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ABOUT A SPECIAL CLASS OF NONCONVEX OPTIMIZATION PROBLEMS

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Summary. The article deals with certain nonconvex optimization problems which have features analogous to those of the linear optimization problems. We can find their absolute extrema and the set of all optimal points of such nonconvex optimization problem represents the closure of a face of a spherical polyhedron which is its feasible set.

Keywords: A nonconvex optimization problem, a spherical polyhedron, a face of a spherical polyhedron.

AMS Classification: 90C30.

Convex optimization problems have two features which are advantageous for their numerical calculation, namely that any of their local extrema is at the same time their absolute extremum, and that the optimal solution set (the set of all optimal points) of a convex optimization problem is convex. Linear optimization problems (LO problems) have in addition the property that their optimal solution set is the closure of a face of a convex polyhedron (which is the feasible set of a LO problem).

The aim of this article is to pick out certain nonconvex optimization problems (NO problems) for which we can find their absolute extrema and, in addition, whose optimal solution sets have features analogous to those of the optimal solution set of a LO problem. Instead of considering a feasible set in the form of a convex polyhedron, as is the case with LO problems, we have now a fesible set in the form of a so-called spherical polyhedron (which is the intersection of a polyhedral cone with a hypersphere, and thus a nonconvex set). We search for a nonconvex objective function so that the optimal solution set of a NO problem should represent the closure of a face of a spherical polyhedron. The formulation of the NO problems dealt with in the article was inspired by the geometrical idea described in Corollary 6. We are looking for either the smallest or the largest angle between the projection of a vector of a spherical polyhedron in a certain plane and a certain vector of the plane. Corollary 7 implies that there exist some other nonconvex functions as well as some other nonconvex feasible sets such that the corresponding NO problems have features mentioned above. As an example of a spherical polyhedron we can imagine such a region on the globe from which we can watch a certain complex of stars simultaneously at a fixed moment t. Indeed, at the time t any star of the complex is visible within just one hemisphere. For various stars of the complex the corresponding hemispheres are different. That is why the places on the globe from which all the complex can be observed simultaneously at t form a spherical polyhedron. Similarly, the territory on the globe on which we can receive signals from more stationary satellites located above one hemisphere forms (approximately) a spherical polyhedron. The inaccuracy is caused by the fact that the radius of the globe, when compared with the distances of the stationary satellites from the globe, is not negligible. The objective function considered in this article is connected with the determination of the time period (by the rotation of the globe), during which, e.g., we can observe the above considered complex of stars. Namely, the objective function, if equal to a constant, forms a halfplane the boundary of which is the globe axis and whose intersection with the surface of the globe is a meridian. Thus, to determine the time we are looking for means to find the minimum and the maximum of the objective function in the spherical polyhedron. Another problem like the NO problem treated in this article can arise when determining a proper admissible direction by gradient methods for solving nonlinear optimization problems.

INTRODUCTION

Let $\boldsymbol{a}_0, \boldsymbol{a}_1$ be certain orthonormal vectors in the euclidean space $\mathbb{E}_n$. Let us denote

$$(1.1)_a \qquad \boldsymbol{R}_0 := \{\boldsymbol{x} \in \mathbb{E}_n \mid (\boldsymbol{a}_0, \boldsymbol{x}) = 0\},$$

$$(1.1)_b \qquad \boldsymbol{H}_0^- := \{\boldsymbol{x} \in \mathbb{E}_n \mid (\boldsymbol{a}_0, \boldsymbol{x}) < 0\},$$

$$(1.2) \qquad \boldsymbol{L}_{n-2} := \{\boldsymbol{x} \in \mathbb{E}_n \mid (\boldsymbol{a}_0, \boldsymbol{x}) = 0, \quad (\boldsymbol{a}_1, \boldsymbol{x}) = 0\},$$

$$(1.3) \qquad f(\boldsymbol{x}) := \arccos (\boldsymbol{a}_1, \boldsymbol{x}) [(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2]^{-1/2}, \quad \boldsymbol{x} \in \mathbb{E}_n \setminus \boldsymbol{L}_{n-2}.$$

Lemma 1. *Let $\boldsymbol{x}_0 \in \boldsymbol{H}_0^-$ be an arbitrary point and let us define $\alpha := f(\boldsymbol{x}_0)$. Then the set*

$$(1.4) \qquad \boldsymbol{P}_\alpha := \{\boldsymbol{x} \in \boldsymbol{H}_0^- \mid f(\boldsymbol{x}) = \alpha\}$$

is the open halfhyperplane in $\mathbb{E}_n$ and its boundary is the set $\boldsymbol{L}_{n-2}$ from (1.2). For the hyperplane

$$(1.5)_a \qquad \boldsymbol{R}_\alpha := \{\boldsymbol{x} \in \mathbb{E}_n \mid (\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha, \boldsymbol{x}) = 0\} \text{ [1]}$$

[1]) Since $\|\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha\| = 1$, it follows that $\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha \neq \boldsymbol{o}$.

and the halfspaces

$$(1.5)_b \qquad \boldsymbol{H}_\alpha^+ := \{\boldsymbol{x} \in \mathbb{E}_n | \; (\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha, \boldsymbol{x}) > 0\} \,,$$

$$\boldsymbol{H}_\alpha^- := \{\boldsymbol{x} \in \mathbb{E}_n | \; (\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha, \boldsymbol{x}) < 0\}$$

we have

$$\boldsymbol{P}_\alpha \subset \boldsymbol{R}_\alpha \,,$$

$$0 < f(\boldsymbol{x}_1) < f(\boldsymbol{x}_0) < f(\boldsymbol{x}_2) < \pi \quad \textit{for any} \quad \boldsymbol{x}_1 \in \boldsymbol{H}_0^- \cap \boldsymbol{H}_\alpha^+ \quad \textit{and}$$

$$\boldsymbol{x}_2 \in \boldsymbol{H}_0^- \cap \boldsymbol{H}_\alpha^- \,.$$

Proof. Obviously $\alpha \in (0, \pi)$. For any point $\boldsymbol{x} \in \boldsymbol{P}_\alpha$ we obtain (by (1.3) and (1.4))

$$\arccos (\boldsymbol{a}_1, \boldsymbol{x}) \left[(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2\right]^{-1/2} = \alpha \,.$$

