Ján Filo

On solutions of a perturbed fast diffusion equation

Aplikace matematiky, Vol. 32 (1987), No. 5, 364–380

Persistent URL: `http://dml.cz/dmlcz/104268`

32 (1987) APLIKACE MATEMATIKY No. 5, 364—380

ON SOLUTIONS OF A PERTURBED FAST DIFFUSION EQUATION

JÁN FILO

(Received April 18, 1986)

Summary. The paper concerns the (local and global) existence, nonexistence, uniqueness and some properties of nonnegative solutions of a nonlinear density dependent diffusion equation with homogeneous Dirichlet boundary conditions.

Keywords: Nonlinear diffusion, method of lines, local existence, finite extinction time.

AMS Classification: 35K65.

0. INTRODUCTION

This paper deals with the (local and global) existence, nonexistence and some properties of nonnegative solutions of the initial-boundary value problem

$$(0.1)\qquad \begin{aligned} &v_t - \Delta(v^m) = f(v) && \text{in} \quad D \times (0, T), \quad T > 0, \\ &v(x, t) = 0 && \text{on} \quad \partial D \times (0, T), \\ &v(x, 0) = v_0(x) && \text{in} \quad D, \end{aligned}$$

where m is a positive constant less than 1, D is a bounded domain in R^N with smooth boundary ∂D, Δ is the Laplacian, f is (in general) a locally Lipschitz continuous function with $f(0) \geqq 0$, and v_0 is nonnegative and bounded. Problems of this kind arise in the theory of plasma physics, where v denotes the plasma density, and so it is natural to consider $v \geqq 0$ only (the case $f \equiv 0$, $m = 1/2$ corresponds to the "Okuda-Dawson diffusion" [3]). Computing the Laplacian $\Delta(v^m) = \operatorname{div}(mv^{m-1} \operatorname{grad} v)$ we see that the diffusion coefficient $K(v) = mv^{m-1}$ $(0 < m < 1)$ tends to infinity as $v \downarrow 0$ which is why we speak about the so-called "fast diffusion" case. The most striking manifestation of this type of the nonlinear density dependent diffusion is the fact that the solution of Problem (0.1) with $f \equiv 0$ (for $f \not\equiv 0$ see Section 4) decays to zero in a finite time T^* depending on the initial data (see [13]). This contrasts with the heat conduction case, $m = 1$, and the "slow diffusion" case, $m > 1$ ($K(v) = mv^{m-1}$ tends to zero as $v \downarrow 0$), where the solutions decay to zero in infinite time. An other serious consequence of the "degeneracy" of this equation is the fact that the solution need not be classical (even if the data are smooth) near the points where $v = 0$, and it is necessary to consider some well defined generalized solutions (see [13]).

In Section 1 of the present paper we solve Problem (0.1) with a global Lipschitz continuous function f using the method of lines, which was applied to the "slow diffusion" case in [7] (see also [4]) and has been intensively studied in [8]. The exact power nonlinearity in the diffusion term seems to be necessary for our way of proofs, but this loss of generality is, in our opinion, counter-balanced by its simplicity and obviousness. The continuous dependence as well as the comparison principle stated in Section 2 are obtained by an adaptation of the method of [1].

In Section 3 we use the "maximum principle" from Section 1 and a uniqueness result (Section 2) to prove the local existence for an arbitrary locally Lipschitz continuous function f, and then the nonexistence of a global solution of Problem (0.1) is established (see [11], where the slow diffusion case is considered). In Section 4 the existence of the finite extinction time for $f \not\equiv 0$ is demonstrated by a simple comparison technique.

1. GLOBAL EXISTENCE FOR SMOOTH INITIAL DATA

In this section we prove the existence of a global solution of Problem (0.1) assuming global Lipschitz continuity of f and smooth initial data. In what follows it is more convenient for us to transform Problem (0.1) by setting $u = |v|^m \operatorname{sgn}(v)$ into

$$(1.1)\qquad \begin{aligned} &(\beta(u))_t - \Delta u = f(\beta(u)) && \text{in} \quad Q_T = D \times (0, T)\,, \\ &u(x, t) = 0 && \text{on} \quad S_T = \partial D \times (0, T)\,, \\ &u(x, 0) = u_0(x) && \text{in} \quad D\,, \end{aligned}$$

where $\beta(u) = |u|^\alpha \operatorname{sgn}(u)$, $\alpha(=1/m) > 1$, and from now on we shall deal only with Problem (1.1).

