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A BOUNDARY MULTIVALUED INTEGRAL "EQUATION" APPROACH TO THE SEMIPERMEABILITY PROBLEM

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Summary. The present paper concerns the problem of the flow through a semipermeable membrane of infinite thickness. The semipermeability boundary conditions are first considered to be monotone; these relations are therefore derived by convex superpotentials being in general nondifferentiable and nonfinite, and lead via a suitable application of the saddle-point technique to the formulation of a multivalued boundary integral equation. The latter is equivalent to a boundary minimization problem with a small number of unknowns. The extension of the present theory to more general nonmonotone semipermeability conditions is also studied. In the last section the theory is illustrated by two numerical examples.

Keywords: approximations of unilateral BVP, mixed and dual variational formulation of unilateral BVP

AMS classification: 65N30, 35J85

INTRODUCTION

The problem of semipermeable media and in particular, that of semipermeable membranes has been recently investigated for monotone semipermeability relations within the framework of convex analysis in [1]. These relations describe the flow through porous media (or, in particular, the behaviour of semipermeable membranes either on the boundary or in the interior of the medium) and give rise to a variational inequality formulation of the problem connected either with the ∇-operator or with other elliptic operators. Similar variational inequalities can be formulated for a class of semipermeable heat conduction, potential and temperature control problems, as well as for pressure control problems in hydraulics provided the controlled quantity obeys monotone semipermeability relations along the boundary or in the interior of the body. In the case of nonmonotone semipermeability relations the variational formulations have the form of hemivariational inequalities which have been studied by the last author in [2] using the new notion of nonsmooth analysis, the generalized gradient. However, from the point of view of numerical treatment of these problems, the formulation in terms of variational and hemivariational inequalities has a disadvantage, the large number of unknowns. Therefore in all unilateral problems, i.e. in all problems having variational expressions in the form of inequalities, there is an effort to obtain variational formulations containing only the unknowns causing the unilateral character in order to get BVPs with a smaller number of unknowns. This effort is based on a mathematical approach using the saddle point properties of the problem [3]–[7] and leads to multivalued boundary integral equations (B.I.E.).

In the present paper the problem of the flow through a semipermeable membrane of infinite thickness is investigated. For such a membrane which allows the free flow of the entering fluid but prevents all outflow of it, monotone semipermeability relations with graphs similar to those in Fig. 1 hold. Such a problem is of great significance since it is applied to the simulation of several practical problems, as is e.g. the operational response of valves under varying pressure of the fluid, the behaviour of several physical membranes in biological systems, etc. The aforementioned monotone semipermeability boundary conditions are derived by convex superpotentials which are generally nondifferentiable and nonfinite; the latter lead to the formulation of variational inequality problems which can be numerically treated by solving the equivalent convex optimization problems. Several numerical techniques of the theory of optimization can be employed to solve the arising quadratic programming (Q.P.) problems directly or indirectly. However, the direct application of a Q.P. algorithm includes certain serious difficulties concerning the limitation in the number of unknowns combined with the often very large computer time necessary. Therefore, in order to overcome these difficulties towards a rational and effective numerical study of the present problem, we formulate a multivalued B.I.E. which is equivalent to a boundary minimization problem with a small number of unknowns. At this point it should be pointed out that the method developed corresponds to the so-called direct method for the formulation of B.I.Es for bilateral problems. In contrast to the so-called indirect method which, when extended to unilateral problems, does not lead, at least for the moment ([8] or [2], p. 160), to numerical problems which can be effectively treated, the present method leads to symmetric Q.P. problems with a small number of unknowns which can be effectively treated. Finally, we will present an extension of the present theory to more general unilateral boundary conditions expressed in terms of general subdifferential laws [1], [2] and representing the

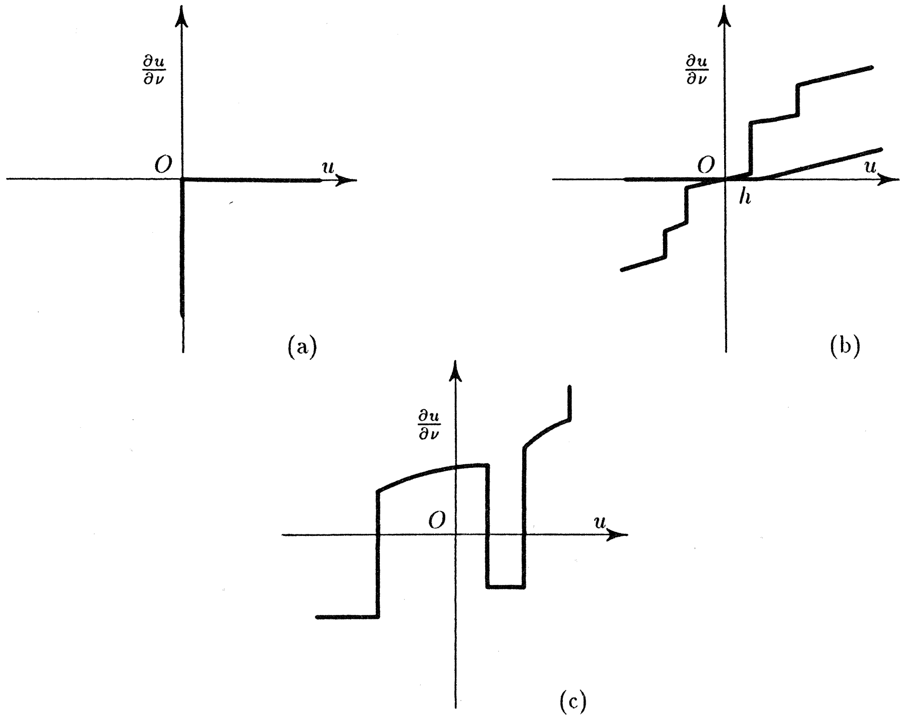

Fig. 1. Monotone and nonmonotone semipermeability relations

most general case of semipermeability and potential temperature or pressure control conditions.

1. Setting of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\bar{\Gamma} = \bar{\Gamma}_u \cup \bar{\Gamma}_c$, where Γ_u and Γ_c are nonempty, nonoverlapping parts of Γ. We shall consider the following unilateral boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_u \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} \geqslant 0, \quad u \geqslant 0, \quad \frac{\partial u}{\partial \nu} \cdot u = 0 \quad \text{on } \Gamma_c. \tag{1.3}$$

In order to give the variational form of the unilateral B.V.P. (1.1)–(1.3), the following functional sets are first introduced:

$$V(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \quad \text{on } \Gamma_u\} \tag{1.4}$$

$$K(\Omega) = \{v \in V(\Omega) \mid v \geqslant 0 \quad \text{on } \Gamma_c\}. \tag{1.5}$$

By $H^k(\Omega)$ ($k \geqslant 0$ integer) we denote the usual Sobolev space of functions which are square integrable over Ω together with their generalized derivatives up to the order k. The norm (seminorm) in $H^k(\Omega)$ will be denoted by $\|.\|_k$ ($|.|_k$).