Hence

$$(\boldsymbol{a}_1, \boldsymbol{x}) = \left[(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2\right]^{1/2} \cos \alpha \Rightarrow \operatorname{sg} (\boldsymbol{a}_1, \boldsymbol{x}) = \operatorname{sg} \cos \alpha \,,$$

$$(1 - \cos^2 \alpha) (\boldsymbol{a}_1, \boldsymbol{x})^2 = -(\boldsymbol{a}_0, \boldsymbol{x})^2 \cos^2 \alpha \,,$$

$$(\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha, \boldsymbol{x}) = 0 \,.$$

We have therefore $\boldsymbol{x} \in \boldsymbol{R}_\alpha$ and thus $\boldsymbol{P}_\alpha \subset \boldsymbol{H}_0^- \cap \boldsymbol{R}_\alpha$. The statement $\boldsymbol{H}_0^- \cap \boldsymbol{R}_\alpha \subset \boldsymbol{P}_\alpha$ can be proved analogously. The set $\boldsymbol{P}_\alpha = \boldsymbol{H}_0^- \cap \boldsymbol{R}_\alpha$ is therefore the open halfhyperplane in $\mathbb{E}_n$.

For the boundary $\partial \boldsymbol{P}_\alpha := \mathrm{cl} \boldsymbol{P}_\alpha \setminus \boldsymbol{P}_\alpha$ we obtain from $(1.1)_a$, $(1.5)_a$

$$\partial \boldsymbol{P}_\alpha = \boldsymbol{R}_\alpha \cap \boldsymbol{R}_0 = \{\boldsymbol{x} \in \mathbb{E}_n | \; (\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha, \boldsymbol{x}) = 0 \,,$$

$$(\boldsymbol{a}_0, \boldsymbol{x}) = 0\} = \boldsymbol{L}_{n-2} \,.$$

For an arbitrary point $\boldsymbol{x}_1 \in \boldsymbol{H}_0^- \cap \boldsymbol{H}_\alpha^+$ let us define $\alpha_1 := f(\boldsymbol{x}_1)$. Then from the equality

$$\alpha_1 = \arccos (\boldsymbol{a}_1, \boldsymbol{x}) \left[(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2\right]^{-1/2}$$

we get by (1.3) and $(1.1)_b$

$$(\boldsymbol{a}_1 \sin \alpha_1 + \boldsymbol{a}_0 \cos \alpha_1, \boldsymbol{x}_1) = 0 \,.$$

Hence

$$-\frac{(\boldsymbol{a}_1, \boldsymbol{x}_1)}{(\boldsymbol{a}_0, \boldsymbol{x}_1)} = \frac{\cos \alpha_1}{\sin \alpha_1} = \cot \alpha_1 \,.$$

Since $\boldsymbol{x}_1 \in \boldsymbol{H}_\alpha^+$, we get (by $(1.5)_b$) also $(\boldsymbol{a}_1 \sin \alpha + \boldsymbol{a}_0 \cos \alpha, \boldsymbol{x}_1) > 0$, and therefore

$$-\frac{(\boldsymbol{a}_1, \boldsymbol{x}_1)}{(\boldsymbol{a}_0, \boldsymbol{x}_1)} > \frac{\cos \alpha}{\sin \alpha} = \cot \alpha \,.$$

So, we have $\cot \alpha_1 > \cot \alpha$, and thus $\alpha_1 = f(\boldsymbol{x}_1) < \alpha = f(\boldsymbol{x}_0)$. The statement $f(\boldsymbol{x}_0) < f(\boldsymbol{x}_2)$ for an arbitrary point $\boldsymbol{x}_2 \in \boldsymbol{H}_0^- \cap \boldsymbol{H}_\alpha^-$ can be proved analogously.

If $\boldsymbol{b}_i \neq \mathbf{o}$ $(i = 1, \ldots, m)$ are arbitrary vectors in $\mathbb{E}_n$ $(n \geqq 3)$, then the set

$$(2.1) \qquad \boldsymbol{K} := \{\boldsymbol{x} \in \mathbb{E}_n | \, (\boldsymbol{b}_i, \boldsymbol{x}) \leqq 0 \quad (i = 1, \ldots, m)\}$$

is evidently a polyhedral cone in $\mathbb{E}_n$ with a vertex at the origin point $\mathbf{o}$. Obviously $\emptyset \neq \boldsymbol{K} \neq \mathbb{E}_n$.

Convention. Throughout this article let us suppose that the polyhedral cone $\boldsymbol{K}$ from (2.1) is not a linear subspace in $\mathbb{E}_n$.

Definition 1. *The set*

$$(2.2) \qquad \boldsymbol{M} := \boldsymbol{Q} \cap \boldsymbol{K},$$

where

$$(2.3) \qquad \boldsymbol{Q} := \{\boldsymbol{x} \in \mathbb{E}_n | \, \|\boldsymbol{x}\| = \varrho\}, \quad \varrho > 0,$$

is called a spherical polyhedron (of the hypersphere $\boldsymbol{Q}$*).*

Corollary 1. Under our supposition, $\dim \boldsymbol{K} \geqq 1$, and therefore $\boldsymbol{M} \neq \emptyset$.

Corollary 2. Because the set $\boldsymbol{L}$ of all vertices of the polyhedral cone $\boldsymbol{K}$ is a linear subspace in $\mathbb{E}_n$ [2]), $\dim \boldsymbol{L} = 0$ if and only if $\boldsymbol{L} = \{\mathbf{o}\}$.

