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ON TIME-HARMONIC MAXWELL EQUATIONS WITH NONHOMOGENEOUS CONDUCTIVITIES: SOLVABILITY AND FE-APPROXIMATION

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Summary. The solvability of time-harmonic Maxwell equations in the 3D-case with nonhomogeneous conductivities is considered by adapting Sobolev space technique and variational formulation of the problem in question. Moreover, a finite element approximation is presented in the 3D-case together with an error estimate in the energy norm. Some remarks are given to the 2D-problem arising from geophysics.

Keywords: time-harmonic Maxwell equations, solution theory, 3D-finite element approximation

AMS Classification: 78A25, 65N30, 35R05.

1. INTRODUCTION

This paper is a supplement to papers [13] and [16]. We consider the Maxwell equations

$$
\begin{aligned}
\frac{\partial \mathscr{D}}{\partial t} &= \operatorname{rot} \mathscr{H} - \mathscr{I}, \\
\frac{\partial \mathscr{B}}{\partial t} &= -\operatorname{rot} \mathscr{E},
\end{aligned}
\tag{1.1}
$$

in a three-dimensional bounded region Ω, where $\mathscr{D} = \varepsilon \mathscr{E}$ is the electric induction, $\mathscr{B} = \mu \mathscr{H}$ is the magnetic induction, $\mathscr{E}$ is the electric field, $\mathscr{H}$ is the magnetic field, ε is the electric dielectricity, μ is the magnetic permeability, $\mathscr{I} = \sigma \mathscr{E}$ is the electric current and σ is the 3×3 matrix of electric conductivities. We assume $\mathscr{E}$ and $\mathscr{H}$ to be time-harmonic with the low angular frequency $\omega \in (0, 1)$, i.e.,

$$
\begin{aligned}
\mathscr{E}(x_1, x_2, x_3, t) &= \operatorname{Re}\left(E(x_1, x_2, x_3) \exp(i\omega t)\right), \\
\mathscr{H}(x_1, x_2, x_3, t) &= \operatorname{Re}\left(H(x_1, x_2, x_3) \exp(i\omega t)\right),
\end{aligned}
\tag{1.2}
$$

where E and H are complex-valued vector functions independent of time and $\operatorname{Re}(v_1, v_2, v_3) = (\operatorname{Re} v_1, \operatorname{Re} v_2, \operatorname{Re} v_3)$. Moreover, we shall assume that μ is a real positive constant in the whole region. (For instance in geophysical computations, the permeability of almost all rocks is nearly equal to the vacuum permeability [14].) On the other hand, the conductivity σ may essentially vary (see Section 5). As $\varepsilon \approx 10^{-11} - 10^{-9}$ $[\mathrm{Fm}^{-1}]$ we see that $\partial \mathcal{D}/\partial t \doteq 0$. Under the above assumptions, the relations (1.1) and (1.2) enable us to deal with the following system without time

$$\begin{aligned} \operatorname{rot} H &= \sigma E \quad \text{in} \quad \Omega\,, \\ \operatorname{rot} E &= -i\omega\mu H \quad \text{in} \quad \Omega\,. \end{aligned} \tag{1.3}$$

From here we see that H can be directly computed from the knowledge of E,

$$H = -\frac{\operatorname{rot} E}{i\omega\mu}\,,$$

and for E we get the equation

$$\operatorname{rot}\operatorname{rot} E + i\omega\mu\sigma E = 0 \quad \text{in} \quad \Omega\,. \tag{1.4}$$

Note that H formally satisfies the same equation at those parts of Ω where σ is constant

$$\operatorname{rot}\operatorname{rot} H + i\omega\mu\sigma H = 0\,.$$

On the boundary of Ω we prescribe the boundary condition

$$n \times E = n \times \tilde{E} \quad \text{on} \quad \partial\Omega\,, \tag{1.5}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to $\partial\Omega$ and $\tilde{E}$ is a given vector function which is for convenience defined over the whole domain Ω. Since $\operatorname{div}\operatorname{rot} \equiv 0$ (cf. (2.11)), the first equation of (1.3) yields $\operatorname{div}\sigma E = 0$ in Ω ($\operatorname{div}\sigma E$ will always mean $\operatorname{div}(\sigma E)$). So we shall look for a complex-valued vector function $u = E - \tilde{E}$ such that

$$\left\{\begin{aligned} \operatorname{rot}\operatorname{rot} u + i\omega\mu\sigma u &= -\operatorname{rot}\operatorname{rot}\tilde{E} - i\omega\mu\sigma\tilde{E} && \text{in} \quad \Omega\,, \\ \operatorname{div}\sigma u &= -\operatorname{div}\sigma\tilde{E} && \text{in} \quad \Omega\,, \\ n \times u &= 0 && \text{on} \quad \partial\Omega\,. \end{aligned}\right. \tag{1.6}$$

Here we assume that $u, \tilde{E} \in (C^2(\bar{\Omega}))^3$, and $\sigma \in (C^1(\bar{\Omega}))^{3\times 3}$ is a real symmetric and uniformly positive definite matrix, i.e., (1.6) is fulfilled in the classical sense. Later we shall prescribe weaker assumptions upon the regularity of u, $\tilde{E}$ and σ.

The paper is organized as follows. In Section 2 we introduce the spaces $H(\operatorname{div}\sigma; \Omega)$ and $H_0(\operatorname{rot}; \Omega)$ which are appropriate for a variational formulation of the problem (1.6). Although we shall work in complex valued spaces, we employ some assertions from the real analysis (to be understood as they are applied to the real and imaginary parts of the functions in question). In Section 3 we introduce the variational formulation of (1.6) necessary to derive the finite element approximation and error estimates in Section 4. In Section 5 we present some remarks for the 2D-problem and give a numerical example, where a new C^1-element is employed.

2. FUNCTION SPACES OF COMPLEX-VALUED FUNCTIONS

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with a Lipschitz boundary. The normal n to $\partial\Omega$ thus exists almost everywhere (see [15, p. 88]). By $H^k(\Omega)$, $k = 0, 1, 2, \ldots$, we will mean the Sobolev spaces of complex-valued functions. The standard norm in $H^k(\Omega)$ and also in $(H^k(\Omega))^3$ will be denoted by $\|\cdot\|_k$ (or $\|\cdot\|_{k,\Omega}$ if necessary). Further, $H^{1/2}(\Omega)$ is the space of traces of functions from $H^1(\Omega)$, and $H_0^1(\Omega)$ consists of functions from $H^1(\Omega)$ with vanishing traces. The space $(L^2(\Omega))^p = (H^0(\Omega))^p$, $p = 1, 2, 3$, will be endowed with the usual scalar product

$$(v, w)_0 = \sum_{j=1}^{p} \int_\Omega (\operatorname{Re} v_j + i \operatorname{Im} v_j)(\operatorname{Re} \omega_j - i \operatorname{Im} w_j)\,\mathrm{d}x$$

for $v = (v_1, \ldots, v_p)$, $w = (w_1, \ldots, w_p) \in (L^2(\Omega))^p$.

The notation $P_k(\Omega)$ is used for the space of complex-valued polynomials of degree at most k.