Then the variational form of the B.V.P. (1.1)–(1.3) reads as follows:

$$\begin{gathered} \text{Find } u \in K(\Omega) \text{ such that} \\ S(u) \leqslant S(v) \quad \forall v \in K(\Omega), \end{gathered} \tag{1.6}$$

where

$$S(v) = \frac{1}{2}|v|_1^2 - (f, v)_0, \tag{1.7}$$

or equivalently:

$$\begin{gathered} \text{Find } u \in K(\Omega) \text{ such that} \\ \left(\nabla u,\ \nabla(v-u)\right)_0 \geqslant (f, v-u)_0 \quad \forall v \in K(\Omega). \end{gathered} \tag{1.8}$$

In the above problems $f \in L^2(\Omega)$, $\nabla u \equiv \operatorname{grad} u$ and the symbol $(.,.)_0$ stands for $L^2(\Omega)$ or $\left(L^2(\Omega)\right)^2$-scalar product. In practice we often meet situations that the knowledge of the gradient of u is more important than the solution u itself. To this end, another variational formulation for the problem (1.6) is more convenient, [7]. Let

$$U_f^+(\Omega) = \left\{q \in [L^2(\Omega)]^2 \mid (q, \nabla v)_0 \geqslant (f, v)_0 \quad \forall v \in K(\Omega)\right\} \tag{1.9}$$

and

$$J(q) = \frac{1}{2}\|q\|^2_{[L^2(\Omega)]^2} \equiv \frac{1}{2}\|q\|_0^2 \quad \forall q \in \left[L^2(\Omega)\right]^2. \tag{1.10}$$

By expressing the B.V.P. (1.6) in terms of the gradient, the following problem is obtained:

$$\begin{gathered} \text{Find } \lambda \in U_f^+(\Omega) \text{ such that} \\ J(\lambda) \leqslant J(q) \quad \forall q \in U_f^+(\Omega). \end{gathered} \tag{1.11}$$

The relation between the solution of problems (1.6) and (1.11) is given by the following proposition [7].

Proposition 1.1. *There exists a unique λ solving problem* (1.11). *Moreover,*

$$\lambda = \nabla u,$$

where $u \in K(\Omega)$ solves problem (1.6).

Remark 1.1. Using Green's formula in the definition of $U_f^+(\Omega)$, we obtain that

$$q \in U_f^+(\Omega) \iff \begin{cases} \operatorname{div} q + f = 0 & \text{in } \Omega \\ q \cdot \nu \geqslant 0 & \text{on } \Gamma_c. \end{cases} \tag{1.12}$$

Let

$$U_f(\Omega) = \left\{ q \in \left[L^2(\Omega)\right]^2 \mid (q, \nabla v)_0 = (f, v)_0 \quad \forall v \in H_0^1(\Omega) \right\}, \tag{1.13}$$

i.e. $q \in U_f(\Omega)$ if and only if $\operatorname{div} q + f = 0$ in Ω.

Comparing the definition of $U_f^+(\Omega)$ and $U_f(\Omega)$, we see that

$$q \in U_f^+(\Omega) \iff q \in U_f(\Omega) \quad \text{and} \quad q \cdot \nu \geqslant 0 \quad \text{on } \Gamma_c. \tag{1.14}$$

Let us denote by $H^{\frac{1}{2}}(\Gamma_c)$ the space of traces of all functions $v \in V(\Omega)$ and by $H^{-\frac{1}{2}}(\Gamma_c)$ the corresponding dual space over $H^{\frac{1}{2}}(\Gamma_c)$. The duality pairing between $H^{-\frac{1}{2}}(\Gamma_c)$ and $H^{\frac{1}{2}}(\Gamma_c)$ will be denoted by the symbol $\langle .,. \rangle$. The following result is evident:

Lemma 1.1. *Let $q \in U_f(\Omega)$. Then*

$$\sup_{v \in K(\Omega)} \left\{ -\langle q \cdot \nu, v \rangle \right\} = \begin{cases} 0 & \text{if } q \in U_f^+(\Omega) \\ +\infty & \text{elsewhere,} \end{cases} \tag{1.15}$$

i.e. (1.15) *is the indicator function of $U_f^+(\Omega)$.*

Using Lemma 1.1, problem (1.11) can be written in the following equivalent form:

$$\inf_{q \in U_f^+(\Omega)} J(q) = \inf_{q \in U_f(\Omega)} \sup_{v \in K(\Omega)} L(q, v), \tag{1.16}$$

where $L(q, v) = \frac{1}{2}\|q\|_0^2 - \langle q \cdot \nu, v \rangle$ is the Lagrange function define on $U_f(\Omega) \times K(\Omega)$.

On the basis of Proposition 1.1, one can easily prove the following proposition [7]:

Proposition 1.2. *There exists a unique saddle-point* (λ^*, u^*) *of* L *on* $U_f(\Omega) \times K(\Omega)$ *and*

$$\lambda^* = \nabla u \quad \textit{in } \Omega \tag{1.17}$$
$$u^* = u \quad \textit{in } \Omega,$$

where $u \in K(\Omega)$ *solves problem* (1.6). *Moreover,*

$$L(\lambda^*, u^*) = \min_{q \in U_f(\Omega)} \sup_{v \in K(\Omega)} L(q, v) = \max_{v \in K(\Omega)} \inf_{q \in U_f(\Omega)} L(q, v). \tag{1.18}$$

The corresponding min, max *are attained for* λ^*, u^*, *respectively.*

2. Formulation of a minimum problem on the boundary

Let $v \in K(\Omega)$ be given and let us introduce the notation

$$\tilde{\Phi}(v) = \inf_{q \in U_f(\Omega)} L(q, v). \tag{2.1}$$

Then the problem

$$\text{Find } u^* \in K(\Omega) \text{ such that}$$
$$\tilde{\Phi}(u^*) \leqslant \tilde{\Phi}(v) \quad v \in K(\Omega) \tag{2.2}$$

will be called the dual problem to (1.11).

We now derive the explicit form of $\tilde{\Phi}$.