Definition 2. *The intersection of the hypersphere* $\boldsymbol{Q}$ *with a* $(d + 1)$*-dimensional face of the polyhedral cone* $\boldsymbol{K}$ $(d \geqq 0)$ *is called a d-dimensional face of the spherical polyhedron* $\boldsymbol{M}$ *from* (2.2). *In the special case of* $d = 0$ *we call it a vertex, and for* $d = 1$ *an edge of the spherical polyhedron* $\boldsymbol{M}$.

Lemma 2. *Let* $\boldsymbol{K}$ *be the polyhedral cone from* (2.1) *and* $\boldsymbol{L}$ *the set of all its vertices. Then there exists a vector* $\boldsymbol{a}_0 \in \operatorname{rel.int} \boldsymbol{K}^p$ *such that for the closed halfspace*

$$(2.4) \qquad \operatorname{cl} \boldsymbol{H}_0^- := \{\boldsymbol{x} \in \mathbb{E}_n | \, (\boldsymbol{a}_0, \boldsymbol{x}) \leqq 0\}$$

we have $\boldsymbol{K} \subset \operatorname{cl} \boldsymbol{H}_0^-$ *and for the hyperplane* $\boldsymbol{R}_0$ *of* $(1.1)_a$ *we have* $\boldsymbol{K} \cap \boldsymbol{R}_0 = \boldsymbol{L}$.

Proof. If we denote by

$$\boldsymbol{K}^p := \{\boldsymbol{x} \in \mathbb{E}_n | \, (\boldsymbol{x}, \boldsymbol{y}) \leqq 0, \quad \boldsymbol{y} \in \boldsymbol{K}\}$$

the polar cone belonging to the polyhedral cone $\boldsymbol{K}$ at its vertex $\mathbf{o}$, then

$$(2.5) \qquad \operatorname{rel.int} \boldsymbol{K}^p = \{\boldsymbol{x} \in \mathbb{E}_n | \, (\boldsymbol{x}, \boldsymbol{y}) = 0 \text{ for } \boldsymbol{y} \in \boldsymbol{L}, \ (\boldsymbol{x}, \boldsymbol{y}) < 0 \text{ for } \boldsymbol{y} \in \boldsymbol{K} \setminus \boldsymbol{L}\} \text{ [3])}.$$

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[2]) See [1], Theorem 4.1.

[3]) See [1], Theorem 4.13.

For any vector $\boldsymbol{a}_0 \in \text{rel.int}\, \boldsymbol{K}^p$ we have, by the theorem of Farkàs[4]), $\boldsymbol{K} \subset \text{cl}\, \boldsymbol{H}_0^-$ (cl $\boldsymbol{H}_0^-$ defined as in (2.4)) and, moreover $\boldsymbol{K} \cap \boldsymbol{R}_0 = \boldsymbol{L}$, by (2.5).

Corollary 3. If $\dim \boldsymbol{L} = 0$, then $\boldsymbol{M} \subset \boldsymbol{H}_0^-$ for any vector $\boldsymbol{a}_0 \in \text{rel.int}\, \boldsymbol{K}^p$.

Lemma 3. *Let $\boldsymbol{L}_{n-2}$ be the set from (1.2), let $\boldsymbol{K}$ be the polyhedral cone defined by (2.1) and $\boldsymbol{a}_0, \boldsymbol{a}_1$ arbitrary orthonormal vectors in $\mathbb{E}_n$ with $\boldsymbol{a}_0 \in \text{rel.int}\, \boldsymbol{K}^p$. Let us define*

$$\boldsymbol{R}_1 := \{\boldsymbol{x} \in \mathbb{E}_n \mid (\boldsymbol{a}_1, \boldsymbol{x}) = 0\}\,.$$

Then

$$\boldsymbol{M} \cap \boldsymbol{L}_{n-2} = \emptyset \Leftrightarrow \boldsymbol{L} = \{\boldsymbol{o}\} \quad \textit{or} \quad \dim \boldsymbol{L} = 1 \quad \textit{and} \quad \boldsymbol{L} \not\subset \boldsymbol{R}_1\,.$$

Proof. Since $\boldsymbol{o} \in \boldsymbol{K} \cap \boldsymbol{L}_{n-2}$, then $\boldsymbol{K} \cap \boldsymbol{L}_{n-2} \neq \emptyset$ and (by Lemma 2)

$$\boldsymbol{K} \cap \boldsymbol{L}_{n-2} = \{\boldsymbol{x} \in \boldsymbol{K} \mid (\boldsymbol{a}_0, \boldsymbol{x}) = 0\,, (\boldsymbol{a}_1, \boldsymbol{x}) = 0\} = \boldsymbol{K} \cap \boldsymbol{R}_0 \cap \boldsymbol{R}_1 = \boldsymbol{L} \cap \boldsymbol{R}_1\,.$$

Taking into account that $\boldsymbol{L}$ and $\boldsymbol{R}_1$ are linear subspaces in $\mathbb{E}_n$ ($\dim \boldsymbol{R}_1 = n - 1$, $0 \leqq \dim \boldsymbol{L} \leqq n - 1$), we see that the set $\boldsymbol{L} \cap \boldsymbol{R}_1$ is also a linear subspace where $\boldsymbol{o} \in \boldsymbol{L} \cap \boldsymbol{R}_1$ and $0 \leqq \dim \boldsymbol{K} \cap \boldsymbol{L}_{n-2} = \dim \boldsymbol{L} \cap \boldsymbol{R}_1 \leqq n - 1$. For the hyperspehere $\boldsymbol{Q}$ it follows from (2.3) that $\boldsymbol{L} \cap \boldsymbol{R}_1 \cap \boldsymbol{Q} = \boldsymbol{M} \cap \boldsymbol{L}_{n-2} = \emptyset$ if and only if $\boldsymbol{L} \cap \boldsymbol{R}_1 = \{\boldsymbol{o}\}$. But the last equation holds if and only if $\boldsymbol{L} = \{\boldsymbol{o}\}$ or $\dim \boldsymbol{L} = 1$ and $\boldsymbol{L} \not\subset \boldsymbol{R}_1$.

