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SOLUTION OF A MATHEMATICAL MODEL OF A SINGLE PISTON PUMP WITH A MORE DETAILED DESCRIPTION OF THE VALVE FUNCTION

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Summary. In this paper a mathematical model of a fluid flow in a tube with a valve and a pump is solved. The function of the valve is described in more detail than in [3], thus making the model more complete.

Keywords: fluid flow in a tube, linear hyperbolic equation.

AMS classification: 76N, 35L.

1. INTRODUCTION

The purpose of the present paper is the mathematical treatment of the flow of a fluid in a pipeline with a piston pump. We use a model suggested by V. Kolarčík [1], which will appear in a more general version in [2]. In [3] we have investigated the same problem without taking into account the resistance of the open valve. This new feature is now considered, thus making the mathematical description of a fluid flow in the pipeline more complete. This research has arisen from a co-operation between Mathematical Institute of the Czechoslovak Academy of Sciences and Research Institute of Concern SIGMA – Olomouc, Czechoslovakia. Our paper is divided into five sections. In Sec. 2 the problem is formulated as a boundary value problem for a linearized Euler system of two partial differential equations. This problem is transformed into a more appropriate form in Sec. 3. In Sec. 4 the necessary notation is introduced. Finally, in Sec. 5 the initial boundary value problem from Sec. 3 is solved.

2. FORMULATION OF THE PROBLEM

We consider the same configuration of the elements pump, valve, accumulator, pipeline and tank as in [3]. We also use the same partial differential equations for the pressure $p = p(x, t)$ and the rate of flow $Q = Q(x, t)$ in the pipeline:

$$p_t + \frac{\varrho_0 c_0^2}{F} Q_x = 0\,,$$

(2.1)

$$Q_t + \frac{F}{\varrho_0} p_x + kQ = 0\,, \quad 0 \leqq x \leqq l\,, \quad t \geqq t_1\,.$$

Here ϱ_0 and c_0 are the density and the sound speed of the still medium, respectively, F and l are the cross-section and the length of the pipeline, respectively, t_1 is the initial time (say, when the valve is closed) and $k = \lambda r F_0 / 4 R_h F T$, where λ is the coefficient of the friction of the fluid near the pipeline walls, R_h is the hydraulic radius, r – the radius of the crank, F_0 – the surface of the piston and T is the period of one cycle of the pump. If the valve, placed at $x = 0$, is closed, we use the boundary condition [3]

(2.2) $$C_1\, p_t(0, t) + Q(0, t) = 0\,, \quad t_1 \leqq t < t_2$$

where C_1 is the capacity of the accumulator and t_2 is the time of opening the valve. On the other hand, if the valve is open then we consider the boundary condition

(2.3) $$C_0 R_1 [C_1\, p_{tt}(0, t) + Q_t(0, t)] + (C_0 + C_1)\, p_t(0, t) + Q(0, t) = f_0(t)\,,$$

$$t_2 \leqq t < t_1 + T,$$

where C_0 is the capacity of the working compartment of the pump, R_1 is the resistance of the open valve and $f_0(t)$ is a smooth function characterizing the motion of the piston measured in the volume. The boundary condition (2.3) arises from the balance of mass and momentum between cuts in the pump-valve-accumulator-pipe system. The presence of higher derivatives is caused by the elimination of the redundant quantities and seems to be rather unnatural. This question will be settled in Sec. 3 by an appropriate reformulation of the problem. On the opposite side $x = l$ of the pipeline we suppose a constant pressure, i.e., we set

(2.4) $$p(l, t) = \text{const.},\ t \geqq t_1\,.$$

The equations (2.1)–(2.4) are to be complemented by initial conditions

(2.5) $$p(x, 0) = p_0(x)\,,$$

(2.6) $$Q(x, 0) = Q_0(x)\,, \quad x \in [0, l]\,.$$

In both cases we suppose that the boundary conditions (2.2), (2.3) are prolonged periodically for $t \in [t_1, \infty)$ (or $t \in R = (-\infty, \infty)$ if necessary) assuming that the valve opens any time when $t = t_2 + nT$ and closes at $t = t_1 + nT$, n being an integer. Thus, for (2.2), (2.3) we substitute the boundary condition

(2.7) $$\nu(t)\, C_0 R_1 [C_1\, p_{tt}(0, t) + Q_t(0, t)] +$$

$$+ (\nu(t)\, C_0 + C_1)\, p_t(0, t) + Q(0, t) = f(t),\ t \geqq t_1\,,$$

where

$$v(t) = 0 \quad \text{for} \quad t_1 \leqq t < t_2\,,$$

$$v(t) = 1 \quad t_2 \leqq t < t_1 + T\,,$$

$$v(t) = v(t + T)\,, \quad t \in R$$

and

$$f(t) = v(t) f_0(t)\,. \quad t \in R\,.$$

3. REFORMULATION OF THE PROBLEM

In this section we carry out some arrangements of equations (2.1), (2.4)–(2.7) which enable us to use methods for differential equations in Hilbert spaces. First, excluding from the system (2.1) pressure p we get for Q one equation

(3.1) $$Q_{tt} + 2\gamma Q_t - c_0^2 Q_{xx} = 0\,, \quad x \in [0, l]\,, \quad t \geqq t_1\,,$$

where

$$\gamma = \frac{\lambda r F_0}{8 R_h T F}\,.$$

Secon dly, inserting the first equation in (2.1) into (2.7), (2.4) we find

(3.2) $$v(t)\, C_0 R_1 \left[-C_1 \frac{\varrho_0 c_0^2}{F}\, Q_{xt}(0, t) + Q_t(0,t) \right] -$$

$$- \left(v(t)\, C_0 + C_1\right) \frac{\varrho_0 c_0^2}{F}\, Q_x(0, t) + Q(0, t) = v(t) f_0(t)\,,$$

$$Q_x(l, t) = 0\,, \quad t \geqq t_1\,.$$

From (2.5), (2.6) we get the initial conditions

(3.3) $$Q(x, t_1) = Q_0(x)\,,$$

(3.4) $$Q_t(x, t_1) = -\frac{F}{\varrho_0}\, p_0'(x) - k\, Q_0(x) \equiv Q_1(x)\,, \quad x \in [0, l]\,.$$

If $t \in [t_1 + nT, t_2 + nT)$, n integer, then $v(t) = 0$ and the problem (3.1)–(3.4) is solved by the formulas given in [3]. A more detailed account is given in Sec. 5. Let $t \in [t_2 + nT, t_1 + (n + 1)\, T)$. After obvious arrangements the problem (3.1) to (3.4) can be written as follows:

(3.5) $$u_{tt} + 2\gamma u_t - c_0^2 u_{xx} = 0\,, \quad x \in (0, l)\,,$$

$$t \in (t_2 + nT, t_1 + (n + 1)\, T)$$

$$u_{xt}(0, t) - \alpha_1 u_t(0, t) + \alpha_2 u_x(0, t) - \alpha_3 u(0, t) = f_1(t), \tag{3.6}$$

$$u_x(l, t) = 0, \quad t \in [t_2 + nT, t_1 + (n + 1) T], \tag{3.7}$$

$$u(x, t_2 + nT +) = u(x, t_2 + nT-), \tag{3.8}$$

$$u_t(x, t_2 + nT+) = u_t(x, t_2 + nT-), \quad x \in [0, l]. \tag{3.9}$$

We have put here

$$Q(x, t) = u(x, t), \quad u(x, t_1) = Q_0(x) \equiv u_0(x),$$

$$u_t(x, t_1) = - \frac{F}{\varrho_0} p_0'(x) - k Q_0(x) \equiv u_1(x),$$

$$\alpha_1 = \frac{F}{C_1 \varrho_0 c_0^2}, \quad \alpha_2 = \frac{C_0 + C_1}{C_0 C_1 R_1}, \quad \alpha_3 = \frac{F}{C_0 C_1 R_1 \varrho_0 c_0^2}, \tag{3.10}$$

$$f_1(t) = \frac{F \cdot f_0(t)}{C_0 C_1 R_1 \varrho_0 c_0^2}. \tag{3.11}$$

The conditions (3.8), (3.9) mean that the solution and its derivative with respect to time are continuously connected to the solution on the preceding interval $[t_1 + nT, t_2 + nT]$. The boundary condition (3.6) is not yet in the proper form. It can be written as

$$u_{xt}(0, t) - \alpha_1 u_t(0, t) + \alpha_2[u_x(0, t) - \alpha_1 u(0, t)] + (\alpha_2\alpha_1 - \alpha_3) u(0, t) = f_1(t). \tag{3.12}$$

To be definite, set $n = 0$. As

$$u_{xt}(0, t) - \alpha_1 u_t(0, t) = \frac{d}{dt} [u_x(0, t) - \alpha_1 u(0, t)]$$

we get from (3.12) an ordinary differential equation for $u_x(0, t) - \alpha_1 u(0, t)$, so that (3.12) is equivalent to

$$u_x(0, t) - \alpha_1 u(0, t) = e^{-\alpha_2(t - t_2)}[u_x(0, t_2-) - \alpha_1 u(0, t_2-)] + \int_{t_2}^{t} e^{-\alpha_2(t-\tau)} [(\alpha_3 - \alpha_1\alpha_2) u(0, \tau) + f_1(\tau)] d\tau. \tag{3.13}$$

According to (3.2) we have

$$- C_1 \frac{\varrho_0 c_0^2}{F} u_x(0, t_2-) + u(0, t_2-) = 0,$$

i.e. (cf. (3.10))

$$u_x(0, t_2-) - \alpha_1 u(0, t_2-) = 0 .$$

Thus we have found the final form of the boundary condition (3.2) on the interval $[t_2, t_1 + T]$:

$$u_x(0, t) - \alpha_1 u(0, t) = \int_{t_2}^{t} e^{-\alpha_2(t-\tau)}[(\alpha_3 - \alpha_1\alpha_2) u(0, \tau) + f_1(\tau)] \, d\tau .$$

Generally, instead of (3.6) we now have

$$u_x(0, t) - \alpha_1 u(0, t) = \int_{t_2+nT}^{t} e^{-\alpha_2(t-\tau)}[(\alpha_3 - \alpha_1\alpha_2) u(0, \tau) + f_1(\tau)] \, d\tau , \tag{3.14}$$

$$t \in [t_2 + nT, t_1 + (n + 1) T] , \quad n = 0, 1, \ldots$$

4. NOTATION

Let $I \subset R$ be an interval, $k \geqq 0$ an integer, B a Banach space. Then $C^k(I; B)$ denotes the space of all functions from I into B which are continuous together with their derivatives up to order k with respect to the norm in B. $H^k(I; B)$ is the Sobolev space of functions from I into B which are square integrable together with their all distributional derivatives up to order k, $H^k(I; R) = H^k(I)$, $H^0(I; B) = L^2(I; B)$. If A: $D(A) \subset B \to B$ is a closed linear operator then for $v \in D(A)$ we denote by $\|v\|_{D(A)} = \|v\|_B + \|Av\|_B$ the graph norm of A under which $D(A)$ is a Banach space. In particular, if $B = H$ is a Hilbert space and A is positive definite, i.e. $\|Av\|_A \geqq m\|v\|_H$, $v \in D(A)$, where $m > 0$ is a constant, then we denote $\|v\|_{D(A)} = \|Av\|$ for $v \in D(A)$. For a selfadjoint positive operator A in H we take $m = \inf \sigma(A)$, where $\sigma(A)$ is the spectrum of the operator A.

5. THE INITIAL VALUE PROBLEM

In this section we solve the initial value problem given by equations (3.5), (3.14), (2.7), (3.8) and (3.9). As we want to use an abstract method for the construction of a solution we must prepare an appropriate notion of the solution. Let us start with the definition of a solution on the intervals $[t_1 + nT, t_2 + nT)$, n integer. To be concise, set $n = 0$. In this case our problem reads

$$u_{tt} + 2\gamma u_t - c_0^2 u_{xx} = 0 , \quad x \in (0, l) , \quad t \in (t_1, t_2) , \tag{5.1}$$

$$u_x(0, t) - \alpha_1 u(0, t) = 0 , \quad u_x(l, t) = 0 , \quad t \in [t_1, t_2] ,$$

$$u(x, t_1) = u_0(x) ,$$

$$u_t(x, t_1) = u_1(x) , \quad x \in [0, l] .$$

Note that if we wanted u to satisfy these equations in the classical sense the compatibility conditions

$$u_0'(0) - \alpha_1 u_0(0) = 0\,, \quad u_0'(l) = 0\,, \qquad (5.2)$$
$$u_1'(0) - \alpha_1 u_1(0) = 0\,, \quad u_1'(l) = 0$$

would be necessary.

Let $H = L^2(0, l)$ with the norm $\|\cdot\|$ and let A be a positive selfadjoint operator in H the square of which is defined as follows:

$$D(A^2) = \{v \in H^2(0, l)\,;\ \ v'(0) - \alpha_1 v(0) = 0,\ v'(l) = 0\}\,, \qquad (5.3)$$
$$(A^2 v)(x) = -c_0^2\, v''(x)\,, \quad x \in (0, l)\,, \quad v \in D(A^2)\,.$$

It has been shown in [3] that this definition is correct and that $D(A) = H^1(0, l)$. We interpret the problem (5.1) as a set of equations in H:

$$u''(t) + 2\gamma\, u'(t) + A^2\, u(t) = h(t)\,, \qquad (5.4)$$
$$u(t_1) = u_0\,,$$
$$u'(t_1) = u_1\,,$$

where in this case $h(t) \equiv 0$ (otherwise h: $[t_1, t_2] \to H$) and $u_0, u_1 \in H$.