Remark. The function spaces we use are rather familiar and we omit the definitions (see e.g. [8], [9]). In the sequel we shall adopt the notation $\int_D u(t)\, \varphi(t) = \int_D u(x, t)\, \varphi(x, t)\, dx$. Nonnegative constants will be denoted by C, which may stand for various constants even in the same discussion.

Our result reads as follows:

Theorem 1.2. *Let $T > 0$ be arbitrarily fixed. Suppose that $u_0 \in H_0^1(D) \cap L^\infty(D)$, $u_0 \geqq 0$, $f(0) \geqq 0$ and f is globally Lipschitz continuous with a constant K.*

Then Problem (1.1) admits a (strong) nonnegative solution $u(x, t)$ such that

$$u \in C([0, T]\,;\, L^2(D)) \cap L^\infty([0, T]\,;\, H_0^1(D)) \cap L^\infty(Q_T)\,,$$
$$u^{(\alpha+1)/2} \in H^1([0, T]\,;\, L^2(D))\,,$$

and the equation is satisfied in the sense that

$$(1.3)\qquad \int_D ((\beta(u(t)))_t\, w + \nabla u(t)\, \nabla w) = \int_D f(\beta(u(t)))\, w\,, \quad u(0) = u_0$$

holds for any test function $w \in H_0^1(D)$ *and a.e. on* $[0, T]$.

Moreover, the following estimates hold:

$$(1.4)\qquad \|u(t)\|_{L^\infty(D)} \leqq (\|u_0\|_{L^\infty(D)} + (f(0)/\varepsilon)^m) \exp((K + \varepsilon)\, mt)$$

(*the "maximum principle"*) *for* $0 \leqq t \leqq T$, $0 < \varepsilon < \infty$,

$$(1.5)\qquad \frac{4\alpha}{(\alpha + 1)^2} \int_0^t \|(u^{(\alpha+1)/2})_t\|^2_{L^2(D)} + V(u(t)) \leqq V(u_0)$$

for $0 \leqq t \leqq T$, *where* $V(\xi) = \frac{1}{2} \int_D |\nabla \xi|^2 - \int_D \int_0^\xi f(\beta(r))\, \mathrm{d}r$, *and*

$$(1.6)\qquad \|u(t) - u(s)\|_{L^2(D)} \leqq C|t - s|^{1/(\alpha+1)}$$

for $0 \leqq t,\ s \leqq T$.

The key idea for solving Problem (1.1) is to replace the t-derivative by a difference quotient of the given step size $\Delta_n t$, where

$$\Delta_n t = T/n \quad \text{for} \quad n \geqq n_0$$

(without loss of generality we may assume $n_0 \geqq KT$).

Therefore, let us first treat the problems

$$(1.7)\qquad \begin{aligned} &\frac{\beta(u_i^n) - \beta(u_{i-1}^n)}{\Delta_n t} - \Delta u_i^n = f_{i-1}^n \quad \text{in} \quad D\,, \\ &u_i^n = 0 \quad \text{on} \quad \partial D\,, \quad i = 1, 2, \dots \end{aligned}$$

where $f_{i-1}^n = f(\beta(u_{i-1}^n))$ and $u_0^n = u_0$ is given.

