transfer on Γ_2, $\alpha \in L_\infty(\Gamma_2)$, where, in the case $\Gamma_1 = \emptyset$, we suppose $\alpha > 0$ on a nonempty open subset of Γ, $\beta \geqslant 0$ is the coefficient of radiation heat transfer on Γ_2, $\beta \in L_\infty(\Gamma_2)$.

The boundary condition on Γ_2 is called *the Stefan-Boltzmann condition.* The function u, which statisfies the above conditions, is called *the classical solution* of the problem (1.1)–(1.2).

There exist many different methods how to treat analytically and numerically the Stefan-Boltzmann condition, see e.g. [1, 3, 6, 7, 10, 12, 13, 15, 16]. In the present paper, we slightly modify a variational approach, which is briefly described in [14]. Then we will examine the associated finite element approximations.

Let us set

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1\},$$

where $H^1(\Omega)$ is the standard Sobolev space with the norm $\|.\|_{1,2}$ and the seminorm $|.|_{1,2}$. Multiplying (1.1) by an arbitrary function $v \in V$ and integrating over Ω we obtain from Green's theorem:

$$\int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} v \,\mathrm{d}x - \int_{\Gamma_2} (\nu^\top \lambda \operatorname{grad} u)\, v \,\mathrm{d}s = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in V.$$

Employing the boundary conditions (1.2), we get

$$\int_\Omega \lambda \operatorname{grad} u \cdot \operatorname{grad} v \,\mathrm{d}x + \int_{\Gamma_2} (\alpha u + \beta u^4) v \,\mathrm{d}s = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in V. \tag{1.3}$$

So every classical solution of (1.1)–(1.2), if it exists, satisfies the equation (1.3).

Now we shall show that all integrals in (1.3) are well-defined even for u, v from $H^1(\Omega)$. This is clear for the first and the third ones. The following theorem is presented in [11, p. 86]:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. Then for every real number $q \geqslant 1$ there exists a single continuous linear mapping Z: $H^1(\Omega) \to L_q(\Gamma)$ which satisfies $Zv = v$ on Γ for all $v \in C^\infty(\overline{\Omega})$.*

From Theorem 1.1 we see that the second integral in (1.3) is also well-defined for arbitrary $u, v \in H^1(\Omega)$.

For the sake of simplicity set

$$a(v, w) = \int_\Omega \lambda \operatorname{grad} v \cdot \operatorname{grad} w \,\mathrm{d}x + \int_{\Gamma_2} \alpha v w \,\mathrm{d}s, \qquad b(v) = \int_\Omega f v \,\mathrm{d}x.$$

Now the equation (1.3) can be rewritten as

$$a(u,v) + \int_{\Gamma_2} \beta u^4 v \, ds = b(v) \quad \forall v \in V. \tag{1.4}$$

Let us define

$$U = \{v \in H^1(\Omega) \mid v \geqslant 0 \ \text{ in } \Omega, v = g \text{ on } \Gamma_1\}.$$

The set U is called *the set of admissible functions.* Note that U is convex and closed with respect to the norm $\|.\|_{1,2}$.

Definition 1.2. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. A function $u_0 \in U$ is called a weak solution of the problem (1.1)–(1.2) if

$$a(u_0, v - u_0) + \int_{\Gamma_2} \beta u_0^4 (v - u_0) \, ds \geqslant b(v - u_0) \quad \forall v \in U. \tag{1.5}$$

For the functions u_0 and $v \in U$ we have $v - u_0 \in V$. So the condition (1.5) is weaker than that in (1.4).

Consider the functional

$$\Pi(v) = \frac{1}{2} a(v,v) + \frac{1}{5} \int_{\Gamma_2} \beta v^5 \, ds - b(v). \tag{1.6}$$

By Theorem 1.1 this functional is well-defined and continuous in $H^1(\Omega)$. Moreover, Π is clearly G-differentiable, strictly convex and coercive in U. That is why there exists precisely one element $\overline{u} \in U$ in which Π attains its minimum over the set U. This element fulfils

$$\mathrm{D}\Pi(\overline{u}; v - \overline{u}) \geqslant 0 \quad \forall v \in U, \tag{1.7}$$

where

$$\mathrm{D}\Pi(v; w) = a(v,w) + \int_{\Gamma_2} \beta v^4 w \, ds - b(w) \tag{1.8}$$

is the Gâteaux derivative of Π at the point v in the direction w. We see the inequality (1.7) is the same as that in (1.5), so $\overline{u}$ is the only solution of (1.1)–(1.2) in the sense of Definition 1.2.