In the three-dimensional case we will assume that the electric conductivity $\sigma \in (L^\infty(\Omega))^{3\times 3}$ is a real symmetric and uniformly positive definite matrix. Consequently, there are constants $m, M > 0$ such that

$$0 < m\|\xi\|^2 \leqq (\sigma(x)\,\xi, \xi) \leqq M\|\xi\|^2 \quad \forall \xi \in \mathbb{C}^3, \quad \xi \neq 0, \tag{2.1}$$

holds for a.e. $x \in \Omega$. Here the symbols $\|\cdot\|$ and $(\cdot, \cdot)$ stand for the standard norm and scalar product in $\mathbb{C}^3$, respectively. Introduce the space

$$\begin{gathered} H(\operatorname{div} \sigma; \Omega) = \{v \in (L^2(\Omega))^3 \mid \exists F \in L^2(\Omega): \\ (\sigma v, \operatorname{grad} z)_0 = -(F, z)_0 \quad \forall z \in \mathscr{D}(\Omega)\} \\ (= \{v \in (L^2(\Omega))^3 \mid \operatorname{div} \sigma v \in L^2(\Omega)\}), \end{gathered}$$

where $\mathscr{D}(\Omega)$ is the space of complex infinitely differentiable functions with a compact support in Ω and the function F is called the divergence of σv in the sense of distributions. We write only $H(\operatorname{div}; \Omega)$ when σ is the unit matrix. Let us set

$$H(\operatorname{div}^0; \Omega) = \{v \in H(\operatorname{div}; \Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega\},$$

$$H(\operatorname{div}^0 \sigma; \Omega) = \{v \in H(\operatorname{div} \sigma; \Omega) \mid \operatorname{div} \sigma v = 0 \text{ in } \Omega\}.$$

Recall (see [7, p. 27]) that the function $v \mapsto n \cdot v|_{\partial\Omega}$ defined on $(C^\infty(\bar{\Omega}))^3$ can be extended by continuity to a linear continuous mapping from the space $H(\operatorname{div}; \Omega)$ into $H^{-1/2}(\partial\Omega)$, the latter being the dual space to $H^{1/2}(\partial\Omega)$. Now the Green formula may be rewritten as

$$(\operatorname{div} v, z)_0 + (v, \operatorname{grad} z)_0 = \langle n \cdot v, z \rangle_{\partial\Omega} \quad \forall v \in H(\operatorname{div}; \Omega) \quad \forall z \in H^1(\Omega), \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. We set

$$H_0(\operatorname{div}^0; \Omega) = \{v \in H(\operatorname{div}^0; \Omega) \mid n \cdot v = 0 \text{ on } \partial\Omega\}.$$

Further, we extend the range of definition of the operator

$$\text{rot } z = \left(\frac{\partial z_3}{\partial x_2} - \frac{\partial z_2}{\partial x_3}, \frac{\partial z_1}{\partial x_3} - \frac{\partial z_3}{\partial x_1}, \frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2}\right), \tag{2.3}$$

$z \in (H^1(\Omega))^3$, introducing the space (see [6])

$$H(\text{rot}; \Omega) = \{v \in (L^2(\Omega))^3 \mid \exists G \in (L^2(\Omega))^3 : (v, \text{rot } z)_0 = (G, z)_0 \quad \forall z \in (\mathscr{D}(\Omega))^3\}$$
$$(= \{v \in (L^2(\Omega))^3 \mid \text{rot } v \in (L^2(\Omega))^3\}),$$

where the function G is called the rotation of v in the sense of distributions. We equip this space with the norm

$$\|v\|_{H(\text{rot};\Omega)} = (\|v\|_0^2 + \|\text{rot } v\|_0^2)^{1/2}, \quad v \in H(\text{rot}; \Omega). \tag{2.4}$$

According to [7, p. 34], the Green formula reads

$$(\text{rot } v, z)_0 - (v, \text{rot } z)_0 = \langle n \times v, z \rangle_{\partial\Omega} \tag{2.5}$$
$$\forall v \in H(\text{rot}; \Omega) \quad \forall z \in (H^1(\Omega))^3,$$

where the vector product $n \times v$ is from $(H^{-1/2}(\partial\Omega))^3$ and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ stands for the duality pairing between $(H^{-1/2}(\partial\Omega))^3$ and $(H^{1/2}(\partial\Omega))^3$. Let us introduce some subspaces of $H(\text{rot}; \Omega)$

$$H_0(\text{rot}; \Omega) = \{v \in H(\text{rot}; \Omega) \mid n \times v = 0 \text{ on } \partial\Omega\},$$
$$H(\text{rot}^0; \Omega) = \{v \in H(\text{rot}; \Omega) \mid \text{rot } v = 0 \text{ in } \Omega\},$$
$$H_0(\text{rot}^0; \Omega) = H_0(\text{rot}; \Omega) \cap H(\text{rot}^0; \Omega).$$

Since we shall later need to employ (2.5) also for functions z which are not from $(H^1(\Omega))^3$, we prove an auxiliary lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary. Then*

$$(\text{rot } s, v)_0 = (s, \text{rot } v)_0 \quad \forall s \in H(\text{rot}; \Omega) \quad \forall v \in H_0(\text{rot}; \Omega). \tag{2.6}$$

Proof: Let $s \in H(\text{rot}; \Omega)$ and $v \in H_0(\text{rot}; \Omega)$ be given. Since $(\mathscr{D}(\Omega))^3$ is dense in $H_0(\text{rot}; \Omega)$ with respect to the norm (2.4) – see e.g. [7, p. 32], there exists a sequence $\{v_j\}_{j=0}^{\infty} \subset (\mathscr{D}(\Omega))^3$ such that

$$\|v - v_j\|_{H(\text{rot};\Omega)} \to 0 \quad \text{as} \quad j \to \infty.$$

Thus we conclude that

$$(\text{rot } s, v_j)_0 \to (\text{rot } s, v)_0 \tag{2.7}$$

and

$$(s, \text{rot } v_j)_0 \to (s, \text{rot } v)_0. \tag{2.8}$$

By the Green formula (2.5) we obtain

$$(\text{rot } s, v_j)_0 - (s, \text{rot } v_j)_0 = \langle n \times s, v_j \rangle_{\partial\Omega} = 0, \quad j = 1, 2, \ldots$$

This together with (2.7) and (2.8) yields (2.6). ∎

Note that from the density $\overline{\mathscr{D}(\Omega)} = H_0^1(\Omega)$, (2.2) and (2.5) we can easily derive that

(2.9) $$\operatorname{grad} z \in H(\operatorname{rot}^0; \Omega) \quad \text{for} \quad z \in H^1(\Omega)\,,$$

(2.10) $$\operatorname{grad} z \in H_0(\operatorname{rot}^0; \Omega) \quad \text{for} \quad z \in H_0^1(\Omega)\,,$$

(2.11) $$\operatorname{rot} v \in H(\operatorname{div}^0; \Omega) \quad \text{for} \quad v \in H(\operatorname{rot}; \Omega)\,,$$

(2.12) $$\operatorname{rot} v \in H_0(\operatorname{div}^0; \Omega) \quad \text{for} \quad v \in H_0(\operatorname{rot}; \Omega)\,.$$

3. VARIATIONAL FORMULATION OF 3D-PROBLEM

Let us introduce the space of test functions

$$V = H(\operatorname{div} \sigma; \Omega) \cap H_0(\operatorname{rot}; \Omega)$$

equipped with the norm

$$|||v|||_\Omega = (\|\operatorname{div} \sigma v\|_{0,\Omega}^2 + \|\operatorname{rot} v\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2)^{1/2}\,, \quad v \in V\,.$$

Suppose that $u \in V$ (with an appropriate smoothness) satisfies (1.6). Using (2.6) for $s = \operatorname{rot} u$ and then for $s = \operatorname{rot} \tilde{E}$, we find that for any $v \in V$

$$(\operatorname{rot} v, \operatorname{rot} u)_0 + (v, i\omega\mu\sigma u)_0 = -(\operatorname{rot} v, \operatorname{rot} \tilde{E})_0 - (v, i\alpha\mu\sigma\tilde{E})_0\,,$$
$$(\operatorname{div} \sigma v, \operatorname{div} \sigma u)_0 = -(\operatorname{div} \sigma v, \operatorname{div} \sigma\tilde{E})_0\,.$$