Theorem 1. *Let $\boldsymbol{K}$ be the polyhedral cone as in (2.1) and $\boldsymbol{a}_0, \boldsymbol{a}_1$ arbitrary orthonormal vectors in $\mathbb{E}_n$ with $\boldsymbol{a}_0 \in \text{rel.int}\, \boldsymbol{K}^p$. If the condition*

(2.6) $\quad \boldsymbol{L} = \{\boldsymbol{o}\} \quad \textit{or} \quad \dim \boldsymbol{L} = 1 \quad \textit{and} \quad \boldsymbol{L} \not\subset \boldsymbol{R}_1$

is fulfilled, then the function $f(\boldsymbol{x})$ defined in (1.3) is continuous on the set $\boldsymbol{M}$ in (2.2).

Proof. The function $f(\boldsymbol{x})$ from (1.3) is evidently continuous in $\mathbb{E}_n \setminus \boldsymbol{L}_{n-2}$. The assertion of our theorem follows from (2.6) by Lemma 3.

A SPECIAL NONCONVEX OPTIMIZATION PROBLEM

Let us consider the problems

$(3.1)_a \quad \min \{f(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{M}\}!\,,$

$(3.1)_b \quad \max \{f(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{M}\}!\,,$

where $\boldsymbol{M}$ is as in (2.2) and $f(\boldsymbol{x})$ as in (1.3).

Theorem 2. *If the assumptions of Theorem 1 hold, then solutions of the problems $(3.1)_{a,b}$ exist.*

Proof. The function $f(\boldsymbol{x})$ from (1.3) is, by Theorem 1, continuous on the set $\boldsymbol{M}$, which is nonempty, closed and bounded.

[4]) See [1], Theorem 4.9.

Theorem 3. *Under the assumptions of Theorem 1 let $\mathbf{x}_0$ be an optimal solution of the problem* $(3.1)_a$ *or* $(3.1)_b$. *Let us define* $\alpha := f(\mathbf{x}_0)$. *Then the hyperplane*

$$\mathbf{R}_\alpha = \{\mathbf{x} \in \mathbb{E}_n | (\mathbf{a}_1 \sin \alpha + \mathbf{a}_0 \cos \alpha, \mathbf{x}) = 0\}$$

from $(1.5)_a$ *is the supporting hyperplane of the set* $\mathbf{M}$ *at its point* $\mathbf{x}_0$.

Proof. By Lemma 2 and (2.2), $\mathbf{M} \subset \operatorname{cl} \mathbf{H}_0^- = \mathbf{H}_0^- \cup \mathbf{R}_0$ ($\operatorname{cl} \mathbf{H}_0^-$ is defined as in (2.4), $\mathbf{R}_0$, $\mathbf{H}_0^-$ as in $(1.1)_{a,b}$).

If $\mathbf{x}_0 \in \mathbf{M} \cap \mathbf{R}_0$, then $(\mathbf{a}_0, \mathbf{x}_0) = 0$, $\mathbf{x}_0 \in \mathbf{L}$ (by Lemma 2) and, by Lemma 3, $(\mathbf{a}_1, \mathbf{x}_0) \neq 0$. In this case, $f(\mathbf{x}_0)$ is equal to 0 or π (by (1.3)). Therefore (by $(1.1)_a$)

$$\mathbf{R}_\alpha = \{\mathbf{x} \in \mathbb{E}_n | (\mathbf{a}_0, \mathbf{x}) = 0\} = \mathbf{R}_0 ,$$

and since $\mathbf{M} \subset \operatorname{cl} \mathbf{H}_0^-$, $\mathbf{R}_\alpha$ is the supporting hyperplane of the set $\mathbf{M}$ at its point $\mathbf{x}_0$.

If $\mathbf{x}_0 \in \mathbf{M} \cap \mathbf{H}_0^-$, then $\mathbf{x}_0 \in \mathbf{R}_\alpha$ (by Lemma 1) and thus $\mathbf{x}_0 \in \mathbf{M} \cap \mathbf{R}_\alpha$. For the open halfspaces $\mathbf{H}_\alpha^+$, $\mathbf{H}_\alpha^-$ from $(1.5)_b$ belonging to the hyperplane $\mathbf{R}_\alpha$ the statement $\mathbf{M} \subset \operatorname{cl} \mathbf{H}_\alpha^-$ or $\mathbf{M} \subset \operatorname{cl} \mathbf{H}_\alpha^+$ holds by Lemma 1.

Consequence. An optimal solution $\mathbf{x}_0$ of the problem $(3.1)_a$ or $(3.1)_b$ is a boundary point of the set $\mathbf{M}$.

Theorem 4. *Under the assumptions of Theorem 1 let $\mathbf{x}_0$ be an optimal solution of the problem* $(3.1)_a$ *or* $(3.1)_b$ *belonging to a k-dimensional face $\mathbf{S}_k$ of the spherical polyhedron $\mathbf{M}$* ($k \geqq 1$). *Then any point of the closure* $\operatorname{cl} \mathbf{S}_k$ *is an optimal solution of the problem* $(3.1)_a$ *or* $(3.1)_b$.

Proof. If $\mathbf{x}_0 \in \mathbf{M} \cap \mathbf{R}_0$ then $\dim \mathbf{S}_k = 0$, and therefore such a case is not possible. Thus we have $\mathbf{x}_0 \in \mathbf{M} \cap \mathbf{H}_0^-$. By Definition 2, $\mathbf{x}_0 \in \mathbf{Z}_{k+1}$, where $\mathbf{Z}_{k+1}$ is a $(k+1)$-dimensional face of the polyhedral cone $\mathbf{K}$ with $\mathbf{S}_k \subset \mathbf{Z}_{k+1}$. Let us consider the linear envelope $\mathbf{L}_{k+1}$ of the face $\mathbf{Z}_{k+1}$. Evidently $\mathbf{L}_{k+1} \subset \mathbf{R}_\alpha$, $\mathbf{R}_\alpha$ having the sense from $(1.5)_a$. Since by Lemma 1 the equality $f(\mathbf{x}) = f(\mathbf{x}_0)$ holds for any point $\mathbf{x} \in \mathbf{H}_0^- \cap \mathbf{R}_\alpha$, this equality holds also for any point $\mathbf{x} \in \operatorname{cl} \mathbf{S}_k$.