2. Some properties of the weak solution

It is clear that a classical solution is also a weak solution. Now we shall formulate conditions ensuring that the weak solution is the classical one.

Definition 2.1. The set Γ_1 is said to possess the property (K) if it is the union of a finite number of connected and relatively open sets in Γ such that their closures are disjoint.

Let us set

$$\Omega'_\delta = \{x \in \mathbb{R}^2 \mid \operatorname{dist}(\Gamma_1, x) < \delta\}, \ \Omega_\delta = \Omega'_\delta \cap \Omega. \tag{2.1}$$

The next lemma follows from a more general proposition which is presented in [4, p. 618].

Lemma 2.2. *Let Γ_1 have the property* (K). *Then for an arbitrary function $u \in V$ there exists a sequence of functions $\{v_n\} \subset V \cap C^\infty(\overline{\Omega})$ such that $v_n \to v$ in the norm of $H^1(\Omega)$ and $v_n = 0$ on $\Omega_{\frac{1}{n}}$.*

For an arbitrary real constant $r > 0$ define

$$W(r) = \{v \in C^\infty(\overline{\Omega}) \mid \|v\|_{C(\overline{\Omega})} \leqslant r, \ v = 0 \text{ on } \Gamma_1\}.$$

Theorem 2.3. *Let Γ_1 have the property* (K) *and let u_0 be the weak solution of* (1.1)–(1.2). *Let $u_0 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and let there exist a constant $C_{u_0} > 0$ such that $u_0 \geqslant C_{u_0}$ holds in the domain Ω. Then u_0 is also the classical solution of* (1.1)–(1.2).

Proof. The weak solution $u_0 \in U$ satisfies the condition

$$\mathrm{D}\Pi(u_0; w - u_0) \geqslant 0 \quad \forall w \in U.$$

For every function $v \in W(C_{u_0})$ we have $u_0 + v \in U$, $u_0 - v \in U$. Therefore

$$\mathrm{D}\Pi(u_0; u_0 + v - u_0) = \mathrm{D}\Pi(u_0; v) \geqslant 0, \quad \mathrm{D}\Pi(u_0; u_0 - v - u_0) = \mathrm{D}\Pi(u_0; -v) \geqslant 0.$$

From linearity of $\mathrm{D}\Pi(u_0; .)$ and density of $V \cap C^\infty(\overline{\Omega})$ in V (cf. Lemma 2.2) we have $\mathrm{D}\Pi(u_0; v) = 0$. Hence by (1.8) we obtain

$$a(u_0, v) + \int_{\Gamma_2} \beta u_0^4 v \, \mathrm{d}s = b(v) \tag{2.2}$$

for all $v \in V$. From here we arrive at

$$-\int_\Omega \operatorname{div}(\lambda \operatorname{grad} u_0) v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in H_0^1(\Omega),$$

$$-\int_{\Gamma_2} (\nu^\mathsf{T} \lambda \operatorname{grad} u_0) v \,\mathrm{d}s = \int_{\Gamma_2} (\alpha u_0 + \beta u_0^4) v \,\mathrm{d}s \quad \forall v \in V.$$

Since the space $H_0^1(\Omega)$ is dense in $L_2(\Omega)$ and traces of the functions from V are dense in $L_2(\Gamma_2)$ (cf. [11, p. 87]), u_0 fulfils (1.1)–(1.2). □

Theorem 2.4. *The weak solution u_0 of (1.1)–(1.2) depends continuously on the functions α, β a f.*