Consequently,

(3.1) $$a(v, u) = b(v) \quad \forall v \in V\,,$$

where a is the sesquilinear form

(3.2) $$a(v, w) = (\operatorname{div} \sigma v, \operatorname{div} \sigma w)_0 + (\operatorname{rot} v, \operatorname{rot} w)_0 - i\omega\mu(v, \sigma w)_0\,, \quad v, w \in V\,,$$

and b is the linear form defined by

(3.3) $$b(v) = -(\operatorname{div} \sigma v, \operatorname{div} \sigma\tilde{E})_0 - (\operatorname{rot} v, \operatorname{rot} \tilde{E})_0 + i\omega\mu(v, \sigma\tilde{E})_0\,, \quad v \in V\,.$$

In Remark 3.4 we further show that if u is sufficiently smooth and fulfils (3.1) then u fulfils also (1.6). Hence, we are justified to call (3.1) the variational formulation of the problem (1.6).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a connected Lipschitz boundary $\partial\Omega$ and let $\tilde{E} \in H(\operatorname{div} \sigma; \Omega) \cap H(\operatorname{rot}; \Omega)$. Then there exists a unique $u \in V$ such that (3.1) holds.*

Before we prove this theorem we introduce some definitions and two lemmas. Let

$$Q = H_0(\mathrm{div}^0; \Omega) \cap H(\mathrm{rot}; \Omega) .$$

By S we denote the orthocomplement of the space

$$\mathscr{H}_{\mathscr{D}} = H_0(\mathrm{div}^0; \Omega) \cap H(\mathrm{rot}^0; \Omega)$$

in Q with respect to the scalar product $(\cdot, \cdot)_0 + (\mathrm{rot}\cdot, \mathrm{rot}\cdot)_0$. One may easily show that $\mathscr{H}_{\mathscr{D}}$ is a closed subspace of Q in the corresponding norm (2.4). Note that (see [13, p. 310]) the space $\mathscr{H}_{\mathscr{D}}$ is trivial if and only if Ω is simply connected. (When Ω is e.g. a torus axisymmetric with respect to the axis x_3, then for

$$v(x_1, x_2, x_3) = \left(\frac{x_2}{x_1^2 + x_2^2}, \quad -\frac{x_1}{x_1^2 + x_2^2}, 0\right)$$

one can directly verify that $v \in \mathscr{H}_{\mathscr{D}} \neq \{0\}$.) The next lemma assigns a special vector potential function (stream function) to a divergence-free function.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a connected Lipschitz boundary and let $\psi \in H(\mathrm{div}^0; \Omega)$ be given. Then there exists exactly one stream function $s \in S$ such that*

$$\psi = \mathrm{rot}\, s .$$

Moreover,

$$\|s\|_0 \leqq c\|\mathrm{rot}\, s\|_0 , \tag{3.4}$$

where $c > 0$ is independent of s (and ψ).

Proof. Let $\psi \in H(\mathrm{div}^0; \Omega)$ be arbitrary. Since $\partial\Omega$ is connected, by [7, p. 45] there exists a stream function $q' \in H(\mathrm{div}^0; \Omega) \cap (H^1(\Omega))^3$ (not uniquely determined) such that $\psi = \mathrm{rot}\, q'$. According to (2.2), we get $\langle n\cdot q', 1\rangle_{\partial\Omega} = 0$. Hence, there exists a weak solution $\varphi \in H^1(\Omega)$ (unique apart from a constant) of the Neumann problem

$$-\Delta\varphi = 0 \quad \text{in} \quad \Omega ,$$

$$\frac{\partial\varphi}{\partial n} = n\cdot q' \quad \text{on} \quad \partial\Omega .$$

Then clearly the function $q = q' - \mathrm{grad}\, \varphi$ is from Q (see (2.9)) and q is also a stream function to ψ, that is

$$\psi = \mathrm{rot}\, q ,$$

(cf. [2], [20]). However, by [13, p. 310], the stream function $q \in Q$ is still not unique when Ω is multiply connected (e.g. toroidal). Therefore, we use the orthogonal decomposition

$$Q = S \oplus \mathscr{H}_{\mathscr{D}} \tag{3.5}$$

with respect to the scalar product $(\cdot, \cdot)_0 + (\mathrm{rot}\cdot, \mathrm{rot}\cdot)_0$. Thus by (3.5) we get $q =$

$= s + h$, where $s \in S$ and $h \in \mathscr{H}_{\mathscr{D}}$, and from the definition of $\mathscr{H}_{\mathscr{D}}$ we obtain

$$\psi = \operatorname{rot} s .$$

Further, let us suppose that $\psi = \operatorname{rot} s^1 = \operatorname{rot} s^2$ for some $s^1, s^2 \in S$. Then $s^1 - s^2 \in S \cap \mathscr{H}_{\mathscr{D}}$ and (3.5) yields that $s^1 - s^2 = 0$, i.e. the stream function s to ψ exists unique in S. Consequently, by (2.11) the liner operator

$$\operatorname{rot}\colon S \to H(\operatorname{div}^0; \Omega) \tag{3.6}$$

is a one-to-one mapping. It is obvious that S with the norm (2.4) and $H(\operatorname{div}^0; \Omega)$ with the $\|\cdot\|_0$-norm are Banach spaces. As the operator (3.6) is continuous, that is

$$\|\operatorname{rot} s\|_0 \leqq C\|s\|_{H(\operatorname{rot};\, \Omega)} ,$$

we conclude by the Theorem on Isomorphism (see [10, p. 216]) that the inverse operator is continuous, too. Thus

$$\|s\|_0 \leqq \|s\|_{H(\operatorname{rot};\, \Omega)} \leqq c\|\operatorname{rot} s\|_0 , \quad \blacksquare$$

The next lemma is known (see [12], [17], [21]) for Ω convex or $\partial\Omega$ smooth, and under some restrictions upon σ.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary. Then there exists a constant $C > 0$ such that*

$$\|v\|_0 \leqq C(\|\operatorname{div} \sigma v\|_0 + \|\operatorname{rot} v\|_0) \quad \forall v \in V = H(\operatorname{div} \sigma; \Omega) \cap H_0(\operatorname{rot}; \Omega) \tag{3.7}$$

if and only if $\partial\Omega$ is connected.

Proof. Let the set $\partial\Omega$ be not connected and let Γ be one of its components. Consider the weak solution $z \in H^1(\Omega)$ of the problem

$$\begin{aligned} -\operatorname{div}(\sigma \operatorname{grad} z) &= 0 \quad \text{in} \quad \Omega ,\\ z &= 1 \quad \text{on} \quad \Gamma ,\\ z &= 0 \quad \text{on} \quad \partial\Omega\backslash\Gamma . \end{aligned}$$

Then we easily find that $v = \operatorname{grad} z \in H(\operatorname{div}^0 \sigma; \Omega) \cap H_0(\operatorname{rot}^0; \Omega)$, whereas $\|v\|_0 \neq 0$, i.e., (3.7) cannot hold.