Consequence. If the assumptions of Theorem 1 are fulfilled, then the optimal solution set of the problem $(3.1)_a$ or $(3.1)_b$ is equal to the closure of a certain face of the spherical polyhedron $\mathbf{M}$. Among all vertices of the spherical polyhedron $\mathbf{M}$ there exists at least one vertex corresponding to the optimal solutions of the problems $(3.1)_{a,b}$.

Corollary 4. The book [2][5]) contains a rather advantageous method for calculating all vertices and all edges of a polyhedral cone $\mathbf{K}$, and we can use it when solving problems like $(3.1)_{a,b}$. The algorithm consists in the determination of all vertices and all edges of the polyhedral cone $\mathbf{K}_j := \{\mathbf{x} \in \mathbb{E}_n | (\mathbf{b}_i, \mathbf{x}) \leqq 0 \, (i = 1, \ldots, j)\}$

[5]) Page 252.

if we know all vertices and all edges of the polyhedral cone $\boldsymbol{K}_{j-1} := \{\boldsymbol{x} \in \mathbb{E}_n | (\boldsymbol{b}_i, \boldsymbol{x}) \leqq \leqq 0 \ (i = 1, \ldots, j-1)\}$, $(j \geqq 1)$.

Example. Let

$$\boldsymbol{K} = \{\boldsymbol{x} \in \mathbb{E}_4 | \ x_1 - x_2 \leqq 0, \ x_3 + x_4 \leqq 0, \ x_2 + x_4 \leqq 0, \\ x_1 - x_3 - x_4 \leqq 0\},$$

then $\boldsymbol{L} = \{\boldsymbol{o}\}$ and $\boldsymbol{h}_1 = (0, 1, 1, -1)$, $\boldsymbol{h}_2 = (-1, -1, -2, 1)$, $\boldsymbol{h}_3 = (0, 0, 1, -1)$, $\boldsymbol{h}_4 = (-1, -1, -1, 1)$ are vectors in the directions of all its edges. In the case $\varrho = 1$, the points $\boldsymbol{x}_1 = (0, 1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, $\boldsymbol{x}_2 = (-1/\sqrt{7}, -1/\sqrt{7}, -2/\sqrt{7}, 1/\sqrt{7})$, $\boldsymbol{x}_3 = (0, 0, 1/\sqrt{2}, -1/\sqrt{2})$, $\boldsymbol{x}_4 = (-1/2, -1/2, -1/2, 1/2)$ are all vertices of the corresponding spherical polyhedron $\boldsymbol{M}$. Let us choose $\boldsymbol{a}_0 = (2/\sqrt{5}, 0, 0, 1/\sqrt{5})$, $\boldsymbol{a}_1 = (0, -1/\sqrt{5}, 2/\sqrt{5}, 0)$. For the function $f(\boldsymbol{x})$ from (1.3) we have $f(\boldsymbol{x}_1) = = \arccos 1/\sqrt{2} = 45°$, $f(\boldsymbol{x}_2) = \arccos -3/\sqrt{10} \doteq 161{,}5°$, $f(\boldsymbol{x}_3) = \arccos 2/\sqrt{5} \doteq \doteq 26{,}5°$, $f(\boldsymbol{x}_4) = \arccos -1/\sqrt{2} = 135°$. Thus the vertex $\boldsymbol{x}_3$ gives the minimum and the vertex $\boldsymbol{x}_2$ the maximum of the function $f(\boldsymbol{x})$ over the set $\boldsymbol{M}$.

OPTIMALITY CRITERION

Let $\boldsymbol{H}_0^-$ be the halfspace from $(1.1)_b$ and $\boldsymbol{Q}$ the hypersphere from (2.3). For any point $\boldsymbol{x}_0 \in \boldsymbol{H}_0^- \cap \boldsymbol{Q}$ and any unit vector $\boldsymbol{v}$ with the property $(\boldsymbol{x}_0, \boldsymbol{v}) = 0$, the set

$$(4.1) \qquad \boldsymbol{C}(\boldsymbol{x}_0; \boldsymbol{v}) := \{\boldsymbol{x} \in \mathbb{E}_n | \ \boldsymbol{x} = \boldsymbol{x}_0 \cos t + \varrho \boldsymbol{v} \sin t, \ t \in (0, 2\pi\rangle\}$$

is a main circle of the hypersphere $\boldsymbol{Q}$ with a parametrical description. The intersection $\boldsymbol{C}(\boldsymbol{x}_0; \boldsymbol{v}) \cap \boldsymbol{R}_0$, where $\boldsymbol{R}_0$ is as in $(1.1)_a$, defines the unique value of the parameter $t_0 \in (0, 2\pi)$, so that

$$(4.2) \qquad \boldsymbol{B}(\boldsymbol{x}_0; \boldsymbol{v}) := \boldsymbol{C}(\boldsymbol{x}_0; \boldsymbol{v}) \cap \boldsymbol{H}_0^- = \\ = \{\boldsymbol{x} \in \mathbb{E}_n | \ \boldsymbol{x} = \boldsymbol{x}_0 \cos t + \varrho \boldsymbol{v} \sin t, \ t \in (0, t_0)\}.$$

Definition 3. *The set $\boldsymbol{B}(\boldsymbol{x}_0; \boldsymbol{v})$ of (4.2) is called the arc of the main circle $\boldsymbol{C}(\boldsymbol{x}_0; \boldsymbol{v})$ originating at the given point $\boldsymbol{x}_0$ in the direction $\boldsymbol{v}$ and belonging to the halfspace $\boldsymbol{H}_0^-$.*

Lemma 4. *The function $f(\boldsymbol{x})$ from (1.3) is along the set $\boldsymbol{B}(\boldsymbol{x}_0; \boldsymbol{v})$ from (4.2) strictly increasing, strictly decreasing or constant according to whether the determinant*

$$(4.3) \qquad D := \begin{vmatrix} (\boldsymbol{a}_1, \boldsymbol{v}) & (\boldsymbol{a}_0, \boldsymbol{v}) \\ (\boldsymbol{a}_1, \boldsymbol{x}_0) & (\boldsymbol{a}_0, \boldsymbol{x}_0) \end{vmatrix}$$

is positive, negative or equal to zero, respectively.