Under our assumptions we are able to solve Porblems (1.7) recursively for u_i^n by the already known u_{i-1}^n and thereby construct sequences of step approximate "solutions" $\{\bar{u}_n\}$, $\{\bar{u}_n^*\}$, defined by

$$(1.8)\qquad \bar{u}_n (= \bar{u}_n(x, t)) = u_i^n(x) \quad \text{for} \quad i\Delta_n t \leqq t < (i + 1)\, \Delta_n t\,, \quad i = 0, 1, \dots,$$

$$(1.9)\qquad \bar{u}_n^* (= \bar{u}_n^*(x, t)) = \beta(u_i^n(x)) \quad \text{for} \quad i\Delta_n t \leqq t < (i + 1)\, \Delta_n t\,, \quad i = 0, 1, \dots,$$

and piecewise linear approximate "solutions" $\{u_n^*\}$,

$$(1.10)\qquad \begin{aligned} &u_n^* (= u_n^*(x, t)) = \beta(u_i^n(x)) + \frac{t - i\Delta_n t}{\Delta_n t} (\beta(u_{i+1}^n) - \beta(u_i^n))\, (x) \quad \text{for} \\ &i\Delta_n t \leqq t \leqq (i + 1)\, \Delta_n t\,, \quad i = 0, 1, \dots. \end{aligned}$$

Their convergence to a desired solution u (or $\beta(u)$) of Problem (1.1) in the sense given later on will be shown. We start with

Definition 1.11. *A function u_i^n is said to be a solution of (1.7) if $u_i^n \in H_0^1(D) \cap \cap L^{\alpha+1}(D)$ and the following identity holds for all $w \in H_0^1(D) \cap L^{\alpha+1}(D)$:*

$$\int_D \left(\frac{\beta(u_i^n) - \beta(u_{i-1}^n)}{\Delta_n t} w + \nabla u_i^n \nabla w \right) = \int_D f_{i-1}^n w . \tag{1.12}$$

Lemma 1.13. *There exists a unique solution u_i^n of Problem (1.7) for any positive integers i, $n \geqq n_0$.*

Moreover, each $u_i^n \in L^\infty(D)$ and $u_i^n \geqq 0$ on D.

Proof. By induction with respect to i ($n \geqq n_0$ arbitrary). By the assumption we have $u_0^n = u_0 \in H_0^1(D) \cap L^\infty(D)$, $u_0 \geqq 0$. Suppose the assertion is true for $i = 1, 2, \ldots, k - 1$. Let us now prove it for $i = k$.

Existence. Let

$$J(v) = \int_D \left(2^{-1}|\nabla v|^2 + (\Delta_n t(\alpha + 1))^{-1} |v|^{\alpha+1} - f_{k-1}^n v - (\Delta_n t)^{-1} \beta(u_{i-1}^n) v\right) .$$

Then it is easy to see that J is a continuous, strictly convex and coercive functional over $H_0^1(D) \cap L^{\alpha+1}(D)$. The existence of a solution of Problem (1.7) for $i = k$ now follows immediately if the classical results concerning the minimization of J,namely the existence of the minimum and the characterization of solutions, are taken into account (see e.g. [6]).

Nonnegativity. By contradiction. Let $u_k^n < 0$ on $E \subset D$ and meas $(E) > 0$. Putting $w = (u_k^n)^-$ ($=\min(0, u_k^n)$) into (1.12) we obtain

$$0 < (\Delta_n t)^{-1} \int_E (\beta(u_k^n) u_k^n + |\nabla u_k^n|^2) = \int_E (f_{k-1}^n + (\Delta_n t)^{-1} \beta(u_{k-1}^n)) u_k^n \leqq 0 ,$$

because $f_{k-1}^n + (\Delta_n t)^{-1} \beta(u_{k-1}^n) \geqq 0$ (by the induction hypothesis $u_{k-1}^n \geqq 0$ and $K \leqq (\Delta_n t)^{-1}$). This contradiction yields $u_k^n \geqq 0$.

Boundedness. Suppose the contrary, that is, there exists $\{c_j\}_{j=1}^\infty$, $c_j \leqq c_{j+1}$, $c_j \to \infty$ for $j \to \infty$ and $K_j = \{x \in D, u_k^n > c_j\}$ with meas $(K_j) > 0$. Putting

$$u_k^n|^j = \begin{cases} u_k^n & \text{for} \quad x \in D \setminus K_j , \\ c_j & \text{for} \quad x \in K_j \end{cases}$$

we easily obtain $J(u_k^n|^j) < J(u_k^n)$ for sufficiently large j, which contradicts the minimum property of u_k^n.