Conversely, let $\partial\Omega$ be connected, let $v \in V$ be arbitrary and let $z \in H_0^1(\Omega)$ be a weak solution of the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(\sigma \operatorname{grad} z) &= -\operatorname{div} \sigma v \quad \text{in} \quad \Omega ,\\ z &= 0 \qquad\qquad \text{on} \quad \partial\Omega . \end{aligned} \tag{3.8}$$

Thus

$$\|z\|_1 \leqq C_1\|\operatorname{div} \sigma v\|_0 . \tag{3.9}$$

From (2.2) and (2.10) we see that $\operatorname{grad} z \in H(\operatorname{div} \sigma; \Omega) \cap H_0(\operatorname{rot}^0; \Omega)$, i.e., by (3.8) we have

$$w = v - \operatorname{grad} z \in H(\operatorname{div}^0 \sigma; \Omega) \cap H_0(\operatorname{rot}; \Omega)\,. \tag{3.10}$$

Since $\sigma w \in H(\operatorname{div}^0; \Omega)$, by Lemma 3.2 there exists a stream function $s \in S$ such that

$$\sigma w = \operatorname{rot} s\,. \tag{3.11}$$

By (3.11), (2.1), (2.6) and (3.4) we come to

$$\begin{aligned} \|w\|_0^2 &= \|\sigma^{-1} \operatorname{rot} s\|_0^2 \leqq m\|\sigma^{-1/2} \operatorname{rot} s\|_0^2 = \\ &= m(\sigma^{-1/2} \operatorname{rot} s, \sigma^{-1/2} \operatorname{rot} s)_0 = m(\operatorname{rot} s, w)_0 = \\ &= m(s, \operatorname{rot} w)_0 \leqq m\|s\|_0 \|\operatorname{rot} w\|_0 = \\ &= mc\|\operatorname{rot} s\|_0 \|\operatorname{rot} w\|_0 \leqq C_2\|\sigma^{-1} \operatorname{rot} s\|_0 \|\operatorname{rot} w\|_0 = C_2\|w\|_0 \|\operatorname{rot} w\|_0\,, \end{aligned} \tag{3.12}$$

where $\sigma^{-1/2}$ denotes the square root of the real positive definite matrix σ^{-1}. Thus from (3.10), (3.9), (3.12) and (2.9) we get

$$\begin{aligned} \|v\|_0 &\leqq \|\operatorname{grad} z\|_0 + \|w\|_0 \leqq C_1\| \operatorname{div} \sigma v\|_0 + C_2\|\operatorname{rot} w\|_0 \\ &\leqq C(\|\operatorname{div} \sigma v\|_0 + \|\operatorname{rot} v\|_0)\,, \end{aligned}$$

whence the result as required. ∎

Proof of Theorem 3.1. It is easy to show that V is a Hilbert space with the scalar product

$$(v, w)_\Omega = (\operatorname{div} \sigma v, \operatorname{div} \sigma w)_0 + (\operatorname{rot} v, \operatorname{rot} w)_0 + (v, w)_0\,, \quad v, w \in V\,,$$

for which $|||v|||_\Omega^2 = (v, v)_\Omega$, $v \in V$. Employing (3.7) to (3.2), we arrive at

$$\begin{aligned} |a(v, v)| &\geqq \operatorname{Re} a(v, v) = \|\operatorname{div} \sigma v\|_0^2 + \|\operatorname{rot} v\|_0^2 \geqq \\ &\geqq \frac{1}{4C^2} \|v\|_0^2 + \tfrac{1}{2}\|\operatorname{div} \sigma v\|_0^2 + \tfrac{1}{2}\|\operatorname{rot} v\|_0^2 \geqq c|||v|||_\Omega^2 \quad (c > 0) \end{aligned} \tag{3.13}$$

for any $v \in V$, i.e., the sequilinear form $a(\cdot, \cdot)$ is V-elliptic. We immediately see that $a(\cdot, \cdot)$ is continuous, $|a(v, w)| \leqq C|||v|||_\Omega\, |||w|||_\Omega$ for all $v, w \in V$, and that the linear form $b(\cdot)$ is also continuous when $\tilde{E} \in H(\operatorname{div} \sigma; \Omega) \cap H(\operatorname{rot}; \Omega)$. Thus the rest of the proof follows from the Lax-Milgram lemma (see [15, p. 38]). ∎

Remark 3.4. Let u, $\tilde{E}$ and σ satisfy the regularity assumptions stated just after (1.6) and let $u \in V$ satisfy (3.1). Then u satisfies all the equations of (1.6). The fulfilment of the condition $n \times u = 0$ on $\partial\Omega$ is obvious, since $u \in V \subset H_0(\operatorname{rot}; \Omega)$. Next we show that

$$\operatorname{div} \sigma u = -\operatorname{div} \sigma \tilde{E} \quad \text{in} \quad \Omega\,. \tag{3.14}$$

So let $f \in L^2(\Omega)$ be arbitrary and let us consider the Dirichlet problem

$$(3.15) \qquad -\operatorname{div}\sigma \operatorname{grad} z - i\omega\mu z = -f \quad \text{in} \quad \Omega\,, \qquad z = 0 \quad \text{on} \quad \partial\Omega\,.$$

By (2.1) and Friedrichs' inequality [15, p. 20] the associated sesquilinear form is $H_0^1(\Omega)$-elliptic as

$$|(\operatorname{grad} z, \sigma \operatorname{grad} z)_0 - i\omega\mu(z, z)_0| \geqq (\operatorname{grad} z, \sigma \operatorname{grad} z)_0 \geqq$$
$$\geqq m(\operatorname{grad} z, \operatorname{grad} z)_0 \geqq C\|z\|_1^2 \quad \forall z \in H_0^1(\Omega)\,,$$

where $C > 0$ is independent of z. Hence, there exists a weak solution $z \in H_0^1(\Omega)$ of the problem (3.15). Setting $v = \operatorname{grad} z$, we find by (2.10) and (3.15) that $v \in V$. Consequently, from (3.1), (3.2), (3.3), (2.9), (2.2) and (3.15) we get

$$0 = a(v, u) - b(v) = (\operatorname{div}\sigma v, \operatorname{div}\sigma(u + \tilde{E}))_0 - i\omega\mu(\operatorname{grad} z, \sigma(u + \tilde{E}))_0 =$$
$$= (\operatorname{div}\sigma \operatorname{grad} z, \operatorname{div}\sigma(u + \tilde{E}))_0 + i\omega\mu(z, \operatorname{div}\sigma(u + \tilde{E}))_0 =$$
$$= (f, \operatorname{div}\sigma(u + \tilde{E}))_0\,.$$

Thus (3.14) is valid.

Further, let $v \in (\mathscr{D}(\Omega))^3$ be arbitrary. We see that $\operatorname{div}\sigma v \in C(\bar{\Omega}) \subset L^2(\Omega)$ and thus $v \in V$. Due to (2.6), (3.2), (3.1), (3.14) and (3.3) we obtain

$$(v, \operatorname{rot}\operatorname{rot} u + i\omega\mu\sigma u)_0 = (\operatorname{rot} v, \operatorname{rot} u)_0 - i\omega\mu(v, \sigma u)_0 =$$
$$= a(v, u) - (\operatorname{div}\sigma v, \operatorname{div}\sigma u)_0 = b(v) + (\operatorname{div}\sigma v, \operatorname{div}\sigma\tilde{E})_0 =$$
$$= -(\operatorname{rot} v, \operatorname{rot}\tilde{E})_0 + i\omega\mu(v, \sigma\tilde{E})_0 = -(v, \operatorname{rot}\operatorname{rot}\tilde{E} + i\omega\mu\sigma\tilde{E})_0\,.$$

Since $(\mathscr{D}(\Omega))^3$ is dense in $(L^2(\Omega))^3$, the first equation of (1.6) holds.