Definition 5.1. *The function*

$$u(t) = e^{-\gamma(t-t_1)} \cos\left[(t - t_1)(A^2 - \gamma^2)^{1/2}\right] u_0 + \qquad (5.5)$$
$$+ e^{-\gamma(t-t_1)} (A^2 - \gamma^2)^{-1/2} \sin\left[(t - t_1)(A^2 - \gamma^2)^{1/2}\right] (u_1 + \gamma u_0) +$$
$$+ \int_{t_1}^{t} e^{-\gamma(t-\tau)} (A^2 - \gamma^2)^{-1/2} \sin\left[(t - \tau)(A^2 - \gamma^2)^{1/2}\right] h(\tau)\, d\tau\,,$$
$$t \in [t_1, t_2]$$

is called a solution of (5.4) (*or* (5.1), *if* $h \equiv 0$) *on* $[t_1, t_2]$ *if* $u \in C^1([t_1, t_2]; L^2(0, l)) \cap C([t_1, t_2]; H^1(0, l))$.

Lemma 5.2. *For* $u_0 \in D(A) = H^1(0, l)$, $u_1 \in H = L^2(0, l)$ *and* $h \in C([t_1, t_2]; D(A))$ *the formula* (5.5) *represents the unique solution of* (5.4)

Proof. This is a standard result. See e.g. [4] or [3]. Note that the formula (5.5) can be wtitten in the form

$$u(t) = [K_t(t, t_1) + \gamma K(t, t_1)]\, u_0 + K(t, t_1)(u_1 + \gamma u_0) + \int_{t_1}^{t} K(t, \tau)\, h(\tau)\, d\tau\,, \qquad (5.6)$$
$$t \in [t_1, t_2]\,,$$

where

$$K(t, \tau) = e^{-\gamma(t-\tau)}(A^2 - \gamma^2)^{-1/2} \sin [(t - \tau)(A^2 - \gamma^2)^{1/2}] , \tag{5.7}$$

$$t_1 \leqq \tau \leqq t \leqq t_2 , \quad K_t(t, \tau)\, u_0 \quad \text{means} \quad d/dt[K(t, \tau)\, u_0] .$$

If $\varphi \in H$ then the function $v(t) = K(t, \tau)\, \varphi$, $t_1 \leqq \tau \leqq t \leqq t_2$ is a solution of the initial value problem

$$\begin{aligned} v''(t) + 2\gamma\, v'(t) + A^2\, v(t) &= 0 , \\ v(\tau) &= 0 , \\ v'(\tau) &= \varphi , \quad t_1 \leqq \tau \leqq t \leqq t_2 . \end{aligned} \tag{5.8}$$

Lemma 5.3. *Let $m = \inf \sigma(A) > \gamma$. Then for $\varphi \in H$ we have*

$$\|K(t, \tau)\, \varphi\| \leqq (m^2 - \gamma^2)^{-1/2}\, e^{-\gamma(t-\tau)} \|\varphi\| , \tag{5.9}$$

$$\|K(t, \tau)\, \varphi\|_{D(A)} \leqq \left(\frac{m^2 + \gamma^2}{m^2 - \gamma^2}\right)^{1/2} e^{-\gamma(t-\tau)}\, \|\varphi\| , \tag{5.10}$$

$$\|K_t(t,\tau)\, \varphi\| \leqq \left(1 + \frac{\gamma}{(m^2 - \gamma^2)^{1/2}}\right) e^{-\gamma(t-\tau)}\, \|\varphi\| , \tag{5.11}$$

$$t_1 \leqq \tau \leqq t \leqq t_2 .$$

For the proof see [3], formulas (58) and above. Let us note that for A defined by (5.3) we have $m^2 = c_0^2 \,.\, \lambda_m$, where λ_m is the least eigenvalue of the boundary value problem

$$\begin{aligned} - v''(x) &= \lambda\, v(x) , \quad x \in (0, l) , \\ v'(0) - \alpha_1\, v(0) &= 0 , \\ v'(l) &= 0 . \end{aligned}$$

It can be shown that for particular values of the physical constants in question we have $c_0^2 \lambda_m \gg \gamma^2$. This is due to the multiplication by the square of sound speed c_0 which in water – our usual medium – is sufficiently large.

The above considerations are easily modified to any interval $[t_1 + nT, t_2 + nT]$, n integer.

Now we are going to interpret and find the solution of the problem given by (3.5), (3.14), (3.7), (3.8) and (3.9) on the intervals $[t_2 + nT, t_1 + (n + 1)\, T]$, n integer.

Again, we suppose $n = 0$ with an easy modification for n general. On $[t_2, t_1 + T]$ the problem reads

$$u_{tt} + 2\gamma u_t - c_0^2 u_{xx} = 0 , \quad x \in (0, l) , \quad t \in (t_2, t_1 + T) , \tag{5.12}$$

$$(5.13)\quad u_x(0, t) - \alpha_1 u(0, t) = \int_{t_2}^{t} e^{-\alpha_2(t-\tau)}[(\alpha_3 - \alpha_1\alpha_2) u(0, \tau) + f_1(\tau)] \, d\tau ,$$

$$(5.14)\quad u_x(l, t) = 0 , \quad t \in [t_2, t_1 + T] ,$$

$$(5.15)\quad u(x, t_2) = u(x, t_2 -) = \bar{u}_0(x) ,$$

$$(5.16)\quad u_t(x, t_2) = u_t(x, t_2 -) \equiv \bar{u}_1(x) , \quad x \in [0, l] .$$

First, investigate the problem given by (5.12), (5.14)–(5.16) and

$$(5.17)\quad u_x(0, t) - \alpha_1 u(0, t) = g(t) , \quad t \in [t_2, t_1 + T]$$

with a given function g which is, say, twice continuously differentiable. Setting

$$(5.18)\quad v(x, t) = u(x, t) + \frac{1}{\alpha_1} g(t)$$

we get for v the problem

$$(5.19)\quad v_{tt} + 2\gamma v_t - c_0^2 v_{xx} = \frac{1}{\alpha_1}[g''(t) + 2 \gamma g'(t)] , \quad x \in (0, l), \quad t \in (t_2, t_1 + T) ,$$

$$v_x(0, t) - \alpha_1 v(0, t) = 0 ,$$

$$v_x(l, t) = 0 , \quad t \in [t_2, t_1 + T] ,$$

$$v(x, t_2) = \bar{u}_0(x) + \frac{1}{\alpha_1} g(t_2) ,$$

$$v_t(x, t_2) = \bar{u}_1(x) + \frac{1}{\alpha_1} g'(t_2) , \quad x \in [0, l] .$$