Let $\lambda(v) \in U_f(\Omega)$ be the solution of the following problem (2.3), where $v \in K(\Omega)$:

$$L(\lambda(v), v) \leqslant L(q, v) \quad \forall q \in U_f(\Omega). \tag{2.3}$$

It is well-known that problem (2.3) has a unique solution $\lambda(v) \in U_f(\Omega)$ for any $v \in K(\Omega)$. Moreover,

$$\lambda(v) = \nabla w(v), \tag{2.4}$$

where $w(v)$ is the unique solution of the mixed boundary value problem

$$-\Delta w(v) = f \quad \text{in } \Omega \tag{2.5}$$
$$w(v) = 0 \quad \text{on } \Gamma_u \tag{2.6}$$
$$w(v) = v \quad \text{on } \Gamma_c. \tag{2.7}$$

We can split $U_f(\Omega)$ and write

$$U_f(\Omega) = q_0 + U_0(\Omega) \tag{2.8}$$

where $q_0 \in U_f(\Omega)$ is an arbitrary element and

$$U_0(\Omega) = \{q \in [L^2(\Omega)^2] \mid (q, \nabla v)_0 = 0 \quad \forall v \in H_0^1(\Omega)\}. \tag{2.9}$$

In other words

$$q \in U_0(\Omega) \iff \operatorname{div} q = 0 \quad \text{in } \Omega. \tag{2.10}$$

Now

$$\inf_{q \in U_f(\Omega)} L(q, v) = \inf_{q \in U_0(\Omega)} L(q + q_0, v) = \inf_{q \in U_0(\Omega)} H(q) + \frac{1}{2}\|q_0\|_0^2 - \langle q_0 \cdot \nu, v\rangle \tag{2.11}$$

with

$$H(q) = \frac{1}{2}\|q\|_0^2 + (q, q_0)_0 - \langle q \cdot \nu, v\rangle, \quad q \in U_0(\Omega). \tag{2.12}$$

Let $q(v) \in U_0(\Omega)$ be the solution of the problem

$$\begin{aligned} &\text{Find } q(v) \in U_0(\Omega) \text{ such that} \\ &H\big(q(v)\big) \leqslant H(q) \quad \forall q \in U_0(\Omega), \end{aligned} \tag{2.13}$$

or of its equivalent form

$$\begin{aligned} &\text{Find } q(v) \in U_0(\Omega) \text{ such that} \\ &\big(q(v), q\big)_0 + (q_0, q)_0 = \langle q \cdot \nu, v\rangle \quad \forall q \in U_0(\Omega). \end{aligned} \tag{2.14}$$

For the sake of simplicity it is assumed that $q_0 = \nabla \overline{w}$, where $\overline{w}$ is the solution of the homogeneous Dirichlet boundary value problem

$$\begin{aligned} -\Delta \overline{w} &= f \quad \text{in } \Omega \\ \overline{w} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.15}$$

Then

$$(q_0, q)_0 = (\nabla \overline{w}, q)_0 = -(\overline{w}, \operatorname{div} q)_0 + \langle q \cdot \nu, \overline{w}\rangle = 0 \tag{2.17}$$

by virtue of the definition of $\overline{w}$ and $U_0(\Omega)$. With such a choice of q_0, one gets

$$H(q) = \frac{1}{2}\|q\|_0^2 - \langle q \cdot \nu, v \rangle \,. \tag{2.18}$$

The solution $q(v)$ of (2.13) is equal to the gradient of a function $z(v)$ solving the problem

$$\Delta z(v) = 0 \quad \text{in } \Omega \tag{2.19}$$

$$z(v) = 0 \quad \text{on } \Gamma_u \tag{2.20}$$

$$z(v) = v \quad \text{on } \Gamma_c \tag{2.21}$$

(see also (2.5), (2.7)). It is known that

$$H\left(q(v)\right) = \inf_{q \in U_0(\Omega)} H(q) = -\frac{1}{2} \langle q(v) \cdot \nu, v \rangle = -\frac{1}{2}\|q(v)\|_0^2. \tag{2.22}$$

Let $G\colon H^{\frac{1}{2}}(\Gamma_c) \mapsto U_0(\Omega)$ be the linear continuous mapping defined by

$$G(v) = q(v) \quad \forall v \in H^{\frac{1}{2}}(\Gamma_c), \tag{2.23}$$

where $q(v) = \nabla z(v)$ and $z(v)$ solves problem (2.19)–(2.21).

In this way we are led to the following expression for $\tilde{\Phi}(v)$:

$$\begin{aligned} \tilde{\Phi}(v) &= \inf_{q \in U_f(\Omega)} L(q, v) = \inf_{q \in U_0(\Omega)} L(q + q_0, v) \\ &= -\frac{1}{2} \langle G(v) \cdot \nu, v \rangle - \langle q_0 \cdot \nu, v \rangle + \frac{1}{2}\|q_0\|_0^2. \end{aligned} \tag{2.25}$$

Next, the term $\|q_0\|_0^2$ is omitted and thus the following expression is obtained:

$$\max_{v \in K(\Omega)} \tilde{\Phi}(v) = - \min_{v \in K(\Omega)} \left\{ \frac{1}{2} \langle G(v) \cdot \nu, v \rangle + \langle q_0 \cdot \nu, v \rangle \right\} = - \min_{v \in K(\Omega)} \Phi(v), \tag{2.26}$$

where

$$\Phi(v) \equiv \frac{1}{2}\gamma(v, v) - \delta(v) \tag{2.27}$$

with $\gamma(v, v) \equiv \langle G(v) \cdot \nu, v \rangle$ and $\delta(v) \equiv -\langle q_0 \cdot \nu, v \rangle$.

Denote by $H_+^{\frac{1}{2}}(\Gamma_c)$ the subset of $H^{\frac{1}{2}}(\Gamma_c)$ containing only traces of functions belonging to $K(\Omega)$. Clearly

$$\min_{v \in K(\Omega)} \Phi(v) = \min_{v \in H_+^{\frac{1}{2}}(\Gamma_c)} \Phi(v). \tag{2.28}$$

Note that the symbol v on the right-hand side of (2.28) denotes the trace of $v \in K(\Omega)$ appearing on the left-hand side.

Now we consider the following boundary minimization problem:

$$\text{(2.29)} \qquad \begin{aligned} &\text{Find } u^* \in H_+^{\frac{1}{2}}(\Gamma_c) \text{ such that} \\ &\Phi(u^*) = \min_{v \in H_+^{\frac{1}{2}}(\Gamma_c)} \Phi(v). \end{aligned}$$

As has been already mentioned (see Proposition 1.2), problem (2.29) has a unique solution u^*. Moreover,

$$\text{(2.30)} \qquad u^* = u \quad \text{on } \Gamma_c,$$

where $u \in K(\Omega)$ is the solution of problem (2.29). However, the uniqueness of u^*, follows from the $H_+^{\frac{1}{2}}(\Gamma_c)$-ellipticity of the quadratic form γ, which will be established in the sequel:

It is known [3] that $H_+^{\frac{1}{2}}(\Gamma_c)$ can be equipped with the norm

$$\text{(2.31)} \qquad \|\varphi\|_{\frac{1}{2},\Gamma_c} = \inf_{\substack{v \in V(\Omega) \\ v = \varphi \text{ on } \Gamma_c}} |v|_1 = |\bar{u}|_1,$$

where $\bar{u}$ is a solution of the B.V.P.