Proof. Suppose there are two weak solutions u_i of two problems given by functions α_i, β_i, f_i, $i = 0, 1$. From the definition of the weak solution we have

$$\int_\Omega \lambda \operatorname{grad} u_0 \cdot \operatorname{grad}(u_1 - u_0) \,\mathrm{d}x + \int_{\Gamma_2} \alpha_0 u_0 (u_1 - u_0) \,\mathrm{d}s$$
$$+ \int_{\Gamma_2} \beta_0 u_0^4 (u_1 - u_0) \,\mathrm{d}s \geqslant \int_\Omega f_0 (u_1 - u_0) \,\mathrm{d}x,$$
$$\int_\Omega \lambda \operatorname{grad} u_1 \cdot \operatorname{grad}(u_0 - u_1) \,\mathrm{d}x + \int_{\Gamma_2} \alpha_1 u_1 (u_0 - u_1) \,\mathrm{d}s$$
$$+ \int_{\Gamma_2} \beta_1 u_1^4 (u_0 - u_1) \,\mathrm{d}s \geqslant \int_\Omega f_1 (u_0 - u_1) \,\mathrm{d}x.$$

We multiply both inequalities by -1, sum up and obtain

$$\int_\Omega \lambda \,|\operatorname{grad}(u_0 - u_1)|^2 \,\mathrm{d}x + \int_{\Gamma_2} (\alpha_0 u_0 - \alpha_1 u_1)(u_0 - u_1) \,\mathrm{d}s$$
$$+ \int_{\Gamma_2} (\beta_0 u_0^4 - \beta_1 u_1^4)(u_0 - u_1) \,\mathrm{d}s \leqslant \int_\Omega (f_0 - f_1)(u_0 - u_1) \,\mathrm{d}x.$$

Using the relations

$$\alpha_0 u_0 - \alpha_1 u_1 = \alpha_0 u_0 - \alpha_0 u_1 + \alpha_0 u_1 - \alpha_1 u_1 = \alpha_0 (u_0 - u_1) + u_1 (\alpha_0 - \alpha_1),$$

$$\beta_0 u_0^4 - \beta_1 u_1^4 = \beta_0 u_0^4 - \beta_0 u_1^4 + \beta_0 u_1^4 - \beta_1 u_1^4 = \beta_0 (u_0^4 - u_1^4) + u_1^4 (\beta_0 - \beta_1),$$

we have

$$\int_\Omega \lambda \,|\operatorname{grad}(u_0 - u_1)|^2 \,\mathrm{d}x + \int_{\Gamma_2} \alpha_0 (u_0 - u_1)^2 \,\mathrm{d}s$$
$$+ \int_{\Gamma_2} (\beta_0 (u_0^4 - u_1^4) + u_1^4 (\beta_0 - \beta_1))(u_0 - u_1) \,\mathrm{d}s$$
$$\leqslant \int_\Omega (f_0 - f_1)(u_0 - u_1) \,\mathrm{d}x - \int_{\Gamma_2} u_1 (\alpha_0 - \alpha_1)(u_0 - u_1) \,\mathrm{d}s.$$

Since

$$\beta_0(u_0^4 - u_1^4)(u_0 - u_1) = \beta_0(u_0 - u_1)^2(u_0 + u_1)(u_0^2 + u_1^2) \geqslant 0$$

for u_0, $u_1 \in U$, we obtain from the foregoing inequality

$$\int_\Omega \lambda\, |\mathrm{grad}\,(u_0 - u_1)|^2\, \mathrm{d}x + \int_{\Gamma_2} \alpha_0(u_0 - u_1)^2\, \mathrm{d}s \leqslant \int_\Omega |(f_0 - f_1)(u_0 - u_1)|\, \mathrm{d}x$$
$$+ \int_{\Gamma_2} |u_1(\alpha_0 - \alpha_1)(u_0 - u_1)|\, \mathrm{d}s + \int_{\Gamma_2} u_1^4 |(\beta_0 - \beta_1)(u_0 - u_1)|\, \mathrm{d}s.$$

Now we estimate the left-hand side by the Friedrichs inequality and the second and third terms on the right-hand side with help of Theorem 1.1. Then for a sufficiently large constant $C > 0$ we have

$$\|u_0 - u_1\|_{1,2}^2 \leqslant C(\|f_0 - f_1\|_{0,2} + \|\alpha_0 - \alpha_1\|_{0,\infty,\Gamma_2} + \|\beta_0 - \beta_1\|_{0,\infty,\Gamma_2})\|u_0 - u_1\|_{1,2},$$

where C depends only on u_1. □

3. Solution of the variational formulation by the finite element method

For simplicity we shall suppose that $\Omega \subset \mathbb{R}^2$ is a polygonal domain with a Lipschitz boundary and Γ_1 has the property (K). Then $\overline{\Omega}$ can be divided in the sense of [2, p.38] into a finite number of closed triangles K_i, $i \in I$. Such a triangulation will be denoted by T_h, where $h = \max_{i \in I} \operatorname{diam} K_i$.