The solution of this problem will be interpreted in the sense of Definition 5.1, where we set

$$u := v, \; h := \frac{1}{\alpha_1}[g''(t) + 2\gamma g'(t)] \, . \, 1 ,$$

$$t_1 := t_2 , \quad t_2 := t_1 + T ,$$

$$u_0 := \bar{u}_0 + \frac{1}{\alpha_1} g(t_2) \, . \, 1 , \quad u_1 := \bar{u}_1 + \frac{1}{\alpha_1} g'(t_2) \, . \, 1$$

where 1 means the function in $L^2(0, 1)$ identically equal to the constant 1. According to Lemma 5.2 and (5.6), for $\bar{u}_0 \in H^1(0, l)$, $\bar{u}_1 \in L^2(0, l)$ we can write

$$(5.20)\quad v(t) = [K_t(t, t_2) + \gamma K(t, t_2)] \left(\bar{u}_0 + \frac{1}{\alpha_1} g(t_2) \, . \, 1\right) +$$

$$+ K(t, t_2)\left(\bar{u}_1 + \frac{1}{\alpha_1} g'(t_2) . 1 + \gamma \bar{u}_0 + \frac{\gamma}{\alpha_1} g(t_2) . 1\right) +$$

$$+ \frac{1}{\alpha_1} \int_{t_2}^{t} [g''(\tau) + 2\gamma g'(\tau)] K(t, \tau) 1 \, d\tau , \quad t \in [t_2, t_1 + T] ,$$

where $v(t) = v(\cdot, t)$, $v \in C^1([t_2, t_1 + T] ; L^2(0, l)) \cap C([t_2, t_1 + T]; H^1(0, l))$.

In what follows we set

$$(5.21) \quad g(t) = \int_{t_2}^{t} e^{-\alpha_2(t-\tau)}[(\alpha_3 - \alpha_1\alpha_2) u(0, \tau) + f_1(\tau)] \, d\tau , \quad t \in [t_2, t_1 + T] .$$

As $u \in C^1([t_2, t_1 + T]; L^2(0, l)) \cap C([t_2, t_1 + T] ; H^1(0, l))$ implies that $u(0, \cdot)$ belongs to $C([t_2, t_1 + T])$ but not necessarily to $C^1([t_2, t_1 + T])$, we can expect $g \in C^1([t_2, t_1 + T])$ only. That is why we integrate the term $\frac{1}{\alpha_1} \int_{t_2}^{t} g''(\tau) K(t, \tau) 1 \, d\tau$ in (5.20) by parts (take into account that $K_\tau(t, \tau) = -K_t(t, \tau)$ by (5.7)), thus getting the expression which makes sense also for $g \in C^1([t_2, t_1 + T])$:

$$(5.22) \quad u(t) = v(t) - \frac{1}{\alpha_1} g(t) . 1 = [K_t(t, t_2) + \gamma K(t, t_2)] (\bar{u}_0 + \frac{1}{\alpha_1} g(t_2) . 1) +$$

$$+ K(t, t_2) (\bar{u}_1 + \gamma \bar{u}_0 + \frac{\gamma}{\alpha_1} g(t_2) . 1) - \frac{1}{\alpha_1} g(t) . 1 +$$

$$+ \frac{1}{\alpha_1} \int_{t_2}^{t} g'(\tau) [K_t(t, \tau) + 2\gamma K(t, \tau)] 1 \, d\tau , \quad t \in [t_2, t_1 + T] .$$

Inserting (5.21) into (5.22) we can interpret the solution of the problem (5.12–16) as a solution of an integral equation. In the following definition take into account that by (5.21), $g'(t) = -\alpha_2 g(t) + (\alpha_3 - \alpha_1\alpha_2) u(0, t) + f_1(t)$ holds.

Definition 5.4. *By a solution to the problem* (5.12)–(5.16) *we mean a function* $u(x, t)$ *such that*

$$(5.23) \quad u(\cdot, \cdot) \in C^1([t_2, t_1 + T] ; \; L^2(0, l)) \cap C([t_2, t_1 + T] ; \; H^1(0, l))$$

satisfies the integral equation

$$(5.24) \quad u(t) = \Phi(u)(t) \equiv [K_t(t, t_2) + \gamma K(t, t_2)] \bar{u}_0 + K(t, t_2) (\bar{u}_1 + \gamma \bar{u}_0) -$$

$$- \frac{1}{\alpha_1} \int_{t_2}^{t} e^{-\alpha_2(t-\tau)} [(\alpha_3 - \alpha_1\alpha_2) u(0, \tau) + f_1(\tau)] \, d\tau . 1 + \frac{1}{\alpha_1} \int_{t_2}^{t} \Big\{ (\alpha_3 - \alpha_1\alpha_2) u(0, \tau) +$$

$$+ f_1(\tau) - \alpha_2 \int_{t_2}^{\tau} e^{-\alpha_2(\tau-\sigma)} [(\alpha_3 - \alpha_1\alpha_2) u(0, \sigma) + f_1(\sigma)] \, d\sigma \Big\} .$$

$$. [K_t(t, \tau) + 2\gamma K(t, \tau)] 1 \, d\tau , \quad t \in [t_2, t_1 + T] .$$

Lemma 5.5. *For* $\bar{u}_0 \in H^1(0, l)$, $\bar{u}_1 \in L^2(0, l)$ *and* $f_1 \in C([t_2, t_1 + T])$ *there exists a unique solution of the problem* (5.12)–(5.16).

Proof. If suffices to prove the existence of a unique solution $u \in C([t_2, t_1 + T]; H^1(0, l))$ of the integral equation (5.24), since having such a function, we have $u(0, \cdot) \in C([t_2, t_1 + T])$ by the Sobolev embedding theorem which together with the assumption on the data $\bar{u}_0, \bar{u}_1$ and f_1 yields continuous differentiability of the right hand side of (5.24) in the norm of $L^2(0, l)$. So we shall use the Banach fixed point theorem in the Banach space $B_k = C([t_2, t_1 + T]; H^1(0, l))$ with the norm $\|v\|_{B_k} = \sup_{t \in [t_2, t_1 + T]} (e^{-k(t-t_2)} \|v(t)\|_{H^1})$ to the operator Φ defined in (5.24). A constant $k > 0$ is to be chosen so that Φ be a strict contraction in B_k. First, it is clear that Φ maps B_k into itself. Indeed, for $u \in B_k$ we have $u(0, \cdot) \in C([t_2, t_1 + T])$ and the operators $K_t(t, \tau)$, $K(t, \tau)$ map $D(A) = H^1(0, l)$ continuously into itself. As $1 \in H^1(0, l)$ and f_1 is continuous the integrals in (5.24) are continuous functions of t in $H^1(0, l)$. The same may be easily shown for the remaining terms on the right hand side of (5.24). Let us prove that a constant k can be chosen so that Φ is contractive in B_k. Let $w = v_1 - v_2$, where $v_1, v_2 \in B_k$. Then by (5.24) for $t \in [t_2, t_1 + T]$ we have