$$\text{(2.32)} \qquad \Delta \bar{u} = 0 \quad \text{in } \Omega$$

$$\text{(2.33)} \qquad \bar{u} = 0 \quad \text{on } \Gamma_u$$

$$\text{(2.34)} \qquad \bar{u} = \varphi \quad \text{on } \Gamma_c.$$

The relation (2.22) leads to

$$\text{(2.35)} \qquad -H(q(v)) = \frac{1}{2}\|q(v)\|_0^2 = \frac{1}{2}\|\nabla z(v)\|_0^2 = \frac{1}{2}|z(v)|_1^2,$$

where $z(v)$ solves problem (2.19)–(2.22). Comparing problem (2.19)–(2.21) to (2.32)–(2.34) and applying the definition of $\| \cdot \|_{\frac{1}{2},\Gamma_c}$, we see that

$$\text{(2.36)} \qquad -H(q(v)) = \frac{1}{2}|z(v)|_1^2 = \frac{1}{2}\|v\|_{\frac{1}{2},\Gamma_c}^2.$$

Thus $\Phi(v)$ can be written in the form

$$\text{(2.37)} \qquad \Phi(v) = \frac{1}{2}\|v\|_{\frac{1}{2},\Gamma_c}^2 - \delta(v), \quad v \in H_+^{\frac{1}{2}}(\Gamma_c).$$

Before passing to study the discretization of problem (2.3), let us make some remarks concerning the norms in $H^{\frac{1}{2}}(\Gamma_c)$, $H^{-\frac{1}{2}}(\Gamma_c)$.

Let $\|\cdot\|_{\frac{1}{2},\Gamma_c}$ be given by (2.31). Then for $\mu \in H^{-\frac{1}{2}}(\Gamma_c)$, the dual norm is given by

$$\|\mu\|_{-\frac{1}{2},\Gamma_c} = \sup_{\substack{\varphi \in H^{\frac{1}{2}}(\Gamma_c) \\ \varphi \neq 0}} \frac{\langle \mu, \varphi \rangle}{\|\varphi\|_{\frac{1}{2},\Gamma_c}}. \tag{2.38}$$

It can be shown that

$$\|\mu\|_{-\frac{1}{2},\Gamma_c} = \sup_{\substack{q \in U_0(\Omega) \\ q \cdot \nu = \mu}} \|q\|_0 = \|\bar{q}\|_0 = |u(\mu)|_1, \tag{2.39}$$

where $u(\mu)$ is the unique solution of the problem

$$\Delta u(\mu) = 0 \quad \text{in } \Omega \tag{2.40}$$

$$u(\mu) = 0 \quad \text{on } \Gamma_u \tag{2.41}$$

$$\frac{\partial u(\mu)}{\partial \nu} = \mu \quad \text{on } \Gamma_c \tag{2.42}$$

and $\bar{q} = \nabla u(\mu)$.

Lemma 2.1. *Let $\varphi \in H^{\frac{1}{2}}(\Gamma_c)$. Then*

$$\sup_{\substack{q \in U_0(\Omega) \\ q \neq 0}} \frac{\langle q \cdot \nu, \varphi \rangle}{\|q\|_0} = \|q(\varphi)\|_0 \tag{2.43}$$

where $q(\varphi) \in U_0(\Omega)$ is a solution of the problem

$$\left(q(\varphi), q\right)_0 = \langle q \cdot \nu, \varphi \rangle \quad \forall q \in U_0(\Omega). \tag{2.44}$$

Proof. By the definition

$$\|q(\varphi)\|_0 = \sup_{\substack{q \in U_0(\Omega) \\ q \neq 0}} \frac{\left(q(\varphi), q\right)_0}{\|q\|_0} = \sup_{\substack{q \in U_0(\Omega) \\ q \neq 0}} \frac{\langle q \cdot \nu, \varphi \rangle}{\|q\|_0}. \tag{2.45}$$

□

Remark 2.1. We have $q(\varphi) = \nabla u(\varphi)$, where $u(\varphi)$ is the solution of the problem

$$\Delta u(\varphi) = 0 \quad \text{in } \Omega \tag{2.46}$$

$$u(\varphi) = 0 \quad \text{on } \Gamma_u \tag{2.47}$$

$$u(\varphi) = \varphi \quad \text{on } \Gamma_c. \tag{2.48}$$

Thus

$$\|q(\varphi)\|_0 = |u(\varphi)|_1 = \|\varphi\|_{\frac{1}{2},\Gamma_c}. \tag{2.49}$$

3. The general subdifferential boundary condition. Convex and nonconvex case.

In this section we assume that instead of the semipermeability boundary conditions (1.3) a more general boundary condition of the form

$$q_i \nu_i \in \partial j(u) \quad \text{on } \Gamma_c \tag{3.1}$$

holds, where j is a convex lower semicontinuous (l.s.c.) functional on $\mathbb{R}$ taking values in $(-\infty, +\infty]$, $j \not\equiv \infty$, $q = \{q_i\}$ is the flux vector and $\nu = \{\nu_j\}$ is the outward unit normal to Γ_c. Here j is a convex superpotential [9]. We refer to [2], Ch. 3 for the mechanical meaning of (3.1). Note only that the graph $(z_i \nu_i, u)$ in (3.1) is a monotone graph containing finite or infinite complete jumps. According to [4] we are led to the following problem:

$$\begin{aligned} &\text{Find } u^* \in H^{\frac{1}{2}}(\Gamma_c) \text{ such that} \\ &\tilde{\Phi}(u^*) = \inf\left\{\tilde{\Phi}(v) \mid v \in H^{\frac{1}{2}}(\Gamma_c)\right\} \end{aligned} \tag{3.2}$$

where

$$\tilde{\Phi}(v) = \frac{1}{2}\gamma(v, v) - \delta(v) + \mathcal{J}(v) \tag{3.3}$$

with

$$\mathcal{J}(v) = \begin{cases} \int_{\Gamma_c} j(v)\, \mathrm{d}\Gamma & \text{if } j(v) \in L^1(\Gamma_c) \\ \infty & \text{otherwise.} \end{cases} \tag{3.4}$$

Note that $\mathcal{J}$ is a convex l.s.c. functional on $H^{\frac{1}{2}}(\Gamma_c)$ with values in $(\infty, +\infty]$, $\mathcal{J}(v) \not\equiv \infty$. Due to the symmetry of $\gamma(.,.)$, the following proposition can be proved:

Proposition 3.1. *Problem* (3.2) *is equivalent to the variational inequality*

(3.5) *Find* $u \in H^{\frac{1}{2}}(\Gamma_c) \quad \gamma(u, v-u) + \mathcal{J}(v) - \mathcal{J}(u) - \delta(v-u) \geqslant 0 \quad \forall v \in H^{\frac{1}{2}}(\Gamma_c)$

and to the multivalued B.I.E.