Proof. From (1.3) we have for any point $\boldsymbol{x} \in \boldsymbol{H}_0^-$

$$\frac{\partial f}{\partial x_\alpha} = -[1 - (\boldsymbol{a}_1, \boldsymbol{x})^2/((\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2)]^{-1/2} \cdot \frac{\partial}{\partial x_\alpha}(\boldsymbol{a}_1, \boldsymbol{x}) \cdot$$
$$\cdot [(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2]^{-1/2} = 1/(\boldsymbol{a}_0, \boldsymbol{x}) [(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2]^{1/2} \cdot$$
$$\cdot (\boldsymbol{a}_0, \boldsymbol{x}) [a_{1\alpha}(\boldsymbol{a}_0, \boldsymbol{x}) - a_{0\alpha}(\boldsymbol{a}_1, \boldsymbol{x})] \cdot [(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2]^{-3/2} =$$
$$= [a_{1\alpha}(\boldsymbol{a}_0, \boldsymbol{x}) - a_{0\alpha}(\boldsymbol{a}_1, \boldsymbol{x})] \cdot [(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2]^{-1},$$

$(\alpha = 1, \ldots, n)$. Thus

$$\nabla f(\boldsymbol{x}) = [\boldsymbol{a}_1(\boldsymbol{a}_0, \boldsymbol{x}) - \boldsymbol{a}_0(\boldsymbol{a}_1, \boldsymbol{x})] \cdot [(\boldsymbol{a}_1, \boldsymbol{x})^2 + (\boldsymbol{a}_0, \boldsymbol{x})^2]^{-1}, \quad \boldsymbol{x} \in \boldsymbol{H}_0^- .$$

For the function

$$\tilde{f}(t) := f(\boldsymbol{x}(t)) = f(\boldsymbol{x}_0 \cos t + \varrho\, \boldsymbol{v} \sin t), \quad t \in (0, t_0),$$

we have

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} = \left(\nabla f(\boldsymbol{x}(t)), \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right) = ([\boldsymbol{a}_1(\boldsymbol{a}_0, \boldsymbol{x}(t)) - \boldsymbol{a}_0(\boldsymbol{a}_1, \boldsymbol{x}(t))] \cdot$$
$$\cdot [(\boldsymbol{a}_1, \boldsymbol{x}(t))^2 + (\boldsymbol{a}_0, \boldsymbol{x}(t))^2]^{-1}, -\boldsymbol{x}_0 \sin t + \varrho \boldsymbol{v} \cos t) =$$
$$= \varrho[(\boldsymbol{a}_1, \boldsymbol{x}(t))^2 + (\boldsymbol{a}_0, \boldsymbol{x}(t))^2]^{-1} \cdot \begin{vmatrix} (\boldsymbol{a}_1, \boldsymbol{v}) & (\boldsymbol{a}_0, \boldsymbol{v}) \\ (\boldsymbol{a}_1, \boldsymbol{x}_0) & (\boldsymbol{a}_0, \boldsymbol{x}_0) \end{vmatrix},$$

and the assertion of our theorem follows by (4.2).

Theorem 5. *Let* $\boldsymbol{K}$ *be the polyhedral cone from* (2.1), $\boldsymbol{L}$ *the set of all its vertices and* $\boldsymbol{a}_0, \boldsymbol{a}_1$ *arbitrary orthonormal vectors in* $\mathbb{E}_n$ *with* $\boldsymbol{a}_0 \in \operatorname{rel.int} \boldsymbol{K}^p$. *Then*

a) *the spherical polyhedron* $\boldsymbol{M}$ *from* (2.2) *has at least one vertex if and only if* $\dim \boldsymbol{L} \leqq 1$;

b) *if* $\dim \boldsymbol{L} \leqq 1$ *and* $\dim \boldsymbol{K} > 2$, *then the spherical polyhedron* $\boldsymbol{M}$ *has always edges, and the function* $f(\boldsymbol{x})$ *is strictly increasing or strictly decreasing or constant along any edge.*

Proof. By Corollary 1, $\dim \boldsymbol{K} \geqq 1$ and assertion a) follows from Definition 2.

If $\dim \boldsymbol{L} \leqq 1$, then, by assertion a) and by the assumption $\dim \boldsymbol{K} > 2$, the spherical polyhedron $\boldsymbol{M}$ possesses edges. By Definition 2, any of its edges is the intersection of a 2-dimensional face of the polyhedral cone $\boldsymbol{K}$ with the hypersphere $\boldsymbol{Q}$, and therefore an arc of a main circle of $\boldsymbol{Q}$. The inclusion $\boldsymbol{M} \subset \operatorname{cl} \boldsymbol{H}_0^-$ shows that any edge of $\boldsymbol{M}$ is contained either in $\boldsymbol{H}_0^-$ or in $\boldsymbol{R}_0$ ($\operatorname{cl} \boldsymbol{H}_0^-$ having the sense of (2.4), $\boldsymbol{R}_0$, $\boldsymbol{H}_0^-$ from $(1.1)_{\mathrm{a,b}}$). In the former case, assertion b) is proved by Lemma 4. The latter case implies that the corresponding edge is a subset of $\boldsymbol{R}_0$, which contradicts our assumption and Lemma 2.