$$\Phi(v_1)(t) - \Phi(v_2)(t) = \frac{\alpha_1\alpha_2 - \alpha_3}{\alpha_1} \int_{t_2}^{t} e^{-\alpha_2(t-\tau)} w(0, \tau) . 1 +$$

$$+ \frac{\alpha_3 - \alpha_1\alpha_2}{\alpha_1} \int_{t_2}^{t} \left\{ \{ w(0, \tau) - \alpha_2 \int_{t_2}^{\tau} e^{-\alpha_2(t-\sigma)} w(0, \sigma) \, d\sigma \right\} \times$$

$$\times [K_t(t, \tau) + 2\gamma K(t, \tau)] \, 1 \, d\tau .$$

It follows that

$$e^{-k(t-t_2)} \|\Phi(v_1)(t) - \Phi(v_2)(t)\|_{H^1} \leqq \tag{5.25}$$

$$\leqq \frac{|\alpha_1\alpha_2 - \alpha_3|}{\alpha_1} . e^{-k(t-t_2)} . \left\{ \left| \int_{t_2}^{t} |w(0, \tau) \, d\tau \right\| 1 \|_{H^1} + \int_{t_2}^{t} \left[|w(0, \tau)| + \alpha_2 \int_{t_2}^{\tau} |w(0, \sigma)| \, d\sigma \right] . \right.$$

$$\left. . [\|K_t(t, \tau) \, 1\|_{H^1} + 2\gamma \|K(t, \tau) \, 1\|_{H^1}] \, d\tau \right\} .$$

Since the graph norm of A is equivalent to that of $H^1(0, l)$ (see [3], Lemma 2), there exists a constant $a > 0$ such that

$$a^{-1} \|z\|_{H^1} \leqq \|z\|_{D(A)} = \|Az\| \leqq a\|z\|_{H^1} , \quad z \in D(A) = H^1(0, l) . \tag{5.26}$$

Hence using Lemma 5.3 we get

$$\|K_t(t, \tau) \, 1\|_{H^1} \leqq a\|K_t(t, \tau) \, 1\|_{D(A)} = a\|A \, K_t(t, \tau) \, 1\| = a\|K_t(t, \tau) \, A \, 1\| \leqq \tag{5.27}$$

$$\leqq a(1 + \gamma/(m^2 - \gamma^2)^{1/2}) \, e^{-\gamma(t-\tau)} \, \|A \, 1\| \leqq a^2 \left(1 + \frac{\gamma}{(m^2 - \gamma^2)^{1/2}}\right) \|1\|_{H^1} ,$$

(5.28) $\|K(t,\tau)\,1\|_{H^1} \leqq a(m^2-\gamma^2)^{-1/2}\,\mathrm{e}^{-\gamma(t-\tau)}\,\|A\,1\| \leqq a^2(m^2-\gamma^2)^{-1/2}\,\|1\|_{H^1}$.

As $\|1\|_{H^1} = l^{1/2}$, we find from (5.25) that

$$\text{(5.29)}\qquad \mathrm{e}^{-k(t-t_2)}\,\|\Phi(v_1)(t)-\Phi(v_2)(t)\|_{H^1} \leqq$$

$$\leqq \frac{l^{1/2}|\alpha_1\alpha_2-\alpha_3|}{\alpha_1}\,\mathrm{e}^{-k(t-t_2)}\,.\left\{\int_{t_2}^{t}|w(0,\tau)|\,\mathrm{d}\tau + a^2\left(1+\frac{3\gamma}{(m^2-\gamma^2)^{1/2}}\right).\right.$$

$$.\left.\int_{t_2}^{t}\left[|w(0,\tau)| + \alpha_2\int_{t_2}^{\tau}|w(0,\sigma)|\,\mathrm{d}\sigma\right]\mathrm{d}\tau\right\} = \frac{l^{1/2}|\alpha_1\alpha_2-\alpha_3|}{\alpha_1}\,\mathrm{e}^{-k(t-t_2)}\,.$$

$$.\int_{t_2}^{t}\left[1+a^2\left(1+\frac{3\gamma}{(m^2-\gamma^2)^{1/2}}\right)(1+\alpha_2(t-\tau))\right]|w(0,\tau)|\,\mathrm{d}\tau \leqq$$

$$\leqq \frac{l^{1/2}|\alpha_1\alpha_2-\alpha_3|}{\alpha_1}\left[1+a^2\left(1+\frac{3\gamma}{(m^2-\gamma^2)^{1/2}}\right)(1+\alpha_2(t_1+T-t_2))\right].$$

$$\int_{t_2}^{t}\mathrm{e}^{-k(t-\tau)}\,.\,\mathrm{e}^{-k(\tau-t_2)}\,|w(0,\tau)|\,\mathrm{d}\tau\,.$$

By the Sobolev embedding theorem there exists a constant b such that

$$\text{(5.30)}\qquad |w(0,\tau)| \leqq b\,.\,\|w(\cdot,\tau)\|_{H^1}\,,\quad \tau\in[t_2,t_1+T]\,.$$

Denoting

$$M = \frac{bl^{1/2}|\alpha_1\alpha_2-\alpha_3|}{\alpha_1}\left[1+a^2\left(1+\frac{3\gamma}{(m^2-\gamma^2)^{1/2}}\right).\,(1+\alpha_2(t_1+T-t_2))\right],$$

we find that (5.29), (5.30) yields

$$\text{(5.31)}\qquad \mathrm{e}^{-k(t-t_2)}\,\|\Phi(v_1)(t)-\Phi(v_2)(t)\|_{H_1} \leqq$$

$$M\,.\int_{t_2}^{t}\mathrm{e}^{-k(t-\tau)}\,\mathrm{d}\tau \sup_{\tau\in[t_2,t_1+T]}\mathrm{e}^{-k(\tau-t_2)}\,\|w(\cdot,\tau)\|_{H^1} = \frac{M}{k}(1-\mathrm{e}^{-k(t-t_2)})\,\|w\|_{B_k} \leqq$$

$$\leqq \frac{M}{k}\|w\|_{B_k} = \frac{M}{k}\|v_1-v_2\|_{B_k}\,.$$

Taking $\sup_{t\in[t_2,t_1+T]}$ on the left hand side of (5.31) and a fixed $k > M$ we get the contractivity of Φ in B_k. By the Banach contraction principle there exists a unique $u\in B_k$ satisfying the equation

$$u(t) = \Phi(u)(t)\,,\quad t\in[t_2,t_1+T]\,.$$

This completes the proof.