$$(3.6) \qquad -\delta + \operatorname{grad} \gamma(u, u) \in \partial \mathcal{J}(u).$$

P r o o f. The proof of (3.5) is a direct application of Prop. 2.1 of [10], p. 37. The proof of (3.6) results from the definition of the subdifferential ∂. □

Proposition 3.2. *Problem* (3.2) *admits a unique solution.*

P r o o f. According to the Hahn-Banach theorem we may write

$$(3.7) \qquad \mathcal{J}(v) \geqslant -(c_1 \|v\|_{-\frac{1}{2},\Gamma_c} + c_2) \quad \forall v \in H^{\frac{1}{2}}(\Gamma_c) \quad c_1,\ c_2 \geqslant 0.$$

Thus we get that

$$(3.8) \qquad \tilde{\Phi}(v) \geqslant \frac{1}{2}\|v\|^2_{-\frac{1}{2},\Gamma_c} - c_1'\|v\|_{-\frac{1}{2},\Gamma_c} + c_2' \quad c_1',\ c_2' \text{ constant}$$

and thus

$$(3.9) \qquad \tilde{\Phi}(v) \to \infty \quad \text{for } \|v\|_{-\frac{1}{2},\Gamma_c} \to \infty.$$

By using Prop. 1.2 of [10], p. 35, we obtain the existence of the solution. The strict convexity of $\gamma(.,.)$ guarantees the uniqueness of the solution. □

Further let us assume that instead of (3.1) we have a nonmonotone boundary condition

$$(3.10) \qquad q_i \nu_i \in \bar{\partial} \tilde{j}(u) \quad \text{on } \Gamma_c.$$

Here $\bar{\partial}$ denotes the generalized gradient of Clarke [11] and $\tilde{j}$ is a locally Lipschitz function. The law (3.10) describes nonmonotone semipermeability conditions (Fig. 1c) (see also [12]). We assume that $\tilde{j}$ is obtained as follows:

Let $b \in l^{\infty}_{\mathrm{loc}}(\mathbb{R})$ and for any $\mu > 0$ and $\xi \in \mathbb{R}$ let us define functions

$$(3.11) \qquad \underline{b}_\mu(\xi) = \operatorname{ess} \inf_{|\xi-\xi_1|<\mu} \underline{b}(\xi_1), \quad \bar{b}_\mu(\xi) = \operatorname{ess} \sup_{|\xi-\xi_1|<\mu} \bar{b}(\xi_1)$$

from which, due to their monotonicity, we can get functions

$$\underline{b}(\xi) = \lim_{\mu \to 0_+} \underline{b}_\mu(\xi), \quad \bar{b}(\xi) = \lim_{\mu \to 0_+} \bar{b}_\mu(\xi). \tag{3.12}$$

Now the multivalued function

$$\xi \to \hat{b}(\xi) = [\underline{b}(\xi), \bar{b}(\xi)] \tag{3.13}$$

is defined. The graph $\{\xi, \hat{b}(\xi\}$ results from $\{\xi, b(\xi\}$ by completing it at the points of discontinuity of b with vertical segments. From b a locally Lipschitz function $\bar{j}$ is defined [13] by the relation

$$\bar{j}(\xi) = \int_0^\xi b(t)\, dt \tag{3.14}$$

up to an additive constant, such that $\bar{\partial}\bar{j}(\xi) \subseteq \hat{b}(\xi)$. If, moreover, $b(\xi\pm)$ exists for every $\xi \in \mathbb{R}$, we can write that

$$\partial(\xi) = \bar{\partial}\bar{j}(\xi) \quad \forall \xi \in \mathbb{R}. \tag{3.15}$$

The law (3.10) is a nonconvex superpotential law. This type of nonmonotone mechanical laws and boundary conditions has been introduced by the third author in [14] and leads to a new type of variational expressions, the so-called hemivariational inequalities [15], [16]. We are now led to the following problem (cf. e.g. [6]):

(3.16) Find $u \in H^{\frac{1}{2}}(\Gamma_c)$ such that

$$\gamma(u, v-u) + \int_{\Gamma_c} \bar{j}^0(u, v-u)\, d\Gamma \geqslant \delta(v-u) \quad \forall v \in H^{\frac{1}{2}}(\Gamma_c).$$

Here $\bar{j}^0(.,.)$ denotes the directional derivative of Clarke. We can prove the following proposition:

Proposition 3.3. *Let* $\operatorname{mes}\Gamma_u > 0$. *Then problem* (3.16) *has at least one solution if there* $\exists \bar{\xi} > 0$ *such that*

$$\operatorname*{ess\,sup}_{(-\infty,-\bar{\xi})} b(\xi) \leqslant \operatorname*{ess\,inf}_{(\bar{\xi},\infty)} b(\xi). \tag{3.17}$$

Proof. The proof follows the general pattern used for the existence proof for coercive hemivariational inequalities (cf. e.g. [2] Ch. 8, [15] Ch. 2, [12], [16], [17]). First, let $p \in \mathscr{D}(-1,+1)$, $p \geqslant 0$, with $\int_{-\infty}^{+\infty} p(\xi)\, d\xi = 1$ be a mollifier and let

$$b_\varepsilon = p_\varepsilon * b \quad \text{where} \quad p_\varepsilon(\xi) = \frac{1}{\varepsilon} p\left(\frac{\xi}{\varepsilon}\right). \tag{3.18}$$

Let us also consider a Galerkin basis of $H^{\frac{1}{2}}(\bar{\Gamma})\,|_{\Gamma_c} \cap L^\infty(\Gamma_c)$ and let V_n denote the corresponding n-dimensional subspace. Now we consider the finite dimensional regularized problem

$$\text{(3.19)} \qquad \begin{gathered}\text{Find } u_{\varepsilon n} \in V_n \text{ such that} \\ \gamma(u_{\varepsilon n}, v) + \int_{\Gamma_c} b_\varepsilon(u_{\varepsilon n}) v \, d\Gamma = \delta(v) \quad \forall v \in V_n.\end{gathered}$$

Due to (3.17) we obtain the estimate (cf. e.g. [15], p. 86)

$$\text{(3.20)} \qquad \int_{\Gamma_c} b_\varepsilon(u_{\varepsilon n}) u_{\varepsilon n} \, d\Gamma \geqslant -\varrho_1 \varrho_2 \operatorname{mes} \Gamma_c, \quad \varrho_1 > 0,\ \varrho_2 > 0$$

and applying Brouwer's fixed point theorem we obtain due to the coercivity of $\gamma(.,.)$ that $\|u_{\varepsilon n}\|_{\frac{1}{2},\Gamma_c} < c$. Thus a subsequence can be determined such that

$$\text{(3.21)} \qquad u_{\varepsilon n} \to u \quad \text{weakly in} \quad H^{\frac{1}{2}}(\bar{\Gamma})\,|_{\Gamma_c}.$$