Lemma 1.14. *For any $n \geqq n_0$ and $1 \leqq i \leqq n$, the solution u_i^n of Problem* (1.7) *satisfies*

$$\|u_i^n\|_{L^\infty(D)} \leqq (\|u_0\|_{L^\infty(D)} + (f(0)/\varepsilon)^m) \exp\left(\frac{i\Delta_n t(K+\varepsilon)}{1-\Delta_n t\varepsilon}\right), \tag{1.15}$$

for any ε, $0 < \varepsilon < (\Delta_n t)^{-1}$.

Proof. Let us omit the index n throughout the proof. Putting $w = (u_i)^k$ into (1.12), where a positive integer k is arbitrarily large (it may be easily shown by Lemma 1.13 that $(u_i)^k \in H_0^1(D) \cap L^\infty(D)$), we obtain

$$\int_D (u_i)^{\alpha+k} + \Delta t k \int_D (u_i)^{k-1} |\nabla u_i|^2 = \Delta t \int_D (f_{i-1} + (\Delta t)^{-1} \beta(u_{i-1}))(u_i)^k,$$

which by the Lipschitz continuity of f implies

$$\int_D (u_i)^{\alpha+k} \leqq (1 + K\Delta t) \int_D (u_{i-1})^\alpha (u_i)^k + f(0)\Delta t \int_D (u_i)^k. \tag{1.16}$$

Applying Young's inequality to the last term of (1.16) we further have

$$(1-\varepsilon\Delta t)\int_D (u_i)^{\alpha+k} \leqq (1-\varepsilon\Delta t)\int_D (1+\eta\Delta t)(u_{i-1})^\alpha (u_i)^k + \Delta t\, C(\varepsilon)(f(0))^{(\alpha+k)/\alpha},$$

where $\eta = (K+\varepsilon)/(1-\varepsilon\Delta t)$ and $C(\varepsilon) = (k/\varepsilon(\alpha+k))^{k/\alpha}\alpha \operatorname{meas}(D)/(\alpha+k)$. The above inequality yields (again by Young's inequality)

$$\begin{aligned}\int_D (u_i)^{\alpha+k} &\leqq (1+\eta\Delta t)^{(\alpha+k)/\alpha}\int_D (u_{i-1})^{\alpha+k} + \\ &+ (\alpha - \alpha\varepsilon\Delta t)^{-1}(\alpha+k)\, C(\varepsilon)\Delta t\, f(0)^{(\alpha+k)/\alpha}.\end{aligned} \tag{1.17}$$

From (1.17) we obtain recurrently

$$\int_D (u_i)^{\alpha+k} \leqq (1+\eta\Delta t)^{i(\alpha+k)/\alpha}\left(\int_D (u_0)^{\alpha+k} + (\varepsilon^{-1} f(0))^{(\alpha+k)/\alpha} C'\right), \tag{1.18}$$

where the constant $C' \equiv C'(\alpha, k, \varepsilon)$ is such that its $(\alpha+k)$-th root tends to 1 if $k \to \infty$. Indeed, (1.17) may be formally rewirtten as $y_i \leqq a y_{i-1} + b \leqq \ldots \leqq a^i y_0 + b(a^i - 1)/(a-1) \leqq a^i(y_0 + b/(a-1))$ and going back using $\eta\Delta t((1+\eta\Delta t)^{i(\alpha+k)/\alpha} - 1)^{-1} \leqq 1$ we have (1.18). Now, taking the $(\alpha+k)$-th root of (1.18) and letting $k \to \infty$ we obtain

$$\|u_i\|_{L^\infty(D)} \leqq (1+\eta\Delta t)^{i/\alpha}(\|u_0\|_{L^\infty(D)} + (\varepsilon^{-1} f(0))^m),$$

where we can still estimate $(1+\eta\Delta t)^{i/\alpha}$ by $\exp(i\Delta t\, m(K+\varepsilon)/(1-\varepsilon\Delta t))$. This completes the proof.