Consider a family of triangulations $\mathscr{F} = \{T_h\}$, $h \to 0+$. For a fixed h the triangulation T_h is formed by the triangles K_i^h, $i \in I^h$. Let us define

$$X_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_{K_i^h} \in P_1(K_i^h)\ \forall i \in I^h\}.$$

The spaces X_h contain functions continuous on $\overline{\Omega}$ and linear on each of $K_i^h \in T_h$. Moreover (cf. [9, p. 27]),

$$X_h \subset H^1(\Omega). \tag{3.1}$$

Let $\pi_h : C(\overline{\Omega}) \mapsto X_h$ be the standard linear interpolation operator such that for all functions $v \in C(\overline{\Omega})$ we have $\pi_h v(x) = v(x)$ at the vertices x of the triangles K_i^h, $i \in I^h$.

For simplicity we shall suppose that the function g occurring in the boundary conditions (1.2) and the chosen family of triangulations $\mathscr{F}$ satisfy

$$\exists h_0 > 0 \ \forall h \in (0, h_0) \ g \in X_h. \tag{3.2}$$

It means that the function g is equal to its interpolation $\pi_h g$ for sufficiently small h.

For $h \in (0, h_0)$ let us define

$$U_h = U \cap X_h.$$

Clearly the sets U_h are convex and closed. So for $h \in (0, h_0)$ there exists one and only one solution of the discrete problem

$$\Pi(u_h) = \inf_{v_h \in U_h} \Pi(v_h). \tag{3.3}$$

Now our goal is to prove that under certain assumptions the solutions u_h of the problems (3.3) tend to the weak solution of (1.1)–(1.2).

Definition 3.1. We say that a family of triangulations $\mathscr{F}$ satisfies the maximum angle condition if there exists an angle $\gamma < 180°$ such that every triangle $K_i^h \in T_h \in \mathscr{F}$ has all its angles less or equal to γ.

In the paper [8, p. 225] it is proved:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with a Lipschitz boundary, $\mathscr{F}$ a family of triangulations which fulfils the maximum angle condition. Then there exists a constant $C > 0$ independent of h such that*

$$\|v - \pi_h v\|_{1,2} \leqslant Ch|v|_{2,2} \quad \forall v \in H^2(\Omega).$$

For an arbitrary parameter $\varrho > 0$ we denote by R_ϱ the standard regularization operator which maps $L_1(\Omega)$ into $C^\infty(\overline{\Omega})$ (cf. [11, p. 58]):

$$(R_\varrho v)(x) = \int_{|y|<\varrho} v(x+y)\omega(y, \varrho)\,\mathrm{d}y,$$

where we define $v = 0$ outside Ω and where $\omega(y, \varrho)$ (the regularization kernel) is a smooth non-negative hill function defined in $\mathbb{R}^2$, which vanishes for y such that $|y| \geqslant \varrho$ and fulfils

$$\int_{|y|<\varrho} \omega(y, \varrho)\,\mathrm{d}y = 1.$$

Recall that the sets Ω_δ, Ω'_δ were defined in (2.1).

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. Then for every $\delta > 0$ there exists a function $\theta_\delta \in C^\infty(\overline{\Omega})$, $\theta_\delta = 0$ in $\Omega_{\frac{\delta}{2}}$, $\theta_\delta = 1$ in $\overline{\Omega} - \Omega_\delta$, $\theta_\delta \geqslant 0$ in Ω.*

Proof. Put $\tau = \frac{3}{4}\delta$. Let us define

$$\begin{aligned} \theta'_\delta &= 0 \quad \text{for } x \in \Omega'_\tau, \\ \theta'_\delta &= 1 \quad \text{for } x \in \mathbb{R}^2 - \Omega'_\tau. \end{aligned}$$

If we choose $\varrho \in (0, \frac{\delta}{4})$, we can take $\theta_\delta = R_\varrho \theta'_\delta$. □

The following lemma is to be found in [5, p. 25]:

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. For $v \in H^1(\Omega)$ define*

$$v^+ = \max(0, v), \quad v^- = \max(0, -v).$$

Then the functions v^+, v^- belong to $H^1(\Omega)$ and the mapping $v \mapsto \{v^+, v^-\}$ from $H^1(\Omega)$ into $H^1(\Omega) \times H^1(\Omega)$ is continuous.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary and $v, w \in H^1(\Omega)$. Let $\{v_n\} \subset H^1(\Omega)$ be a sequence of functions, $v_n \to v$ in $H^1(\Omega)$ and let*

$$\hat{v} = \max(v, w), \quad \hat{v}_n = \max(v_n, w).$$

Then $\hat{v}_n \to \hat{v}$ in the norm of the space $H^1(\Omega)$.