Because of the compact imbedding $H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$, we have

$$\text{(3.22)} \qquad u_{\varepsilon n} \to u \quad \text{strongly in} \quad L^2(\Gamma_c)$$

and thus

$$\text{(3.23)} \qquad u_{\varepsilon n} \to u \quad \text{a.e. on} \quad \Gamma_c.$$

Further we show (cf. e.g. [15], p. 87 or [12], [17]) that $\{b_\varepsilon(u_{\varepsilon n})\}$ is weakly precompact in $L^1(\Gamma_c)$ and thus a subsequence can be determined such that

$$\text{(3.24)} \qquad b_\varepsilon(u_{\varepsilon n}) \to \chi \quad \text{weakly in} \quad L^1(\Gamma_c).$$

Since $V_n \subset H^{\frac{1}{2}}(\Gamma)\big|_{\Gamma_c} \cap L^\infty(\Gamma_c)$ which is dense in $H^{\frac{1}{2}}(\bar{\Gamma})\big|_{\Gamma_c}$ for the $H^{\frac{1}{2}}$-norm, we may pass to the limit $\varepsilon \to 0$, $n \to \infty$ in (3.19) thus obtaining

$$\text{(3.25)} \qquad \gamma(u, v) + \int_{\Gamma_c} \chi v \, d\Gamma = \delta(v) \quad \forall v \in H^{\frac{1}{2}}(\bar{\Gamma})\big|_{\Gamma_c}.$$

From (3.23) we see that we can apply Egoroff's theorem, i.e. for any $\alpha > 0$ there is a $\omega \subset \Gamma_c$ with $\operatorname{mes}\omega < \alpha$ such that

$$\text{(3.26)} \qquad u_{\varepsilon n} \to u \quad \text{uniformly on} \quad \Gamma_c - \omega$$

with $u \in L^\infty(\Gamma_c - \omega)$. Exactly as in [15], p. 89 (or as in [12], [17], or in [16], p. 148) we show that

$$\text{(3.27)} \qquad \chi \in [\underline{b}(u), \bar{b}(u)] = \hat{b}(u) \subseteq \bar{\partial}\tilde{j}(u) \quad \text{a.e. on } \Gamma_c$$

and the proof is complete. □

First we treat problem (1.1)–(1.3) in the plane. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $\{\mathcal{T}_h\}$, $h \to 0_+$ a regular family of triangulations of $\bar{\Omega}$. With any $\mathcal{T}_h$ we associate the following sets, constructed by means of finite elements:

$$V_h(\Omega) = \{v_h \in C(\bar{\Omega}) \mid v_h|_{T_i} \in P_1(T_1) \quad \forall T_i \in \mathcal{T}_h,\ v_h = 0 \text{ on } \Gamma_u\} \tag{4.1}$$

$$K_h(\Omega) = \{v_h \in V_h(\Omega) \mid v_h \geqslant 0 \text{ on } \Gamma_c\} \tag{4.2}$$

and

$$U_{0h}(\Omega) = \Big\{q_h \in [L^2(\Omega)]^2 \mid q_h|_{T_i} \in [P_1(T_i)]^2 \quad \forall T_i \in \mathcal{T}_h, \\ \operatorname{div} q_h = 0 \text{ (in the sense of distributions) in } \Omega\Big\}. \tag{4.3}$$

The construction of $V_h(\Omega)$, $K_h(\Omega)$ is classical. Let us briefly describe the construction of $U_{0h}(\Omega)$.

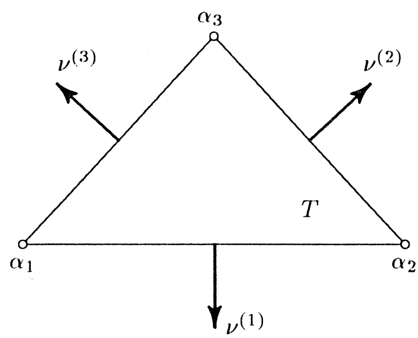

Fig. 2. Concerning the definition of triangle T.

Let T be a non-degenerate triangle with vertices α_1, α_2, α_3 and set $\alpha_4 = \alpha_1$. The length of the side $\overline{\alpha_j\alpha_{j+1}}$ will be denoted by l_j and the corresponding outward unit normal vector by $\nu^{(i)}$ (see Fig. 2).

Denote

$$M(T) = \Big\{q_h^{(T)} \in [P_1(T)]^2 \mid \operatorname{div} q_h^{(T)} = 0 \quad \text{in } T\Big\}. \tag{4.4}$$

Then it is easy to see that (cf. [18])

$$q_h^{(T)} \in M(T) \iff q_h^{(T)} \in [P_1(T)]^2 \ \&\ \sum_{i=1}^{3}(\alpha_i^{(T)} + \beta_i^{(T)})l_i = 0 \tag{4.5}$$

where

$$\alpha_i^{(T)} = q_h^{(T)}(\alpha_i) \cdot \nu^{(i)}, \quad \beta_i^{(T)} = q_h^{(T)}(\alpha_{i+1}) \cdot \nu^{(i)} \tag{4.6}$$

are values of the corresponding fluxes at the indicated nodes. If T' is an adjacent triangle to T with a common side $\overline{\alpha_j\alpha_{j+1}}$, then a function q_h defined on $T \cup T'$ through the relation

$$q_h = \begin{cases} q_h^{(T)} \text{ on } T & \operatorname{div} q_h^{(T)} = 0 \\ q_h^{(T')} \text{ on } T' & \operatorname{div} q_h^{(T')} = 0 \end{cases} \tag{4.7}$$

satisfies $\operatorname{div} q_h = 0$ on $T \cup T'$ if and only if the continuity of fluxes across $\overline{\alpha_j\alpha_{j+1}}$ is guaranteed, i.e.

$$\alpha_i^{(T)} + \alpha_i^{(T')} = 0, \quad \beta_i^{(T)} + \beta_i^{(T')} = 0. \tag{4.8}$$

From this the construction of $U_{0h}(\Omega)$ follows.

Let

$$N(T) = \left\{ q \in \left[L^2(T)\right]^2 \mid \operatorname{div} q = 0 \quad \text{in } T,\ q \cdot \nu \in H^{-\frac{1}{2}+\varepsilon}(\partial T) \right\} \quad {}^{*)} \tag{4.9}$$

for some positive ε.