Corollary 1.19. *The sequences of the approximate "solutions" defined by* (1.8)–(1.10) *are bounded in* $L^\infty(Q_T)$ (*e.g.*, $\{\bar{u}_n\}$ *is bounded by* $L_1 = (\|u_0\|_{L^\infty(D)} + (f(0)/\varepsilon)^m) \exp(Tm(K + \varepsilon)/(1 - \Delta_{n_0} t\varepsilon))$).

Lemma 1.20. *There exist nonnegative constants* L_i *such that*

$$\text{(1.21)}\qquad \begin{aligned} &\text{(i)}\quad \sum_{i=1}^{n} \left\| \frac{(u_i^n)^{(\alpha+1)/2} - (u_{i-1}^n)^{(\alpha+1)/2}}{\Delta_n t} \right\|_{L^2(D)}^2 \Delta_n t \leqq L_2 ,\\ &\text{(ii)}\quad \|\nabla u_i^n\|_{L^2(D)} \leqq L_3 , \quad 1 \leqq i \leqq n ,\\ &\text{(iii)}\quad \sum_{i=1}^{n} \left\| \frac{\beta(u_i^n) - \beta(u_{i-1}^n)}{\Delta_n t} \right\|_{L^2(D)}^2 \Delta_n t \leqq L_4 , \end{aligned}$$

where all estimates are uniform with respect to $n \geqq n_0$ *and* L_i *depend only on the data of our problem and on* T.

Proof. Putting $w = u_i - u_{i-1}$ (again omitting the index n) into (1.12) we obtain

$$\text{(1.22)}\qquad \begin{aligned} &(\Delta t)^{-1} \int_D (\beta(u_i) - \beta(u_{i-1}))(u_i - u_{i-1}) + 2^{-1} \int_D |\nabla u_i|^2 - 2^{-1} \int_D |\nabla u_{i-1}|^2 \leqq \\ &\leqq \int_D \left(\int_{u_{i-1}}^{u_i} f(\beta(r))\, dr + \int_{u_{i-1}}^{u_i} (f(\beta(u_{i-1})) - f(\beta(r)))\, dr \right) \leqq \\ &\leqq \int_D \left(\int_{u_{i-1}}^{u_i} f(\beta(r))\, dr + K(\beta(u_i) - \beta(u_{i-1}))(u_i - u_{i-1}) \right). \end{aligned}$$

In the sequel, the following inequality plays an important role:

$$\text{(1.23)}\qquad \frac{4\alpha}{(\alpha + 1)^2} (y^{(\alpha+1)/2} - z^{(\alpha+1)/2})^2 \leqq (y^\alpha - z^\alpha)(y - z)$$

for $y, z \geqq 0$, which may be verified e.g. by simple calculation with a nonnnegative function $g(\lambda) = (\lambda^\alpha - 1)(\lambda - 1) - (4\alpha/(\alpha + 1)^2)(\lambda^{(\alpha+1)/2} - 1)^2$ for $\lambda \geqq 1$.

So (1.22) by (1.23) yields

$$\begin{aligned} &\frac{4\alpha(1 - K\,\Delta t)}{(\alpha + 1)^2} \int_D \left| \frac{(u_i)^{(\alpha+1)/2} - (u_{i-1})^{(\alpha+1)/2}}{\Delta t} \right|^2 \Delta t + \\ &+ 2^{-1} \int_D |\nabla u_i|^2 - 2^{-1} \int_D |\nabla u_{i-1}|^2 \leqq \int_D \int_{u_{i-1}}^{u_i} f(\beta(r))\, dr . \end{aligned}$$

Summing the above inequality through $i = 1, 2, \ldots, k$ we obtain

$$\text{(1.24)}\qquad \frac{4\alpha(1 - K\,\Delta t)}{(\alpha + 1)^2} \sum_{i=1}^{k} \int_D \left| \frac{(u_i)^{(\alpha+1)/2} - (u_{i-1})^{(\alpha+1)/2}}{\Delta t} \right|^2 \Delta t + \frac{1}{2} \int_D |\nabla u_k|^2 -$$

$$-\int_D\int_0^{u_k} f(\beta(r))\,dr \leqq \frac{1}{2}\int_D |\nabla u_0|^2 - \int_D\int_0^{u_0} f(\beta(r))\,dr\,.$$

Now, Lemma 1.14 and the obvious estimate

$$|\beta(u_i) - \beta(u_{i-1})| \leqq \frac{2\alpha}{(\alpha+1)}(\max(u_i, u_{i-1}))^{(\alpha-1)/2}\,|(u_i)^{(\alpha+1)/2} - (u_{i-1})^{(\alpha+1)/2}|$$

a.e. on D give the assertion (1.21).