Proof.

$$\begin{aligned} \hat{v} &= \max(v, w) = \tfrac{1}{2}((v + w) + |v - w|), \\ \hat{v}_n &= \max(v_n, w) = \tfrac{1}{2}((v_n + w) + |v_n - w|). \end{aligned}$$

Since $v_n - w \to v - w$, it follows from Lemma 3.4 that $|v_n - w| \to |v - w|$. □

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^2$ be a domain. Let $\overline{\Omega} \subset \Omega_1$, where $\Omega_1 \subset \mathbb{R}^2$ is a bounded domain, and let $v \in C(\overline{\Omega_1})$. Then*

$$\lim_{\varrho \to 0} \|v - R_\varrho v\|_{C(\overline{\Omega})} = 0.$$

P r o o f. The proof is similar to that in [11, p. 59], which deals with the case of convergence in the spaces $L_p(\Omega)$. For an arbitrary $x \in \Omega$ and $\varrho < \operatorname{dist}(\partial\Omega, \partial\Omega_1)$ we have

$$\begin{aligned} |v(x) - (R_\varrho v)(x)| &= \Big| v(x) - \int_{|y|<\varrho} v(x+y)\omega(y,\varrho)\,\mathrm{d}y \Big| \\ &= \Big| \int_{|y|<\varrho} (v(x) - v(x+y))\omega(y,\varrho)\,\mathrm{d}y \Big|. \end{aligned}$$

The set $\overline{\Omega_1}$ is compact, so v is uniformly continuous there. Then for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for arbitrary x_1, $x_2 \in \Omega$ we get

$$|x_1 - x_2| < \delta \Rightarrow |v(x_1) - v(x_2)| < \varepsilon.$$

For $\varrho < \frac{\delta}{2}$ this yields

$$|v(x) - (R_\varrho v)(x)| < \varepsilon.$$

□

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with a Lipschitz boundary. Let $\mathscr{F}$ be a family of triangulations which satisfies the maximum angle condition, and let (3.2) hold. Then for all $v \in U$ there exists a sequence of functions $\{v_h\}$, $v_h \in U_h$, such that*

$$\lim_{h\to 0} \|v - v_h\|_{1,2} = 0.$$

P r o o f. The proof is a modified version of the proof of a similar theorem, which is in [5, p. 35]. Let $v \in U$ and let us define $w = v - g$. Lemma 2.2 yields that there exists a sequence of functions $\{w_n\} \subset C^\infty(\overline{\Omega})$, $w_n = 0$ on $\Omega_{\frac{2}{n}}$ and $w_n \to w$ in the space $H^1(\Omega)$.

The function g, which is continuous and piecewise linear in $\overline{\Omega}$, can be easily continuously extended onto $\mathbb{R}^2$. The functions $w_n \in C^\infty(\overline{\Omega})$ can also be continuously extended onto $\mathbb{R}^2$ by Calderon's extension (cf. [11, p.80]), since $C^\infty(\overline{\Omega}) \subset H^2(\Omega)$. We denote the extended functions by the same symbols.

Let us define functions

$$\hat{w}_n = \max(-g, w_n). \tag{3.4}$$

Obviously, $\hat{w}_n \in C(\mathbb{R}^2)$, $\hat{w}_n \geqslant -g$ and $\hat{w}_n = 0$ on $\Omega_{\frac{2}{n}}$. Since $w = \max(-g, w)$, Lemma 3.5 implies that $\hat{w}_n \to w$ in the space $H^1(\Omega)$.