If $q \in N(T)$, one can construct a mapping $\Pi_T \in L\big(N(T), M(T)\big)$ (and write $\Pi_T q = q_h^{(T)}$) be means of the equations

$$\begin{gathered} [q \cdot \nu,\ \tilde{\lambda}_j^{(i)}] = \alpha_i[\lambda_1^{(i)}, \lambda_j^{(i)}]_i + \beta_i[\lambda_2^{(i)}, \lambda_j^{(i)}]_i, \quad j = 1, 2 \\ q_h^{(T)}(\alpha_i) \cdot \nu^{(i)} = \alpha_i,\ q_h^{(T)}(\alpha_{i+1}) \cdot \nu^{(i)} = \beta_i \end{gathered} \tag{4.10}$$

where $[.\,,.]_i$ is the $L^2(\overline{\alpha_i\alpha_{i+1}})$-scalar product, $\lambda_j^{(i)}$, $j = 1, 2$, are linear basic functions on $\overline{\alpha_i\alpha_{i+1}}$, i.e.

$$\lambda_1^{(i)}(\alpha_i) = 1, \quad \lambda_1^{(i)}(\alpha_{j+1}) = 0 \tag{4.11}$$

$$\lambda_2^{(i)}(\alpha_i) = 0, \quad \lambda_2^{(i)}(\alpha_{j+1}) = 1 \tag{4.12}$$

and $\tilde{\lambda}_j^{(i)}$ are extensions of $\lambda_j^{(i)}$ by zero from $\overline{\alpha_i\alpha_{i+1}}$ to the whole ∂T. Let

$$N(\Omega) = \{ q \,|q|_T \in N(T) \quad \forall T \in \mathcal{T}_h \}. \tag{4.13}$$

---

*) $H^{-\frac{1}{2}+\varepsilon}(\partial T)$ denotes the dual space to $H^{\frac{1}{2}-\varepsilon}(\partial T)$ (for the definition see [19], [20]; the corresponding duality pairing will be denoted by [ ]).

Then one can define the "global projector" $\tau_h \in L(N(\Omega), U_{0h}(\Omega))$ by means of the local mappings

$$(\tau_h q)\big|_T = \Pi_T(q\big|_T) \quad \forall q \in N(\Omega) \quad \forall T \in \mathcal{T}_h. \tag{4.14}$$

Approximate properties of τ_h are studied in [18], [21].

Finally, let us define

$$U_{f,h}(\Omega) = q_{0h} + U_{0h}(\Omega) \tag{4.15}$$

and

$$U^+_{f,h}(\Omega) = q_{0h} + U^+_{0h}(\Omega). \tag{4.16}$$

The choice of q_{0h} will be specified later and

$$U^+_{0h}(\Omega) = \{q_h \in U_{0h}(\Omega) \mid \langle q_h \cdot \nu, v_h\rangle \geqslant 0 \quad \forall v_h \in K_h(\Omega)\}. \tag{4.17}$$

We mention here that the corresponding duality is realized by the $L^2(\Gamma)$ scalar product. Now we are able to define the finite element approximations of problems (1.6) and (1.11) respectively as follows:

$$\begin{aligned} &\text{Find } u_h \in K_h(\Omega) \text{ such that} \\ &S(u_h) \leqslant S(v_h) \quad \forall v_h \in K_h(\Omega) \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\text{Find } \lambda_h \in U^+_{f,h}(\Omega) \text{ such that} \\ &J(\lambda_h) \leqslant J(q_h) \quad \forall q_h \in U^+_{f,h}(\Omega). \end{aligned} \tag{4.19}$$

It is easy to verify that both problems (4.18) and (4.19) have unique solutions u_h, λ_h, respectively.

Adopting the same approach as in the continuous case, one can write

$$\inf_{q \in U^+_{f,h}(\Omega)} J(q_h) = \inf_{q_h \in U_{f,h}(\Omega)} \sup_{v_h \in K_h(\Omega)} L(q_h, v_h), \tag{4.20}$$

where

$$L(q_h, v_h) = \frac{1}{2}\|q_h\|_0^2 - \langle q_h \cdot \nu, v_h\rangle\,. \tag{4.21}$$

Theorem 4.1. *There exists a unique saddle-point* (λ_h^*, u_h^*) *of* L *on* $U_{f,h}(\Omega) \times K_h(\Omega)$.

Proof. As

$$L(q_h, v_h) \to \infty \text{ if } \|q_h\|_0 \to \infty \tag{4.22}$$

for any $v_h \in K_h(\Omega)$ and $L(q_h, v_h)$ as a function of q_h is strictly convex, the existence and uniqueness of (λ_h^*, u_h^*) will be consequences of the following implication (see [22]):

$$\langle q_h \cdot \nu, v_h \rangle = 0 \quad \forall q_h \in U_{0h}(\Omega) \iff v_h = 0 \text{ on } \Gamma. \tag{4.23}$$

So we shall verify (4.23). Let α_i, α_{i+1} be nodes of a boundary triangle $T \in \mathcal{T}_h$ (see Fig. 3).

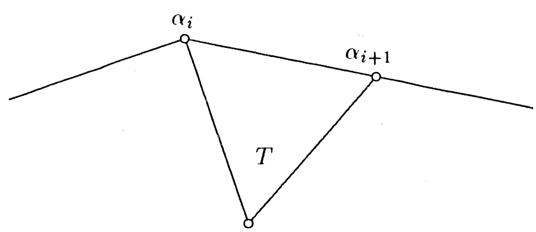

Fig. 3.

By $q_h^{(i)}$ we denote a function belonging to $U_{0h}(\Omega)$ the boundary fluxes of which at α_i, a_{i+1} are equal to 1, -1, respectively and in all the other nodes are equal to zero. Then, in particular,

$$[q_h^{(i)} \cdot \nu, v_h]_i = 0 \implies v_h(\alpha_i) = v_h(\alpha_{i+1}), \tag{4.24}$$

i.e. v_h is a constant on Γ_c. As on the other hand $v_h = 0$ on Γ_u, we obtain that $v_h \equiv 0$ on Γ. Obviously

$$L(\lambda_h^*, u_h^*) = \min_{q_h \in U_{f,h}(\Omega)} \sup_{v_h \in K_h(\Omega)} L(q_h, v_h) = \max_{v_h \in K_h(\Omega)} \inf_{q_h \in U_{f,h}(\Omega)} L(q_h, v_h) \tag{4.25}$$

and the corresponding min, max are attained at λ_h^*, u_h^*, respectively.