Remark 3.5. Let $\sigma = 0$ in some subdomain $\Omega_0 \neq \emptyset$ of Ω (i.e., (2.1) is not valid). Then the problem (1.6) is not uniquely solvable. Putting $E_0 = \operatorname{grad}\varphi$ for some $\varphi \in \{z \in \mathscr{D}(\Omega) \mid z = 0 \text{ in } \Omega \setminus \Omega_0\}$, $\varphi \neq 0$, we find that

$$\operatorname{rot}\operatorname{rot} E_0 + i\omega\mu\sigma E_0 = 0 \quad \text{in} \quad \Omega\,,$$
$$\operatorname{div}\sigma E_0 = 0 \quad \text{in} \quad \Omega\,,$$
$$n \times E_0 = 0 \quad \text{on} \quad \partial\Omega\,.$$

For such a case we refer to the paper of Colton and Päivärinta [5].

Remark 3.6. Lemma 3.3 may be also applied to get the unique solvability of the stationary problem

$$(3.16) \qquad \begin{aligned} \operatorname{div}\sigma u &= F \quad \text{in} \quad \Omega\,, \\ \operatorname{rot} u &= G \quad \text{in} \quad \Omega\,, \\ n \times u &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

The associated sesquilinear form

$$A(v, w) = (\mathrm{div}\, \sigma v, \mathrm{div}\, \sigma w)_0 + (\mathrm{rot}\, v, \mathrm{rot}\, w)_0\,, \quad v, w \in V\,,$$

occurring in the variational formulation of (3.16), is V-elliptic due to Lemma 3.3. Let us further mention that Theorem 3.1 remains valid also when $\omega = 0$.

4. FE-APPROXIMATION OF 3D-PROBLEM

Let Ω be a bounded polyhedral domain with a Lipschitz boundary, $\mathscr{T}_h$ a decomposition of $\bar{\Omega}$ into elements (tetrahedra, prisma, parallelepipeds, etc.) in the usual sense (see [4]), i.e., any face of any $K \in \mathscr{T}_h$ is either a subset of $\partial\Omega$, or a face of another $K' \in \mathscr{T}_h$. Let

$$W_h = \{v_h \in (L^2(\Omega))^3 \mid v_h|_K \in (P_K)^3 \quad \forall K \in \mathscr{T}_h\}\,,$$

where P_K is a space of complex-valued polynomials such that $P_K \supseteq P_1(K)$. A discrete analogue of the problem (3.1) consists in finding $u_h \in V_h$ such that

$$a(v_h, u_h) = b(v_h) \quad v_h \in V_h\,, \tag{4.1}$$

where $V_h = V \cap W_h$. From the V-ellipticity of $a(\cdot, \cdot)$ we see that u_h exists unique if $V_h \neq \emptyset$. First we establish some properties of the space V_h.

Definition 4.1. *Let $v_h \in W_h$. Then the tangential components of v_h are said to be continuous at element interfaces if for any two adjacent elements $K_1, K_2 \in \mathscr{T}_h$ we have*

$$\nu \times v_h|_{K_1} = \nu \times v_h|_{K_2} \quad on \quad S = K_1 \cap K_2\,, \tag{4.2}$$

where ν is a normal to S.

Lemma 4.2. *If $v_h \in V_h$ then the tangential components of v_h are continuous at element interfaces.*

Proof. Let $K_1, K_2 \in \mathscr{T}_h$ be adjacent and let $v_h \in V_h \subset H(\mathrm{rot}; \Omega)$ be arbitrary. Using the Green formula (2.5) for $z \in (\mathscr{D}(K_1 \cup K_2))^3$, we get

$$\int_{K_j} (z\, \mathrm{rot}\, v_h - v_h\, \mathrm{rot}\, z)\, \mathrm{d}x = \int_S (n_{K_j} \times v_h|_{K_j})\, z\, \mathrm{d}s\,, \quad j = 1, 2\,, \tag{4.3}$$

where n_{K_j} denotes the outward unit normal to ∂K_j. Summing (4.3) over $j = 1, 2$, we find by (2.5) that

$$0 = \int_S (n_{K_1} \times v_h|_{K_1} - n_{K_1} \times v_h|_{K_2})\, z\, \mathrm{d}s\,,$$

which holds particularly for all $z \in (\mathscr{D}(S))^3$. Hence (4.2) is valid. ∎

To the end of this section we shall moreover assume that the first derivatives of σ exist and that they are continuous over each element $K \in \mathscr{T}_h$, i.e., for any $v_h \in W_h$ the divergence of σv_h exists on each $K \in \mathscr{T}_h$ in the classical sense.

Definition 4.3. *Let $v_h \in W_h$. Then the normal component of σv_h is said to be continuous at element interfaces if for any two adjacent elements $K_1, K_2 \in \mathscr{T}_h$ we have*

$$(4.4) \qquad \nu \cdot (\sigma v_h)|_{K_1} = \nu \cdot (\sigma v_h)|_{K_2} \quad \text{on} \quad S = K_1 \cap K_2 ,$$

where ν is a normal to S.

Lemma 4.4. *If $v_h \in V_h$ then the normal component of σv_h is continuous at element interfaces.*

The proof is analogous to that of Lemma 4.3 (now the Green formula (2.2) has to be used).

Note that when σ is e.g. a diagonal matrix and $\sigma_{11} = \sigma_{22} = \sigma_{33} \in C^1(\bar{\Omega})$, then any $v_h \in V_h$ is continuous on the whole $\bar{\Omega}$ due to Lemmas 4.2 and 4.4.

Lemma 4.5. *Let* (4.2) *and* (4.4) *be fulfilled for some $v_h \in W_h$ and let $n \times v_h = 0$ on $\partial\Omega$. Then $v_h \in V_h$.*

Proof. We show that $v_h \in V = H_0(\text{rot}; \Omega) \cap H(\text{div}\, \sigma; \Omega)$. Let $G \in (L^2(\Omega))^3$ be defined through the relation

$$G|_K = \text{rot}\, v_h|_K , \quad K \in \mathscr{T}_h .$$

Denoting by n_K the outward unit normal to ∂K, we obtain by (2.5)

$$\int_\Omega v_h \,\text{rot}\, z \,dx = \sum_{K \in \mathscr{T}_h} \int_K v_h \,\text{rot}\, z \,dx = \sum_{K \in \mathscr{T}_h} \left(\int_K z \,\text{rot}\, v_h \,dx - \int_{\partial K} (n_K \times v_h)\, z \,ds\right) =$$

$$= \int_\Omega Gz \,dx \quad \forall z \in (\mathscr{D}(\Omega))^3 ,$$

where the sum of the boundary integral vanishes as $n_{K_1} + n_{K_2} = 0$ on $S = K_1 \cap K_2$ for adjacent elements $K_1, K_2 \in \mathscr{T}_h$. Hence $G = \text{rot}\, v_h$ and $v_h \in H_0(\text{rot}; \Omega)$ because of the assumption $n \times v_h = 0$ on $\partial\Omega$.