Now let us fix n and make the regularization of $\hat{w}_n$. We choose $\varrho \in (0, \frac{1}{n})$. The functions $R_\varrho \hat{w}_n$ vanish on $\Omega_{\frac{1}{n}}$, they are from $C^\infty(\mathbb{R}^2)$ and by Lemma 3.6

$$\lim_{\varrho \to 0} \|\hat{w}_n - R_\varrho \hat{w}_n\|_{C(\overline{\Omega})} = 0. \tag{3.5}$$

From here and (3.4) it follows that for every $\varepsilon > 0$ there exists a sufficiently small $\varrho_0(\varepsilon) \in (0, \frac{1}{n})$ such that for all $\varrho \in (0, \varrho_0)$ the inequality

$$-g - \varepsilon \leqslant \hat{w}_n - \varepsilon < R_\varrho \hat{w}_n < \hat{w}_n + \varepsilon \tag{3.6}$$

holds in $\overline{\Omega}$. In the domain Ω, put

$$w_n^\varepsilon = R_\varrho \hat{w}_n + \varepsilon \theta_{\frac{1}{n}}, \tag{3.7}$$

where $\theta_{\frac{1}{n}}$ is the function from Lemma 3.3. Then $w_n^\varepsilon \in C^\infty(\overline{\Omega})$, $w_n^\varepsilon = 0$ on $\Omega_{\frac{1}{2n}}$, and from (3.6), (3.7) and (3.5) we have

$$w_n^\varepsilon \geqslant -g, \tag{3.8}$$

$$\lim_{\varepsilon \to 0} \|\hat{w}_n - w_n^\varepsilon\|_{1,2} = 0.$$

Since $C^\infty(\overline{\Omega}) \subset H^2(\Omega)$, Theorem 3.2 implies that $\lim_{h \to 0} \|w_n^\varepsilon - \pi_h w_n^\varepsilon\|_{1,2} = 0$.

According to (3.8), the function $\pi_h w_n^\varepsilon + g \in H^1(\Omega)$ is non-negative and, moreover, it is clearly $\pi_h w_n^\varepsilon + g = g$ on Γ_1. Therefore $\pi_h w_n^\varepsilon + g \in U_h$. Obviously,

$$\begin{aligned} \|v - (\pi_h w_n^\varepsilon + g)\|_{1,2} &= \|w + g - \pi_h w_n^\varepsilon - g\|_{1,2} \\ &\leqslant \|w - \hat{w}_n\|_{1,2} + \|\hat{w}_n - w_n^\varepsilon\|_{1,2} + \|w_n^\varepsilon - \pi_h w_n^\varepsilon\|_{1,2}. \end{aligned}$$

The last three norms tend to zero, since we have already proved that

$$\begin{aligned} &\hat{w}_n \to w \quad \text{as } n \to \infty, \\ &w_n^\varepsilon \to \hat{w}_n \quad \text{as } \varepsilon \to 0 \quad \text{for any } n = 1, 2, \dots, \\ &\pi_h w_n^\varepsilon \to w_n^\varepsilon \quad \text{as } h \to 0 \quad \text{for any } n = 1, 2, \dots \text{ and any } \varepsilon > 0. \end{aligned}$$

□

We see that a family of triangulations $\mathscr{F}$ which fulfils the maximum angle condition, allows for sufficiently small h a very precise approximation of a function $v \in U$ by an appropriate function from the space U_h.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with a Lipschitz boundary. Let $\mathscr{F}$ be a family of triangulations which fulfils the maximum angle condition, and let (3.2) hold. Then the functions u_h which are the solutions of the problems (3.3) satisfy*

$$\lim_{h\to 0} \|u_0 - u_h\|_{1,2} = 0,$$

where u_0 is the weak solution of (1.1)–(1.2).

Proof. From Theorem 3.7 we know that there exists a sequence $v_h \in U_h$ such that $\lim_{h\to 0} \|u_0 - v_h\|_{1,2} = 0$. Evidently, it is $\Pi(u_0) \leqslant \Pi(u_h) \leqslant \Pi(v_h)$. Since Π is continuous, we have

$$\lim_{h\to 0} \Pi(u_h) = \Pi(u_0). \tag{3.9}$$

Taylor's expansion of Π yields that

$$\Pi(u_h) = \Pi(u_0) + \mathrm{D}\Pi(u_0; u_h - u_0) + \tfrac{1}{2}\mathrm{D}^2\Pi(u_0 + \theta_h(u_h - u_0); u_h - u_0, u_h - u_0),$$