Lemma 5.5 allows us to construct the solution of the problem defined by the equations (3.5), (3.14), (3.7), (3.8), (3.9) for any integer n. This result combined with Lemma 5.2 makes it possible to define a solution of the problem.

$$(5.32)\qquad u_{tt} + 2\gamma u_t - c_0^2 u_{xx} = 0\,,\quad x \in (0, l)\,,$$

$$t \in (t_1 + nT, t_2 + nT) \cup (t_2 + nT, t_1 + (n+1)\,T)\,,$$

$$u_x(0, t) - \alpha_1\, u(0, t) = v(t) \int_{t_2+nT}^{t} e^{-\alpha_2(t-\tau)} \left[(\alpha_3 - \alpha_1\alpha_2)\, u(0, \tau) + f_1(\tau)\right] d\tau\,,$$

$$u_x(l, t) = 0\,,\quad t \in [t_1 + nT, t_1 + (n+1)\,T]\,,$$

$$u(x, t_i + nT +) = u(x, t_i + nT -)\,,$$

$$u_t(x, t_i + nT +) = u(x, t_i + nT -)\,,\quad x \in [0, l]\,,\quad i = 1, 2\,,$$

$$u(x, t_1 -) = u_0(x)\,,$$

$$u_t(x, t_1 -) = u_1(x)\,,\quad x \in [0, l]\,,$$

$$n = 0, 1, 2, \ldots.$$

Definition 5.6. *By a solution of the problem we mean a function* $u \in C^1([t_1, \infty); L^2(0, l)) \cap C([t_1, \infty); H^1(0, l))$ *satisfying the relations*

$$(5.33)\qquad u(t) = [K_t(t, t_1 + nT) + \gamma\, K(t, t_1 + nT)]\, u(t_1 + nT -) + K(t, t_1 + nT)\,(u'(t_1 + nT -) + \gamma\, u(t_1 + nT -))$$

for $t \in [t_1 + nT, t_2 + nT]$ and

$$(5.34)\qquad u(t) = [K_t(t, t_2 + nT) + \gamma\, K(t, t_2 + nT)]\, u(t_2 + nT -) + K(t, t_2 + nT)\,(u'(t_2 + nT -) + \gamma\, u(t_2 + nT -)) -$$

$$- \frac{1}{\alpha_1} \int_{t_2+nT}^{t} e^{-\alpha_2(t-\tau)} \left[(\alpha_3 - \alpha_1\alpha_2)\, u(0, \tau) + f_1(\tau)\right] d\tau \,.\, 1 +$$

$$+ \frac{1}{\alpha_1} \int_{t_2+nT}^{t} \left\{(\alpha_3 - \alpha_1\alpha_2)\, u(0, \tau) + f_1(\tau) - \alpha_2 \int_{t_2}^{\tau} e^{-\alpha_2(\tau-\sigma)} \,.\, \left[(\alpha_3 - \alpha_1\alpha_2)\, u(0, \sigma) + f_1(\sigma)\right] d\sigma\right\} . \left[K_t(t, \tau) + 2\gamma\, K(t, \tau)\right] 1\, d\tau$$

for $t \in [t_2 + nT, t_1 + (n+1)\,T]$, $n = 0, 1, 2, \ldots$, where we set

$$(5.35)\qquad u(x, t_1 -) = u_0(x)\,,\quad u_t(x, t_1 -) = u_1(x)\,,\quad x \in [0, l]\,.$$

Theorem 5.7. *Let* $u_0 \in H^1(0, l)$, $u_1 \in L^2(0, l)$ *and* $f_1 \in C([t_2 + nT, t_1 + (n+1)\,T])$ *for each* $n = 0, 1, 2, \ldots$. *Then there exists a unique solution of the problem* (5.32).

The proof is a trivial consequence of Lemmas 5.2 and 5.5.

The next theorem establishes the correctness of the problem (5.32) in the sense of continuous dependence of the solution on the data in the appropriate norms.

Theorem 5.8. *There exist real constants* $M > 0$ *and* ω *such that for any* $u_0 \in H^1(0, l)$, $u_1 \in L^2(0, l)$ *and* $f_1 \in C([t_2 + nT, t_1 + (n+1)\,T])$, $f_1(t) = 0$, $t \in$

$\in [t_1 + nT, t_2 + nT)$, $n = 0, 1, 2, \dots$ *the solution of the problem* (5.32) *satisfies the estimates*

$$(5.36) \qquad \|u(t)\|_{H^1} \leqq M\, e^{\omega(t-t_1)} \left(\|u_0\|_{H^1} + \|u_1\|_{L^2} + \int_{t_1}^{t} \nu(\tau)\, |f_1(\tau)|\, d\tau\right),$$

$$\|u'(t)\|_{L^2} \leqq M\, e^{\omega(t-t_1)} \left(\|u_0\|_{H^1} + \|u_1\|_{L^2} + \right.$$

$$\left. + \int_{t_1}^{t} \nu(\tau)\, |f_1(\tau)|\, d\tau\right), \quad t \geqq t_1 .$$

The constants M and ω are independent of u_0, u_1 and f_1.

Proof. Let $u_0 \in H^1(0, l)$, $u_1 \in L^2(0, l)$, $f_1 \in C([t_2 + nT, t_1 + (n + 1) T])$, $f_1(t) = 0$, $t \in [t_1 + nT, t_2 + nT)$, $n = 0, 1, 2, \dots$. To prove the estimates (5.36) we use the formulas (5.33), (5.34). We restrict ourselves to $t \in [t_1, t_1 + T]$, extending the estimate on $[t_1, t_1 + T]$ by induction onto $[t_1, \infty) = \bigcup_{n=0}^{\infty} [t_1 + nT, t_1 + (n + 1) T]$. Let $t \in [t_1, t_2]$. Then by (5.26), (5.33) and Lemma 5.3

$$(5.37) \quad \|u(t)\|_{H^1} \leqq a\|u(t)\|_{D(A)} \leqq a[\|K_t(t, t_1)\, A\, u(t_1)\| + \gamma\|K(t, t_1)\, A\, u(t_1)\| +$$

$$+ \|A\, K(t, t_1)\, u'(t_1)\| + \gamma\|K(t, t_1)\, A\, u(t_1)\|] \leqq$$

$$\leqq a\left[a\left(1 + \frac{\gamma}{(m^2 - \gamma^2)^{1/2}}\right)\|u(t_1)\|_{H^1} + \gamma a(m^2 - \gamma^2)^{-1/2}\|u(t_1)\|_{H^1} + \right.$$

$$\left. + \left(\frac{m^2 + \gamma^2}{m^2 - \gamma^2}\right)^{1/2} \|u'(t_1)\|_{L^2} + \gamma a(m^2 - \gamma^2)^{-1/2}\|u(t_1)\|_{H^1}\right] e^{-\gamma(t-t_1)} \leqq$$

$$\leqq k_1[\|u(t_1)\|_{H^1} + \|u'(t_1)\|_{L^2}]\, e^{-\gamma(t-t_1)},$$

where (and in what follows) $k_1, k_2, \dots$ denote constants. Differentiating (5.33) and using the identity $K_{tt}(t, \tau) = -\gamma\, K_t(t, \tau) - A^2\, K(t, \tau)$, quite analogously we get

$$(5.38) \qquad \|u'(t)\|_{L^2} \leqq k_2[\|u(t_1)\|_{H^1} + \|u'(t_1)\|_{L^2}]\, e^{-\gamma(t-t_1)}.$$