Defining $F \in L^2(\Omega)$ by

$$F|_K = \text{div}\, \sigma v_h|_K , \quad K \in \mathscr{T}_h ,$$

we get by (2.2)

$$\int_\Omega (\sigma v_h) \,\text{grad}\, z \,dx = \sum_{K \in \mathscr{T}_h} \int_K (\sigma v_h) \,\text{grad}\, z \,dx =$$

$$= \sum_{K \in \mathscr{T}_h} \left(- \int_K z \,\text{div}\, \sigma v_h \,dx + \int_{\partial K} n_K \cdot (\sigma v_h)\, z \,ds\right) = - \int_\Omega Fz \,dx \quad \forall z \in \mathscr{D}(\Omega) ,$$

i.e. $v_h \in H(\text{div}\, \sigma; \Omega)$. ∎

Remark 4.6. Let us consider the problem (4.1) in case of linear tetrahedral elements, i.e.

$$(4.5) \qquad P_K = P_1(K) \quad \forall K \in \mathscr{T}_h .$$

We will briefly outline that under certain assumptions the rate of convergence takes the form

$$(4.6) \qquad |||u - u_h|||_\Omega = O(h) \quad \text{as} \quad h \to 0 .$$

We shall suppose that σ is piecewise constant; more precisely, let there exist mutually disjoint polyhedral domains $\Omega_j, j = 1, \ldots, r$, such that

$$\bar{\Omega} = \bigcup_{j=1}^{r} \bar{\Omega}_j$$

and

$$\sigma|_{\Omega_j} \in (P_0(\bar{\Omega}_j))^{3\times 3} \quad \forall j .$$

Then evidently

$$\|\operatorname{div} \sigma v\|_{0,\Omega_j} \leqq C\|v\|_{1,\Omega_j} \quad \forall v \in (H^1(\Omega_j))^3 ,$$

and thus

$$\|\!|\cdot|\!\|_{\Omega_j} \leqq C'\|\cdot\|_{1,\Omega_j} , \tag{4.7}$$

where $C, C' > 0$ do not depend on j.

We shall consider only such decompositions $\mathcal{T}_h$ into tetrahedra for which

$$\partial\Omega_j \cap \operatorname{int} K = \emptyset \quad \forall K \in \mathcal{T}_h . \tag{4.8}$$

Further, let $\{\mathcal{T}_h\}_{h\to 0}$ be a regular family of decompositions of $\bar{\Omega}$ into tetrahedra (i.e., there is a constant $\varkappa > 0$ such that for any decomposition $\mathcal{T}_h$ from this family and for any tetrahedron $K \in \mathcal{T}_h$ there exists a ball B_K with radius ϱ_K such that $B_K \subset K$ and $\varkappa \operatorname{diam} K \leqq \varrho_K$).

Moreover, we will require the following piecewise regularity of $u \in V$:

$$u|_{\Omega_j} \in (H^2(\Omega_j))^3 , \quad j = 1, \ldots, r . \tag{4.9}$$

Note that u need not be continuous across $\partial\Omega_j$, since σ may possess jumps and the normal component of σu has to be continuous at $\partial\Omega_j$. In case of (4.9) it is well-known that

$$\|u - \Pi_h^j u\|_{1,\Omega_j} \leqq Ch\|u\|_{2,\Omega_j} , \quad j = 1, \ldots, r , \tag{4.10}$$

where $\Pi_h^j u \in (H^1(\Omega_j))^3$ is the standard linear interpolant of $u|_{\Omega_j}$.

Let $v_h \in W_h$ be defined as

$$v_h = \Pi_h^j u \quad \text{on} \quad \Omega_j , \quad j = 1, \ldots, r . \tag{4.11}$$

We show now that $v_h \in V_h$. Since $\Pi_h^j u = u$ at the vertices of each tetrahedron $K \subset \subset \bar{\Omega}_j, j \in \{1, \ldots, r\}$, the condition $n \times u = 0$ on any face $S \subset \partial\Omega$ and the relation (4.5) imply $n \times \Pi_h^j u = 0$ on S, that is $n \times v_h = 0$ on $\partial\Omega$. According to the Sobolev imbedding $H^2(K) \hookrightarrow C(K)$, (4.9) and (4.8), u is continuous on every $K \in \mathcal{T}_h$. Thus analogously to the proof of Lemma 4.2 we obtain that

$$v \times u|_{K_1} = v \times u|_{K_2} \quad \text{on} \quad S = K_1 \cap K_2$$

for any two adjacent tetrahedra. $K_1, K_2 \in \mathcal{T}_h$. This equality is valid particularly at the vertices of S and thus from (4.11) it is easy to see that (4.2) holds. The relation (4.4) may be derived in a similar way, since $\sigma|_K$ is constant for every $K \in \mathcal{T}_h$. Therefore, $v_h \in V_h$ by Lemma 4.5.

In the sequel we slightly modify the proof of Cea's lemma (see [4, p. 104]) to the complex case. As $u_h - v_h \in V_h$, we get from (3.1) and (4.1) that

$$a(u_h - v_h, u - u_h) = 0 . \tag{4.12}$$

Now using the V-ellipticity condition (3.13), (4.12) and the continuity of $a(\cdot, \cdot)$, we find that

$$\begin{aligned} c|||u - u_h|||_\Omega^2 &\leqq |a(u - u_h, u - u_h)| \leqq \\ &\leqq |a(u - v_h, u - u_h)| \leqq C|||u - v_h|||_\Omega \, |||u - u_h|||_\Omega . \end{aligned} \tag{4.13}$$

Hence, by (4.13), (4.11), (4.7) and (4.10) we come to

$$\begin{aligned} |||u - u_h|||_\Omega^2 &\leqq \tilde{C}|||u - v_h|||_\Omega^2 = \tilde{C}\sum_{j=1}^{r} |||u - \Pi_h^j u|||_{\Omega_j}^2 \\ &\leqq \bar{C}\sum_{j=1}^{r} \|u - \Pi_h^j u\|_{1,\Omega_j}^2 = O(h^2) , \end{aligned}$$

whence (4.6) as required.

5. SOME REMARKS TO 2D-PROBLEM

A two-dimensional analogue of (1.6) is much simpler. It has been investigated by many authors (see e.g. the reference list in [1]). For error estimates for the FEM to Maxwell-type boundary value problems see [16]. Here we confine ourselfs only to several notes and recommendations.

Let $D \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary and $\alpha, \beta \in \mathbb{R}^1$, $\alpha < \beta$. Assume that E (see (1.3)) defined in $\Omega = D \times (\alpha, \beta)$ is polarized, i.e. $E = (0, 0, E_3)$, where E_3 is independent of $x_3 \in (\alpha, \beta)$. Let σ be diagonal and let σ_{33} be also independent of x_3. Then $\operatorname{div} \sigma E = 0$ is trivially fulfilled and we will write for simplicity only σ instead of σ_{33}. Putting $e(x_1, x_2) = E_3(x_1, x_2, x_3)$ for $(x_1, x_2) \in D$, the equation (1.4) reduces by (2.3) to the Helmholtz equation

$$-\Delta e + i\omega\mu\sigma e = 0 \quad \text{in } D , \tag{5.1}$$

where the conductivity $\sigma \geqq 0$ belongs to $L^\infty(D)$. Contrary to the three-dimensional case, we admit $\sigma = 0$ in some subdomain of D. (For instance, the electric conductivity of the air is zero.)