For $v_h \in K_h$ let us denote

$$\tilde{\Phi}_h(v_h) = \inf_{q_h \in U_{f,h}(\Omega)} L(q_h, v_h). \tag{4.26}$$

Analogously to the continuous case, the problem

$$\text{(4.27)} \qquad \begin{aligned} &\text{Find } u_h^* \in K_h(\Omega) \text{ such that} \\ &\tilde{\Phi}_h(u_h^*) = \max_{v_h \in K_h(\Omega)} \tilde{\Phi}_h(v_h) \end{aligned}$$

is said to be dual to (4.20). Now, in the same way as in Section 1, we derive the explicit form of $\tilde{\Phi}_h$:

$$\text{(4.28)} \qquad \begin{aligned} \inf_{q_h \in U_{f,h}(\Omega)} L(q_h, v_h) &= \inf_{q_h \in U_{0h}(\Omega)} L(q_h + q_{0h}, v_h) \\ &= \inf_{q_h \in U_{0h}(\Omega)} H_h(q_h) + \frac{1}{2}\|q_{0h}\|_0^2 - \langle q_{0h} \cdot \nu, v_h \rangle \end{aligned}$$

where

$$\text{(4.29)} \qquad H_h(q_h) \equiv \frac{1}{2}\|q_h\|_0^2 + (q_h, q_{0h})_0 - \langle q_h \cdot \nu, v_h \rangle \,.$$

The minimizer of H_h over $U_{0h}(\Omega)$ will be denoted by $q_h(v_h)$. It is characterized by the relation

$$\text{(4.30)} \qquad \begin{aligned} &\text{Find } q_h(v_h) \in U_{0h}(\Omega) \text{ such that} \\ &\big(q_h(v_h), q_h\big)_0 + (q_h, q_{0h})_0 = \langle q_h \cdot \nu, v_h \rangle \quad \forall q_h \in U_{0h}(\Omega). \end{aligned}$$

Next we shall suppose that $q_{0h} = \nabla \overline{w}_h$, where $\overline{w}_h$ is the Galerkin approximation of the homogeneous Dirichlet boundary value problem (2.15), (2.16) on the finite element space $V_h(\Omega)$. With such a choice of q_{0h}, we have

$$\text{(4.31)} \qquad (q_h, q_{0h})_0 = (q_h, \nabla \overline{w}_h)_0 = -(\operatorname{div} q_h, \overline{w}_h)_0 + \langle q_h \cdot \nu, \overline{w}_h \rangle = 0$$

due to the fact $q_h \in U_{0h}(\Omega)$ and $\overline{w}_h = 0$ on Γ.

Thus

$$\text{(4.32)} \qquad H_h(q_h) = \frac{1}{2}\|q_h\|_0^2 - \langle q_h \cdot \nu, v_h \rangle \,.$$

The value of H_h at $q_h(v_h)$ (the solution of (4.32)) is

$$\text{(4.33)} \qquad H_h\big(q_h(v_h)\big) = -\frac{1}{2}\|q_h(v_h)\|_0^2 = -\frac{1}{2}\langle q_h(v_h) \cdot \nu, v_h \rangle$$

where q_h is the discrete analogue of q, namely for $v_h \in K_h(\Omega)$, $q_h(v_h) \in U_{0h}(\Omega)$ denotes the solution of (4.30). Finally, we obtain the following expression for $\tilde{\Phi}_h$:

$$\text{(4.34)} \qquad \tilde{\Phi}_h(v_h) = -\frac{1}{2}\langle q_h(v_h) \cdot \nu, v_h \rangle - \langle q_{0h} \cdot \nu, v_h \rangle + \frac{1}{2}\|q_{0h}\|_0^2.$$

We can omit the term $\frac{1}{2}||q_{0h}||_0^2$ and write

$$\max_{v_h \in K_h(\Omega)} \tilde{\Phi}_h(v_h) = - \min_{v_h \in K_h(\Omega)} \left[\frac{1}{2} \langle q_h(v_h) \cdot \nu, v_h \rangle + \langle q_{0h} \cdot \nu, v_h \rangle \right] = - \min_{v_h \in K_h(\Omega)} \Phi_h(v_h) \tag{4.35}$$

where

$$\Phi_h(v_h) = \frac{1}{2} \gamma_h(v_h, v_h) - \delta(v_h) \tag{4.36}$$

with

$$\gamma_h(v_h, v_h) = \langle q_h(v_h) \cdot \nu, v_h \rangle \, , \tag{4.37}$$

$$\delta_h(v_h) = - \langle q_{0h} \cdot \nu, v_h \rangle \, . \tag{4.38}$$

Let $B_h(\Gamma)$ be the space of functions defined on Γ, which are given by restrictions of all $v_h \in K_h(\Omega)$ to the boundary Γ. Then (4.35) is equivalent to

$$\text{Find } u_h^* \in B_h(\Gamma) \text{ such that} \quad \Phi_h(u_h^*) = \min_{v_h \in B_h(\Gamma)} \Phi_h(v_h). \tag{4.39}$$

Analogously to the continuous case (see Lemma 2.1 and Remark 2.1) we can define

$$||v_h||_{\frac{1}{2},h} = \sup_{\substack{q_h \in U_{0h}(\Omega) \\ q_h \neq 0}} \frac{\langle q_h \cdot \nu, v_h \rangle}{||q_h||_0}. \tag{4.40}$$

On the other hand

$$||q_h(v_h)||_0 = \sup_{\substack{q_h \in U_{0h}(\Omega) \\ q_h \neq 0}} \frac{(q_h(v_h), q_h)_0}{||q_h||_0} = \sup_{\substack{q_h \in U_{0h}(\Omega) \\ q_h \neq 0}} \frac{\langle q_h \cdot \nu, v_h \rangle}{||q_h||_0}. \tag{4.41}$$

Comparing (4.40) with (4.41) and taking into account that

$$\gamma_h(v_h, v_h) = \frac{1}{2} ||q_h(v_h)||_0^2, \tag{4.42}$$

we see that $\Phi_h(v_h)$ can be written in the form

$$\Phi_h(v_h) = \frac{1}{2} ||v_h||_{\frac{1}{2},h}^2 - \delta(v_h). \tag{4.43}$$

Concerning the more general convex or nonconvex semipermeability problems, there are several still open questions especially with respect to the nonconvex case. Thus in the case of relations (3.5), (3.6), the numerical calculation requires the derived multivalued B.I.E. (3.6) to be discretized. We can apply either a direct mathematical discretization of the corresponding boundary minimum problem based on Galerkin approximation (c.f. [3], [4], [7]), or a more engineering oriented discretization resulting from the mechanical interpretation of the expression (3.6) which is a simple and direct procedure and thus makes the previously presented method attractive from the numerical point of view. In order to obtain the discrete form of the problem under consideration we consider the domain Ω resulting from the original one by assuming only the kinematical constraints on Γ_u. Then the structure is solved for the unit flux $q_{(1)} = 1$ on the first node of Γ_c and we obtain all m components of the corresponding solution (e.g. temperature, pressure, electric potential) for all m nodes of Γ_c, constituting the first column of matrix B. This procedure is repeated for the second node ($q_{(2)} = 1$), the third ($q_{(3)} = 1$) and so on. Thus a symmetric matrix B is formed. Next, the solution for the nodes of Γ_c for the same domain under null boundary fluxes denoted by g is calculated. Then the solution of the discrete convex programming (C.P.) problem corresponding to (3.3) gives the unknown fluxes on Γ_c. □

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