Theorem 6. *Let the assumptions of Theorem 5 hold, $\mathbf{L} \not\subset \mathbf{R}_1$ and let $\mathbf{v}_i$ $(i = 1, \ldots, N)$ be arbitrary vectors in the directions of all edges $\mathbf{h}_i$ of the spherical polyhedron $\boldsymbol{M}$ originating at its vertex $\mathbf{x}_0$. The vertex $\mathbf{x}_0$ is an optimal solution of the problem $(3.1)_a$ or $(3.1)_b$ if and only if*

$$\begin{vmatrix} (\boldsymbol{a}_1, \mathbf{v}_i) & (\boldsymbol{a}_0, \mathbf{v}_i) \\ (\boldsymbol{a}_1, \mathbf{x}_0) & (\boldsymbol{a}_0, \mathbf{x}_0) \end{vmatrix} \geqq 0 \quad or \quad \begin{vmatrix} (\boldsymbol{a}_1, \mathbf{v}_i) & (\boldsymbol{a}_0, \mathbf{v}_i) \\ (\boldsymbol{a}_1, \mathbf{x}_0) & (\boldsymbol{a}_0, \mathbf{x}_0) \end{vmatrix} \leqq 0 \quad (i = 1, \ldots, N), \tag{4.4}$$

respectively.

Proof. If $\boldsymbol{x}_0$ is an optimal solution of the problem $(3.1)_a$, then, by Theorem 5, the function $f(\boldsymbol{x})$ is either constant or strictly increasing along any edge $\boldsymbol{h}_i$ $(i = 1, \ldots, N)$ of $\boldsymbol{M}$. If $\boldsymbol{x}_0$ is an optimal solution of the problem $(3.1)_b$, then, by Theorem 5, the function $f(\boldsymbol{x})$ is either constant or strictly decreasing along any edge $\boldsymbol{h}_i$ of $\boldsymbol{M}$. But any edge $\boldsymbol{h}_i$ is an arc of a main circle of $\mathbf{Q}$ originating at $\mathbf{x}_0$ in the direction $\boldsymbol{v}_i$ belonging to the halfspace $\boldsymbol{H}_0^-$, which implies, by Lemma 4, the assertion (4.4).

Let (4.4) hold for any vector $\boldsymbol{v}_i$ $(i = 1, \ldots, N)$ and let us consider an arbitrary point $\boldsymbol{x}_1 \in \boldsymbol{M}$, $\boldsymbol{x}_1 \neq \boldsymbol{x}_0$. The points $\boldsymbol{x}_0, \boldsymbol{x}_1$ define a unique main circle $\mathbf{C}(\boldsymbol{x}_0; \mathbf{v})$, where $\boldsymbol{v}$ is its tangential vector originating at the point $\boldsymbol{x}_0$ (it is oriented to the point $\boldsymbol{x}_1$). We have $(\boldsymbol{x}_0, \boldsymbol{v}) = 0$, likewise $(\boldsymbol{x}_0, \boldsymbol{v}_i) = 0$ $(i = 1, \ldots, N)$. The intersection of the hyperplane

$$\boldsymbol{T}(\boldsymbol{x}_0) := \{\boldsymbol{x} \in \mathbb{E}_n \mid (\boldsymbol{x}_0, \boldsymbol{x} - \boldsymbol{x}_0) = 0\}$$

(the tangential hyperplane of the hypersphere $\mathbf{Q}$ at its point $\boldsymbol{x}_0$) with the polyhedral cone $\boldsymbol{K}$ is a convex polyhedron $\boldsymbol{M}(\boldsymbol{x}_0)$. Obviously, the point $\boldsymbol{x}_0$ is a vertex of $\boldsymbol{M}(\boldsymbol{x}_0)$ and $\boldsymbol{v}_i$ $(i = 1, \ldots, N)$ lie in the directions of all edges of $\boldsymbol{M}(\boldsymbol{x}_0)$ originating at $\boldsymbol{x}_0$. Thus, for the vector $\boldsymbol{v}$ we have

$$\boldsymbol{v} \in \boldsymbol{T}(\boldsymbol{x}_0) \quad \text{and} \quad \boldsymbol{v} = \sum_{i=1}^{N} \lambda_i \boldsymbol{v}_i, \quad \lambda_i \geqq 0 \quad (i = 1, \ldots, N).$$

This implies, by (4.4),

$$\begin{vmatrix} (\boldsymbol{a}_1, \boldsymbol{v}) & (\boldsymbol{a}_0, \boldsymbol{v}) \\ (\boldsymbol{a}_1, \boldsymbol{x}_0) & (\boldsymbol{a}_0, \boldsymbol{x}_0) \end{vmatrix} = \sum_{i=1}^{N} \lambda_i \begin{vmatrix} (\boldsymbol{a}_1, \boldsymbol{v}_i) & (\boldsymbol{a}_0, \boldsymbol{v}_i) \\ (\boldsymbol{a}_1, \boldsymbol{x}_0) & (\boldsymbol{a}_0, \boldsymbol{x}_0) \end{vmatrix} \geqq 0$$

or

$$\begin{vmatrix} (\boldsymbol{a}_1, \boldsymbol{v}) & (\boldsymbol{a}_0, \boldsymbol{v}) \\ (\boldsymbol{a}_1, \boldsymbol{x}_0) & (\boldsymbol{a}_0, \boldsymbol{x}_0) \end{vmatrix} = \sum_{i=1}^{N} \lambda_i \begin{vmatrix} (\boldsymbol{a}_1, \boldsymbol{v}_i) & (\boldsymbol{a}_0, \boldsymbol{v}_i) \\ (\boldsymbol{a}_1, \boldsymbol{x}_0) & (\boldsymbol{a}_0, \boldsymbol{x}_0) \end{vmatrix} \leqq 0,$$

and further, by Lemma 4, $f(\boldsymbol{x}_1) \geqq f(\boldsymbol{x}_0)$ or $f(\boldsymbol{x}_1) \leqq f(\boldsymbol{x}_0)$, respectively[6]). Since $\boldsymbol{x}_1 \in \boldsymbol{M}$ was an arbitrary point, the last inequality shows that $\boldsymbol{x}_0$ is an optimal solution of the problem $(3.1)_a$ or $(3.1)_b$, respectively.