Assume that $\tilde{E} = (0, 0, \tilde{E}_3)$ and $\tilde{e}(x_1, x_2) = \tilde{E}_3(x_1, x_2, x_3)$, where $\tilde{E}_3$ does not depend on $x_3 \in (\alpha, \beta)$. Then (1.5) reduces to

$$0 = n \times (E - \tilde{E}) = (n_2 E_3, -n_1 E_3, 0) = (n_2, -n_1, 0)(E_3 - \tilde{E}_3)$$

which yields $E_3 = \tilde{E}_3$ on ∂D. Hence, we can consider the Dirichlet boundary condition

$$e = \tilde{e} \quad \text{on} \quad \partial D , \tag{5.2}$$

where $\tilde{e} \in H^1(D)$ is a given function. A variational formulation of (5.1), (5.2) consists in finding $u = e - \tilde{e} \in H_0^1(D)$ such that

$$a(v, u) = b(v) \quad \forall v \in H_0^1(D)\,, \tag{5.3}$$

where

$$a(v, u) = (\operatorname{grad} v, \operatorname{grad} u)_0 - i\omega\mu(v, \sigma u)_0\,, \tag{5.4}$$

$$b(v) = -(\operatorname{grad} v, \operatorname{grad} \tilde{e})_0 + i\omega\mu(v, \sigma\tilde{e})_0\,. \tag{5.5}$$

The next lemma is in fact standard, but we shall need it in the proof of Lemm a 5.2

Lemma 5.1. *The problem* (5.3) *has a unique solution.*

Proof. The form (5.5) is evidently linear and continuous on $H_0^1(D)$, and (5.4) is a sesquilinear and continuous form on $H_0^1(D) \times H_0^1(D)$. Moreover, $a(\cdot, \cdot)$ is $H_0^1(D)$-elliptic:

$$|a(v, v)| \geqq \operatorname{Re} a(v, v) = \|\operatorname{grad} v\|_0^2 \geqq C\|v\|_1^2 \quad \forall v \in H_0^1(D)\,,$$

where the last inequality is Friedrichs' inequality [15, p. 20] with $C > 0$. Now the rest of the proof follows from the Lax-Milgram lemma [15, p. 38]. ∎

Lemma 5.2. *Let D be convex and $\tilde{e} \in H^2(D)$. Then $u \in H^2(D)$.*

Proof. Using the fact that $e = u + \tilde{e}$, we rewrite (5.1) into the form

$$-\Delta u + i\omega\mu\sigma u = f\,,$$

where $f = \Delta\tilde{e} - i\omega\mu\sigma\tilde{e}$ belongs to $L^2(\Omega)$. Thus for the real and imaginary parts we obtain the system

$$\begin{aligned} -\Delta u_1 &= \omega\mu\sigma u_2 + f_1\,, \\ -\Delta u_2 &= -\omega\mu\sigma u_1 + f_2\,. \end{aligned} \tag{5.6}$$

Since $u \in H_0^1(\Omega)$ due to Lemma 5.1, we get that

$$\|\sigma u\|_0^2 \leqq \operatorname*{vraimax}_{x \in D} \sigma^2(x)\, \|u\|_0^2 \leqq C\|u\|_1^2 < \infty\,,$$

i.e. $\sigma u \in L^2(\Omega)$. From (5.6) we see that u_i satisfies the Poisson equation with a square integrable right-hand side. As D is convex and $u_1 = u_2 = 0$ on ∂D, the lemma follows from [11]. ∎

Remark 5.3. In contrast to the 3D-problem, the solution u from Lemma 5.2 is by the Sobolev imbedding theorem continuous even when σ has jumps (cf. (4.9)). Suppose again that $\tilde{e} \in H^2(D)$. If D is a rectangle and Γ is the union of its two opposite sides, then $u \in H^2(\Omega)$ also for the following mixed boundary conditions (see [8])

$$e = \tilde{e} \quad \text{on} \quad \partial D \setminus \Gamma\,,$$

$$\frac{\partial e}{\partial n} = \frac{\partial \tilde{e}}{\partial n} \quad \text{on} \quad \Gamma\,.$$

Remark 5.4. The H^2-regularity enables us to achieve the $O(h)$-convergence of conforming finite element methods in the $H^1(D)$-norm. Moreover, when $e \in H^2(D)$ one may use C^1-elements for approximations. Then by (1.3) the corresponding approximation of $H = (H_1, H_2, 0)$ (which is sometimes more interesting than E_3) will be continuous.

Numerical test 5.5. We have recomputed a geophysical model example from [3, p. 384], where (cf. (5.1)) $\omega = 2\pi \,.\, 10^{-1}\,[s^{-1}]$, $\mu = 4\pi \,.\, 10^{-7}\,[Hm^{-1}]$, $\sigma_0 = 10^{-3}\,[\Omega^{-1}\,m^{-1}]$, $D = (0,520) \times (-226, 200)$,

$$\sigma(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_2 > 0\,, \\ 100\sigma_0 & \text{if} \quad (x_1, x_2) \in \tilde{D} = (240, 280) \times (-16, -6)\,, \\ \sigma_0 & \text{if} \quad x_2 \leqq 0 \quad \text{and} \quad (x_1, x_2) \in D - \tilde{D}\,, \end{cases}$$

and the values characterizing D and $\tilde{D}$ are given in [km]. The boundary condition (5.2) is defined through the function

$$\tilde{e}(x_1, x_2) = \begin{cases} \alpha x_2 + 1 + i\alpha x_2 & \text{if} \quad x_2 > 0\,, \\ (\cos(\alpha x_2) + i \sin(\alpha x_2)) \exp(\alpha x_2) & \text{if} \quad x_2 \leqq 0\,, \end{cases}$$

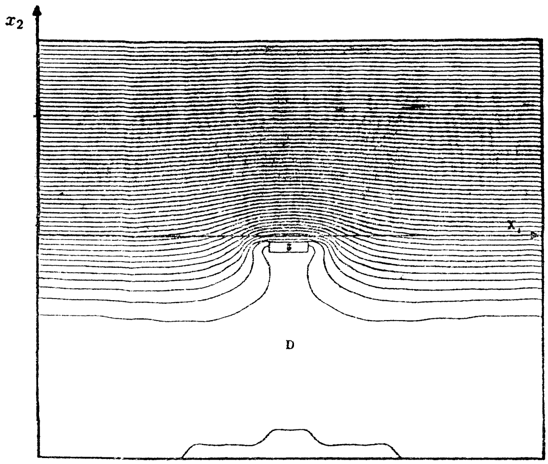
where $\alpha = \sqrt{(\omega\mu\sigma_0/2)}$, i.e., $\tilde{e}$ has been taken so that $\tilde{e} \in H^2(D)$, $\tilde{e}(x_1, 0) = 1$, $\tilde{e}(x_1, x_2) \to 0$ as $x_2 \to -\infty$ and $\tilde{e}$ fulfils the equation

$$-\Delta\tilde{e} + i\omega\mu\sigma_0(\tfrac{1}{2} - \tfrac{1}{2}\operatorname{sign} x_2)\,\tilde{e} = 0 \quad \text{in} \quad D\,.$$

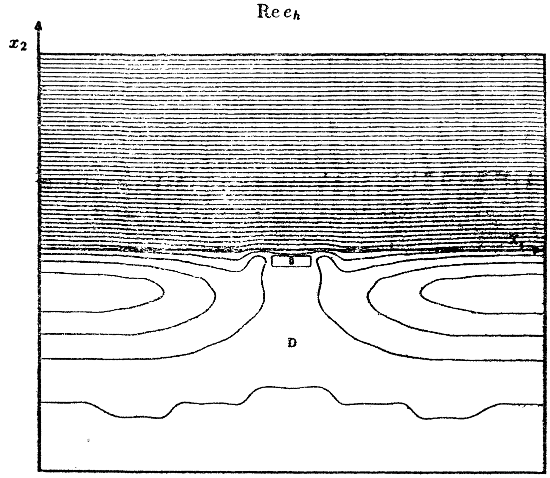
Although σ corresponds to isotropic media, the use of any triangular elements causes "an artificial anisotropy" (especially near the corners of $\tilde{D}$, where σ changes very much). Thus, in this case it is better to employ rectangular elements (which may be of the class C^1 according to Remark 5.4). We have developed a composed rectangular C^1-element which is briefly described in Remark 5.6 below. Its use is compared in Table 5.1 with the standard 5-point finite difference method (FDM) from [3] on the same grid (38×38). (The values of the two last rows in Table 5.1 were taken from a graph in [3].) The real and imaginary parts of the finite element solution e_h are illustrated in Figure 5.1.