Let $t \in [t_2, t_1 + T]$. The first part of (5.34) containing the initial conditions $u(t_2-)$, $u'(t_2-)$ has the same form as (5.33). Thus from (5.34) and following the idea of (5.37) we obtain

$$\|u(t)\|_{H^1} \leqq k_1\, e^{-\gamma(t-t_2)}[\|u(t_2)\|_{H^1} + \|u'(t_2)\|_{L^2}] +$$

$$+ \frac{1}{\alpha_1}\|1\|_{H^1}\int_{t_2}^{t} e^{-\alpha_2(t-\tau)}\, [|\alpha_3 - \alpha_1\alpha_2|\, |u(0, \tau)| + |f_1(\tau)|]\, d\tau +$$

$$+ \frac{1}{\alpha_1}\int_{t_2}^{t}\left\{ |\alpha_3 - \alpha_1\alpha_2|\, |u(0, \tau)| + |f_1(\tau)| + \right.$$

$$+ \alpha_2 \int_{t_2}^{\tau} e^{-\alpha_2(\tau-\sigma)} [|\alpha_3 - \alpha_1\alpha_2| \, |u(0,)|\sigma + |f_1(\sigma)|] \, d\sigma\Big\} \, .$$

$$. [\|K_t(t, \tau) \, 1\|_{H^1} + 2\gamma\|K(t, \tau) \, 1\|_{H^1}] \, d\tau \, .$$

Using (5.27), (5.28), (5.30) and setting $\beta = \min(\gamma, \alpha_2)$ we find

(5.39)
$$\|u(t)\|_{H^1} \leqq k_1 \, e^{-\beta(t-t_2)} [\|u(t_2)\|_{H^1} + \|u'(t_2)\|_{L^2}] +$$
$$+ \frac{b|\alpha_3 - \alpha_1\alpha_2|}{\alpha_1} \|1\|_{H^1} e^{-\beta(t-t_2)} \int_{t_2}^{t} e^{\beta(\tau-t_2)} \|u(\tau)\|_{H^1} \, d\tau +$$
$$+ \frac{1}{\alpha_1} \|1\|_{H^1} e^{-\beta(t-t_2)} \int_{t_2}^{t} e^{\beta(\tau-t_2)} |f_1(\tau)| \, d\tau +$$
$$+ \frac{a^2 b|\alpha_3 - \alpha_1\alpha_2|}{\alpha_1} \left(1 + \frac{3\gamma}{(m^2 - \gamma^2)^{1/2}}\right) \|1\|_{H^1} e^{-\beta(t-t_2)} \int_{t_2}^{t} e^{\beta(\tau-t_2)} \, .$$
$$. \|u(\tau)\|_{H^1} \, d\tau + \frac{a^2}{\alpha_1} \left(1 + \frac{3\gamma}{(m^2 - \gamma^2)^{1/2}}\right) \|1\|_{H^1} \, . \, e^{-\beta(t-t_2)} \int_{t_2}^{t} e^{\beta(\tau-t_2)} |f_1(\tau)| \, d\tau +$$
$$+ \frac{\alpha_2 a^2 b|\alpha_3 - \alpha_1\alpha_2|}{\alpha_1} \left(1 + \frac{3\gamma}{(m^2 - \gamma^2)^{1/2}}\right) \|1\|_{H^1} e^{-\beta(t-t_2)} \int_{t_2}^{t} \int_{t_2}^{\sigma} e^{\beta(\sigma-t_2)} |u(\sigma)| \, d\sigma \, d\tau +$$
$$+ \frac{\alpha_2 a^2}{\alpha_1} \|1\|_{H^1} e^{-\beta(t-t_2)} \int_{t_2}^{t} \int_{t_2}^{\sigma} e^{\beta(\sigma-t_2)} |f_1(\sigma)| \, d\sigma \, d\tau \leqq k_1 \, e^{-\beta(t-t_2)} [\|u(t_2)\|_{H^1} +$$
$$+ \|u'(t_2)\|_{L^2}] + k_3 \, e^{-\beta(t-t_2)} \int_{t_2}^{t} e^{\beta(\tau-t_2)} |f_1(\tau)| \, d\tau + k_4 \, e^{-\beta(t-t_2)} \int_{t_2}^{t} e^{\beta(\tau-t_2)} \|u(\tau)\|_{H^1} \, d\tau \, ,$$

where the double integrals are estimated as e.g.

$$\int_{t_2}^{t} \int_{t_2}^{\sigma} e^{\beta(\sigma-t_2)} \|u(\sigma)\|_{H^1} \, d\sigma \, d\tau = \int_{t_2}^{t} e^{\beta(\sigma-t_2)} \|u(\sigma)\|_{H^1} \left(\int_{\sigma}^{t} d\tau\right) d\sigma \leqq$$
$$\leqq T \int_{t_2}^{t} e^{\beta(\sigma-t_2)} \|u(\sigma)\|_{H^1} \, d\sigma \, .$$

Setting

$$\varphi(t) = e^{\beta(t-t_2)} \|u(t)\|_{H^1} \, ,$$

from (5.37), (5.38) (with $t = t_2$) and (5.39) we find

$$\varphi(t) \leqq k_1(k_1 + k_2) \, e^{-\gamma(t_2-t_1)} [\|u(t_1)\|_{H^1} + \|u'(t_1)\|_{L^2}] +$$
$$+ k_3 \int_{t_2}^{t} e^{\beta(\tau-t_2)} |f_1(\tau)| \, d\tau + k_4 \int_{t_2}^{t} \varphi(\tau) \, d\tau \leqq k_5 \Big\{ e^{-\beta(t_2-t_1)} [\|u(t_1)\|_{H^1} +$$
$$+ \|u'(t_1)\|_{L^2}] + \int_{t_2}^{t} e^{\beta(\tau-t_2)} |f_1(\tau)| \, d\tau \Big\} + k_4 \int_{t_2}^{t} \varphi(\tau) \, d\tau \, .$$

Although $\int_{t_2}^{t} e^{\beta(\tau - t_2)} |f_1(\tau)| \, d\tau$ is not constant, since it is a nondecreasing function the Gronwall lemma can be used as with a constant absolute term to yield

$$\varphi(t) \leqq k_5 \, e^{k_4(t-t_2)} \left\{ e^{-\beta(t_2-t_1)} \left[\|u(t_1)\|_{H^1} + \|u'(t_1)\|_{L^2} \right] + \right.$$

$$\left. + \int_{t_2}^{t} e^{\beta(\tau - t_2)} |f_1(\tau)| \, d\tau \right\}$$

which yields

$$(5.40) \quad \|u(t)\|_{H^1} \leqq k_5 \, e^{k_4(t-t_2)} \left\{ e^{-\beta(t-t_1)} \left[\|u(t_1)\|_{H^1} + \|u'(t_1)\|_{L^2} \right] + \int_{t_2}^{t} |f_1(\tau)| \, d\tau \right\} =$$

$$= \frac{a(t)}{2} \left[\|u(t_1)\|_{H^1} + \|u'(t_1)\|_{L^2} \right] + \frac{b(t)}{2} \int_{t_2}^{t} |f_1(\tau)| \, d\tau \, .$$