where $\theta_h \in (0,1)$. From (1.8) it is easy to derive that for an arbitrary $v \in U$ and $w \in H^1(\Omega)$ we have

$$\mathrm{D}^2\Pi(v; w, w) = a(w,w) + 4\int_{\Gamma_2} \beta v^3 w^2 \,\mathrm{d}s \geqslant C\|w\|_{1,2}^2,$$

where C is a positive constant independent of v. In the last inequality we have used the Friedrichs inequality for the first term and neglected the second term, because it is non-negative. So we have

$$\Pi(u_h) - \Pi(u_0) - \mathrm{D}\Pi(u_0; u_h - u_0) \geqslant C\|u_h - u_0\|_{1,2}^2. \tag{3.10}$$

From the definition of the weak solution we know that

$$\mathrm{D}\Pi(u_0; u_h - u_0) \geqslant 0. \tag{3.11}$$

Using (3.10), (3.9) and (3.11), we obtain

$$\|u_h - u_0\|_{1,2} \to 0.$$

□

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with a Lipschitz boundary. Let $\mathscr{F}$ be a system of triangulations which fulfils the maximum angle condition, and let (3.2) hold. Let the weak solution u_0 of (1.1)–(1.2) be from the space $H^2(\Omega)$ and let there exist a constant C_{u_0} such that $u \geqslant C_{u_0} > 0$ holds in Ω. Then*

$$\|u_0 - u_h\|_{1,2} \leqslant Ch \qquad \text{for } h \to 0,$$

where u_h solve (3.3) and $C > 0$ is a constant depending only on u_0.

Proof. As well as in the proof of Theorem 2.3 the condition $u_0 \geqslant C_{u_0}$ ensures the validity of (2.2) for any $v \in V$. Using this relation, we can easily verify that

$$\Pi(u_0 + v) - \Pi(u_0) = \tfrac{1}{2}a(v, v) + \int_{\Gamma_2} \beta(2u_0^3 v^2 + 2u_0^2 v^3 + u_0 v^4 + \tfrac{1}{5}v^5)\,\mathrm{d}s$$

for every $v \in V$. The integrals over Γ_2 can be estimated with help of Hölder's integral inequality and Theorem 1.1. So we obtain

$$\Pi(u_0 + v) - \Pi(u_0) \leqslant C\|v\|_{1,2}^2 \qquad \forall v \in V,\ \|v\|_{1,2} \leqslant 1, \tag{3.12}$$

where $C > 0$ is a constant depending only on u_0.

If $u_0 \in H^2(\Omega)$ ($\subset C(\overline{\Omega})$, cf. [11, p. 72]), then thanks to (3.2) it is possible to define $\pi_h u_0 \in U_h$. Since u_h solve (3.3), we have

$$\Pi(u_h) \leqslant \Pi(\pi_h u_0). \tag{3.13}$$

Then the relations (1.7), (3.10), (3.13), (3.12) and Theorem 3.2 imply

$$\begin{aligned} C_1\|u_h - u_0\|_{1,2}^2 &\leqslant C_1\|u_h - u_0\|_{1,2}^2 + \mathrm{D}\Pi(u_0; u_h - u_0) \leqslant \Pi(u_h) - \Pi(u_0) \\ &\leqslant \Pi(\pi_h u_0) - \Pi(u_0) \leqslant C_2\|\pi_h u_0 - u_0\|_{1,2}^2 \leqslant C_3 h^2 |u_0|_{2,2}^2, \end{aligned}$$

where C_1, C_2, $C_3 > 0$ are constants independent of h. □

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Souhrn

ŘEŠENÍ STACIONÁRNÍ ÚLOHY VEDENÍ TEPLA S OKRAJOVOU PODMÍNKOU SÁLÁNÍ METODOU KONEČNÝCH PRVKŮ

ZDENĚK MILKA

V článku je uvedena variační formulace stacionární úlohy vedení tepla s okrajovými podmínkami vedení a sálání. Je dokázána existence a jednoznačnost řešení. Dále je úloha řešena metodou lineárních konečných prvků. Je ukázána konvergence přibližných řešení k přesnému a je zkoumána i rychlost této konvergence při jistých požadavcích na hladkost přesného řešení.

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