Table 5.1.

	x_1 in [km]	0	50	100	150	200	250	260
FEM	Re $e_h(x_1, 0)$	1·000	·998	·986	·926	·688	·294	·293
	Im $e_h(x_1, 0)$	·016	·022	·047	·097	·130	·082	·084
FDM	Re $e_h(x_1, 0)$	—	—	·99	·95	·79	·36	·34
	Im $e_h(x_1, 0)$	—	—	·04	·07	·12	·06	·06



Re e_h



Im e_h

Fig. 5.1.

The discrete problem reads

$$\sum_{k=1}^{N} a(v^j, v^k)\, e^k = b(v^j)\,, \quad j = 1, \ldots, N\,, \tag{5.7}$$

where $\{v^j\}_{j=1}^N$ is a basis of $V_h \subset H_0^1(D)$. Note that the complex stiffness matrix $\{a(v^j, v^k)\}_{j,k=1}^N$ from (5.7) is never Hermitian, but it is symmetric provided v^j are real. The necessary and sufficient condition for the convergence of the conjugate gradient method (see [19, p. 203]) is unfortunately not satisfied. Thus we have employed the Gauss-Seidel iterative method which converges in our case (see [18, p. 73]) and enables us to store only non-zero diagonals of the half band of the stiffness matrix. To economize the computer memory it is better to use complex arithmetics for (5.7) than to solve a discrete analogue of the real system (5.6) with the Dirichlet boundary conditions. In the complex case, not only the number of equations is twice smaller than in the real case, but the half bandwidth of the stiffness matrix (and the number of non-zero diagonals) is smaller, too.

Remark 5.6. In [9], Heindl has presented a triangular composed piecewise quadratic C^1-element with only 12 degrees of freedom. (Note that the number of degrees of freedom of any noncomposed C^1-element is greater – see [4].) Here we will introduce a rectangular composed piecewise biquadratic C^1-element (K, P_K, Σ_K), where for simplicity K is the unit square with vertices A_1, A_2, A_3, A_4 – see Figure 5.2.

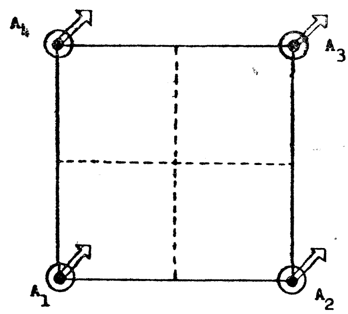


Fig. 5.2.

First of all let us define two piecewise quadratic function $z_1, z_2 \in C^1([0, 1])$,

$$
(5.8) \qquad z_1(x) = \begin{cases} -2x^2 + 1\,, & x \in [0, \frac{1}{2}]\,, \\ 2x^2 - 4x + 2\,, & x \in [\frac{1}{2}, 1]\,, \end{cases}
$$

$$
z_2(x) = \begin{cases} -\frac{3}{2}x^2 + x\,, & x \in [0, \frac{1}{2}]\,, \\ \frac{1}{2}x^2 - x + \frac{1}{2}\,, & x \in [\frac{1}{2}, 1]\,, \end{cases}
$$

for which clearly

$$
(5.9) \qquad z_1(0) = \frac{\partial z_2}{\partial x}(0) = 1\,,
$$

$$
\frac{\partial z_1}{\partial x}(0) = z_1(1) = \frac{\partial z_1}{\partial x}(1) = z_2(0) = z_2(1) = \frac{\partial z_2}{\partial x}(1) = 0.
$$

Further, we set

$$
(5.10) \qquad \begin{aligned} z_3(x) &= z_1(1-x), \quad x \in [0,1], \\ z_4(x) &= z_2(1-x), \quad x \in [0,1]. \end{aligned}
$$

Note that the functions $z_1, \ldots, z_4$ form a basis of the one-dimensional C^1-element analogous to the proposed rectangular element.

Now we define the space P_K of piecewise biquadratic C^1-functions as the linear span (with complex coefficients) of linearly independent functions $p_1, \ldots, p_{16}$ given by

$$
(5.11) \qquad \begin{aligned} & p_{4j+k-4}(x_1, x_2) = z_j(x_1)\, z_k(x_2), \\ & j, k = 1, \ldots, 4, \quad (x_1, x_2) \in K. \end{aligned}
$$

The associated set of degrees of freedom $\Sigma_K = \{\Phi_1, \ldots, \Phi_{16}\}$ may be symbolically written as

$$
(5.12) \qquad \Sigma_K = \left\{ p(A_j), \frac{\partial p}{\partial x_1}(A_j), \frac{\partial p}{\partial x_2}(A_j), \frac{\partial^2 p}{\partial x_1\, \partial x_2}(A_j), j = 1, \ldots, 4 \right\}.
$$

Due to (5.9), (5.10) and (5.11) the set Σ_K can be ordered so that

$$
(5.13) \qquad \Phi_r(p_s) = \delta_{rs}, \quad r, s = 1, \ldots, 16,
$$

i.e., Σ_K is P_K-unisolvent.

Notice that

$$
(5.14) \qquad P_2(K) \subset P_K.
$$

To see this, we find from (5.8) and (5.10) that for any $x \in [0, 1]$

$$
\begin{aligned} 1 &= z_1(x) + z_3(x), \\ x &= z_2(x) + z_3(x) - z_4(x), \\ x^2 &= z_3(x) - 2z_4(x), \end{aligned}
$$

i.e., any quadratic function on $[0, 1]$ is a linear combination of $z_1, \ldots, z_4$. Hence, by (5.11) the space P_K contains all biquadratic functions and (5.14) follows.

Given a rectangular grid, we will construct the finite element space V_h in the usual way [4]. The standard basis functions $v^j \in V_h$ (see (5.7)) are generated via the functions p_s occurring in (5.13). The support of v^j thus consists of four rectangles and each nodal point corresponds to four basis functions (cf. (5.12)). Referring to the properties of $z_1, \ldots, z_4$ and (5.11), we can directly verify that any v^j belongs to $C^1(\bar{D})$, i.e., the finite element (K, P_K, Σ_K) is of the class C^1.

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Souhrn

O ČASOVĚ PERIODICKÝCH MAXWELLOVÝCH ROVNICÍCH S NEHOMOGENNÍMI VODIVOSTMI: ŘEŠITELNOST A APROXIMACE METODOU KONEČNÝCH PRVKŮ

MICHAL KŘÍŽEK, PEKKA NEITTAANMÄKI

V práci se vyšetřují časově periodické Maxwellovy rovnice v trojrozměrném případě s nehomogenními vodivostmi. Navíc se předkládá aproximace metodou konečných prvků v trojrozměrném prostoru spolu s odhadem chyby v energetické normě. Několik poznámek se týká též dvojrozměrného problému, který vzniká v geofyzice.

Резюме

О ПЕРИОДИЧЕСКИХ ВО ВРЕМЕНИ УРАВНЕНИЯХ МАКСВЕЛЛА С НЕОДНОРОДНЫМИ ПРОВОДИМОСТЯМИ: РАЗРЕШИМОСТЬ И АППРОКСИМАЦИЯ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

MICHAL KŘÍŽEK, PEKKA NEITTAANMÄKI

В работе рассматривается вариационная формулировка и разрешимость периодических во времени уравнений Максвелла в трехмерном случае с неоднородными проводимостями. Кроме того предлагается аппроксимация методом конечных элементов в трехмерном пространстве вместе с оценкой погрешности в энергетической норме. Несколько замечаний касается также двумерной задачи, которая возникает в геофизике.

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