By appropriately increasing the constants k_4, k_5 if necessary, quite analogously we get

$$\|u'(t)\|_{L^2} \leqq \frac{a(t)}{2} \left[\|u(t_1)\|_{H^1} + \|u'(t_1)\|_{L^2} \right] + \frac{b(t)}{2} \int_{t_2}^{t} |f_1(\tau)| \, d\tau \, .$$

We conclude that

$$(5.41) \qquad \|u'(t)\|_{L^2} + \|u(t)\|_{H^1} \leqq a(t) \left[\|u'(t_1)\| + \|u(t_1)\|_{H^1} \right] +$$

$$+ \, b(t) \int_{t_1}^{t} v(\tau) \, |f_1(\tau)| \, d\tau \, , \quad t \in [t_1, t_1 + T] \, ,$$

where

$$a(t) = k_6 \, e^{-\gamma(t-t_1)} \, , \quad b(t) = 0 \quad \text{for} \quad t \in [t_1, t_2)$$

and

$$a(t) = k_5 \, e^{k_4(t-t_2)} \, e^{-\beta(t-t_1)} \, , \quad b(t) = k_5 \, e^{k_4(t-t_2)} \quad \text{for} \quad t \in [t_2, t_1 + T] \, .$$

As the estimates (5.37), (5.38), (5.40), (5.41) do not depend on the particular choice of the interval $[t_1 + nT, t_1 + (n+1) T]$ we find

$$(5.42) \qquad \|u'(t)\|_{L^2} + \|u(t)\|_{H^1} \leqq a(t) \left[\|u'(t_1 + nT)\|_{L^2} + \right.$$

$$\left. + \|u(t_1 + nT)\|_{H^1} \right] + b(t) \int_{t_1 + nT}^{t} v(\tau) \, |f_1(\tau)| \, d\tau \, ,$$

$$t \in [t_1 + nT, t_1 + (n+1) T] \, ,$$

where

$$(5.43) \qquad a(t) = k_6 \, e^{-\gamma(t - t_1 - nT)} \, , \quad b(t) = 0 \quad \text{for} \quad t \in [t_1 + nT, t_2 + nT)$$

and

$$(5.44) \qquad a(t) = k_5 \, e^{k_4(t - t_2 - nT)} \, e^{-\beta(t - t_1 - nT)} \, , \quad b(t) = k_5 \, e^{k_4(t - t_2 - nT)}$$

$$\text{for} \quad t \in [t_2 + nT, t_1 + (n+1) T) \, .$$

Set $y_n = \|u'(t_1 + nT)\|_{L^2} + \|u(t_1 + nT)\|_{H^1}$,

$$z_n = \int_{t_1+nT}^{t_1+(n+1)T} \nu(\tau)\,|f_1(\tau)|\,d\tau\,, \quad n = 0, 1, \ldots.$$

Let $t \in (t_1, \infty)$ be arbitrary and let n be the greatest integer such that $t_1 + nT \leqq t \leqq$ $\leqq t_1 + (n + 1)\,T$. By (5.41) we have

$$\|u'(t)\|_{L^2} + \|u(t)\|_{H^1} \leqq a(t)\,y_n + b(t)\int_{t_1+nT}^{t} \nu(\tau)\,|f_1(\tau)|\,d\tau\,. \tag{5.45}$$

Further, it is clear from (5.42) that

$$y_k \leqq a(t_1 + kT)\,y_{k-1} + b(t_1 + kT)\,z_{k-1}\,, \quad k = 1, 2, \ldots \tag{5.46}$$

By (5.44) we have

$$a \equiv a(t_1 + kT) = k_5\,e^{k_4(t_1+T-t_2)}\,e^{-\beta T}\,, \tag{5.47}$$

$$b \equiv b(t_1 + kT) = k_5\,e^{k_4(t_1+T-t_2)}\,.$$

These constants are independent of k. Using (5.46), by induction we get

$$y_n \leqq a^n y_0 + b(a^{n-1}z_0 + a^{n-2}z_1 + \ldots + z_{n-1})\,. \tag{5.48}$$

It is clear from (5.43), (5.44) that there is a constant k_7 such that $a(t) \leqq k_7$, $b(t) \leqq k_7$ independently of $t \in [t_1 + nT, t_1 + (n + 1)\,T]$ and n. Thus (5.45), (5.48) yield

$$\|u'(t)\|_{L^2} + \|u(t)\|_{H^1} \leqq k_7(y_n + \int_{t_1+nT}^{t} \nu(\tau)\,|f_1(\tau)|\,d\tau) \leqq$$

$$\leqq k_7\left[a^n y_0 + b(a^{n-1}z_0 + a^{n-2}z_1 + \ldots + z_{n-1}) + \int_{t_1+nT}^{t} \nu(\tau)\,|f_1(\tau)|\,d\tau\right] \leqq$$

$$\leqq k_7\left[a^n y_0 + b\max(a^{n-1}, 1)\left(\sum_{j=0}^{n-1} z_j + \int_{t_1+nT}^{t} \nu(\tau)\,|f_1(\tau)|\,d\tau\right)\right] =$$

$$= k_7 \exp(n\ln a)\,[\|u_0\|_{H^1} + \|u_1\|_{L^2}] +$$

$$+ k_7 b\max\{\exp((n-1)\ln a), 1\}\,.\int_{t_1}^{t} \nu(\tau)\,|f_1(\tau)|\,d\tau\,.$$

Since $n \leqq (t - t_1)/T$, $\ln a = \ln k_5 + k_4(t_1 + T - t_2) - \beta T$, the estimates (5.36) immediately follow.

References

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Souhrn

ŘEŠENÍ MATEMATICKÉHO MODELU JEDNOPÍSTOVÉHO ČERPADLA S DETAILNĚJŠÍM POPISEM FUNKCE VENTILU

IVAN STRAŠKRABA

V práci je řešen matematický model proudění tekutiny v trubici s ventilem a čerpadlem. Funkce ventilu je popsána detailněji než v práci [3] a činí tak model úplnější.

Резюме

РЕШЕНИЕ МАТЕМАТИЧЕСКОЙ МОДЕЛИ ОДНОПОРШНЕВОГО НАСОСА С БОЛЕЕ ПОДРОБНЫМ ОПИСАНИЕМ ФУНКЦИИ ВЕНТИЛЯ

IVAN STRAŠKRABA

В работе решена математическая модель течения жидкости в трубе с вентилом и насосом. Функция вентиля описана подробнее чем в работе [3] и делает модель более совершенной.

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