Corollary 5. Theorem 6 further gives an idea about how to solve the problems $(3.1)_{a,b}$. If we know an arbitrary vertex $\boldsymbol{x}_0$ of the spherical polyhedron $\boldsymbol{M}$, then we can define a convex polyhedron $\boldsymbol{M}(\boldsymbol{x}_0) := \boldsymbol{K} \cap \boldsymbol{T}(\boldsymbol{x}_0)$, and by the simplex method

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[6]) The assumption $\mathbf{L} \not\subset \mathbf{R}_1$ guarantees that the function $f(\boldsymbol{x})$ is defined on the set $\boldsymbol{M}$.

we look for $\boldsymbol{x}_0$ as for a vertex of $\boldsymbol{M}(\boldsymbol{x}_0)$. From the corresponding simplex table, we read the vectors $\boldsymbol{v}_i$ $(i = 1, \ldots, N)$ in the directions of all edges of $\boldsymbol{M}$ originating at $\boldsymbol{x}_0$, and the inequality (4.4) shows whether $\boldsymbol{x}_0$ is an optimal solution of either $(3.1)_a$ or $(3.1)_b$. If it is not the case, then we shall find, by one step of the simplex method, the vertex $\boldsymbol{x}_1 \in \boldsymbol{M}$ adjacent to the vertex $\boldsymbol{x}_0$ (after its normalization with respect to the value ϱ), with which we repeat the above mentioned process.

Corollary 6. The problems $(3.1)_{a,b}$ in question have, in $\mathbb{E}_3$, a simple geometrical interpretation. Under our assumptions, the vectors $\boldsymbol{a}_0, \boldsymbol{a}_1$ define a plane $\boldsymbol{R}_2$ in $\mathbb{E}_3$. If $\boldsymbol{x}^*$ denotes the projection of a vector $\boldsymbol{x} \in \boldsymbol{M}$ in the plane $\boldsymbol{R}_2$ and $\boldsymbol{a}_x := \boldsymbol{x}^*/\|\boldsymbol{x}^*\|$, then the function $f(\boldsymbol{x})$ represents the angle between the unit vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_x$, and the nonconvex programming problem $(3.1)_a$ or $(3.1)_b$ means to find respectively the smallest or the largest angle between the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_x$ with respect to all $\boldsymbol{x} \in \boldsymbol{M}$.

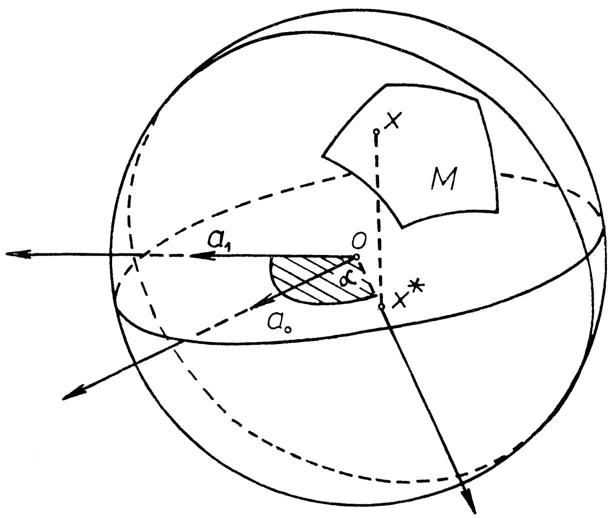

Fig. 1

Corollary 7. We can create even other functions for which Theorems 1–6 also hold. They are e.g. continuous strictly monotone functions of the argument $(\boldsymbol{a}_1, \boldsymbol{x})/(\boldsymbol{a}_0, \boldsymbol{x})$, as artan $(\boldsymbol{a}_1, \boldsymbol{x})/(\boldsymbol{a}_0, \boldsymbol{x})$. Further, we can carry out an extension with regard to the feasible set $\boldsymbol{M}$. It is possible to take into consideration e.g. the set $\boldsymbol{M} := \boldsymbol{K} \cap \boldsymbol{E}$, where

$$\boldsymbol{E} := \{\boldsymbol{x} \in \mathbb{E}_n | \sum_{i=1}^{n} (x_i/c_i)^2 = 1\}, \quad c_i > 0 \quad (i = 1, \ldots, n),$$

or a co-called strictly convex smooth manifold.

References

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Souhrn

O SPECIÁLNÍM TYPU NEKONVEXNÍCH OPTIMALIZAČNÍCH ÚLOH

LIBUŠE GRYGAROVÁ

Článek pojednává o speciálním typu nekonvexních optimalizačních úloh, které mají vlastnosti obdobné vlastnostem optimalizačních úloh lineárních. U nekonvexní optimalizační úlohy tohoto typu umíme najít její absolutní extrém a množina všech jejich optimálních řešení představuje uzávěr stěny sférického polyedru, který je její množinou přípustných řešení.

Резюме

ОБ СПЕЦИАЛЬНОМ ТИПЕ ЗАДАЧ НЕВЫПУКЛОГО ПРОГРАММИРОВАНИЯ

LIBUŠE GRYGAROVÁ

Статья занимается специальным типом задач невыпуклого программирования, свойства которых подобны свойствам задач линейного программирования. У задачи невыпуклого программирования этого типа мы умеем найти её абсолютный экстремум и множество всех её оптимальных решений представляет собой замкнутую стену сферического многогранника, который являестя множеством её допустимых решений.

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