As a result of Lemma 1.20 and Corollary 1.19, using a standard argument (see e.g. [8, Lemma 1.3.13]) we arrive at

Corollary 1.25. (i) *The sequence* $\{\bar{u}_n\}$ (see (1.8)) *is bounded in* $L^\infty([0, T]; H_0^1(D)) \cap \cap L^\infty(Q_T)$ *and there exists a function u such that*

$$\bar{u}_n \to u \quad \textit{weakly* in} \quad L^\infty([0, T];\ H_0^1(D)) \tag{1.26}$$

(through a subsequence).

(ii) *The sequence* $\{u_n^*\}$ (see (1.10)) *is bounded in* $H^1([0, T]; L^2(D)) \cap L^\infty([0, T]; H_0^1(D)) \cap L^\infty(Q_T)$, $\{\bar{u}_n^*\}$ *given by* (1.9) *is bounded in* $L^\infty([0, T];\ H_0^1(D)) \cap L^\infty(Q_T)$, *and*

(1.27)

(i) $u_n^* \to u^*$ *weakly in* $H^1([0, T]; L^2(D))$,

(ii) $u_n^* \to u^*$ *strongly in* $C([0, T]; L^2(D))$,

(iii) $\bar{u}_n^*, u_n^* \to u^*$ *weakly in* $H_0^1(D)$ *for a.e.* $t \in [0, T]$,

(iv) $\bar{u}_n^* \to u^*$ *strongly in* $L^2(Q_T)$,

(v) $\bar{u}_n^*, u_n^* \to u^*$ *a.e. on* Q_T, *(through a subsequence), where* $u^* = \beta(u)$ *and* $u \in C([0, T]; L^2(D))$.

The fact that $u^* = \beta(u)$ follows from the monotonicity of $\beta(\cdot)$, (1.26) and (1.27) (iv).

Indeed, from the monotonicity of β we have

$$0 \leqq \int_0^T\int_D (\bar{u}_n^* - \beta(w))(\bar{u}_n - w)\,,$$

and letting $n \to \infty$ we obtain

$$0 \leqq \int_0^T\int_D (u^* - \beta(w))(u - w) \quad \text{for any} \quad w \in L^\infty(Q_T)\,.$$

Putting $w = u \pm \lambda v$, $\lambda > 0$, the above inequality (after letting $\lambda \to 0$) yields

$$\int_0^T\int_D (u^* - \beta(u))\,v = 0 \quad \text{for any} \quad v \in L^\infty(Q_T) \quad \text{and so} \quad u^* = \beta(u)\,.$$

Now, using our notation, (1.12) may be rewritten in the form

$$\int_D ((u_n^*(t))_t\, w + \nabla\bar{u}_n(t)\,\nabla w) = \int_D f(\bar{u}_n^*(t))\, w\,, \quad 0 < t \leqq T\,.$$

Multiplying this identity by $q(t) \in L^\infty(0, T)$ and integrating over $(0, T)$ we obtain

$$\int_0^T\int_D ((u_n^*)_t\, wq + \nabla\bar{u}_n\,\nabla(wq)) = \int_0^T\int_D f(\bar{u}_n^*)\, wq\,. \tag{1.28}$$

Letting $n \to \infty$ in (1.28) we obtain by (1.27) (i), (1.26), (1.27) (iv) and the global Lipschitz continuity of f

$$\int_0^T \left(\int_D ((\beta(u))_t\, w + \nabla u\,\nabla w - f(\beta(u))\, w\right) q = 0 \quad \text{for any} \quad q \in L^\infty(0, T)\,,$$

hence (1.3).

To prove the estimates (1.4) and (1.5), let now $t \in [0, T]$ be arbitrary but fixed, and for each n let $i(n)$ be such that

$$t_n \leqq t < t_n + \Delta_n t\,, \quad \text{where} \quad t_n = \Delta_n t i(n)\,. \tag{1.29}$$

By (1.29), the estimate (1.15) yields

$$\|u_{i(n)}^n\|_{L^\infty(D)} \leqq (\|u_0\|_{L^\infty(D)} + (f(0)/\varepsilon)^m \exp\left(\frac{tm(K + \varepsilon)}{(1 - \Delta_n t\varepsilon)}\right). \tag{1.30}$$

Taking into account that $\bar{u}_n(t) = u_{i(n)}^n$ on $[t_n, t_n + \Delta_n t)$ for each n and that (1.27) (v) implies $\bar{u}_n \to u$ a.e. on Q_T we have the "maximum principle" (1.4).

To show the energy estimate (1.5) we rewrite (1.24) with the help of (1.29) as

$$C_n \int_0^t\int_D |(u_n^{**})_t|^2 + V(\bar{u}_n(t)) \leqq V(u_0) + C_n \int_{t_n}^t\int_D |(u_n^{**})_t|^2\,,$$

where u_n^{**} is constructed analogously to u_n^* in (1.10) by means of $(u_i^n)^{(\alpha+1)/2}$. Using the estimate (1.21) (i) we obtain the validity of Corollary 1.25 (ii) for u_n^{**} in the same way, and $u^{**} = u^{(\alpha+1)/2}$.

Now, letting $n \to \infty$ we easily obtain (1.5) by virtue of the weak lower semicontinuity of a norm and the fact that $t_n \to t$ as $n \to \infty$ and (1.21) (i) $(C_n \to 4\alpha(\alpha + 1)^{-2})$.

To verify (1.6) we calculate

$$\|u(t) - u(s)\|_{L^2(D)}^2 \leqq \int_D |u^{(\alpha+1)/2}(t) - u^{(\alpha+1)/2}(s)|^{4/(\alpha+1)} \leqq$$

$$\leqq C\left(\int_D\left|\int_s^t (u^{(\alpha+1)/2})_t\right|^2\right)^{2/(\alpha+1)} \leqq C\left(\left(\int_D\int_0^T |(u^{(\alpha+1)/2})_t|^2\right)|t-s|\right)^{2/(\alpha+1)},$$

hence the assertion. This completes the proof of Theorem 1.2.

2. COMPARISON AND CONTINUOUS DEPENDENCE

In this section we follow the corresponding part of [1]. Though the procedure is very similar, we shall indicate its main points because we cannot use the results of [1] for another type of nonlinearity.

Therefore, let us consider the problem

$$(2.1)\qquad \begin{aligned} (\eta(u))_t - \Delta u &= g(u) && \text{in} \quad Q_T\,,\\ u(x,t) &= 0 && \text{on} \quad S_T\,,\\ u(x,0) &= u_0(x) && \text{in} \quad D\,, \end{aligned}$$

where $\eta: R \to R$ is locally Lipschitz continuous and nondecreasing,

$g: R \to R$ is "locally Lipschitz" continuous in the sense that for each bounded subset U of R there exists a constant L_U such that $|g(u) - g(v)| \leqq L_U|\eta(u) - \eta(v)|$ for all $u, v \in U$.

Definition 2.2. *A (weak) solution u of Problem (2.1) on $[0, T]$ is a function u with the following properties:*

(i) $u \in C([0, T]; L^1(D)) \cap L^\infty(Q_T)$,

(ii) *u satisfies*

$$(2.3)\qquad \int_D \eta(u(t))\,\varphi(t) - \iint_{Q_t} (\eta(u)\,\varphi_t + u\,\Delta\varphi + g(u)\,\varphi) = \int_D \eta(u_0)\,\varphi(0)\,,\quad 0 \leqq t \leqq T,$$

for all $\varphi \in C^{2,1}(\bar{Q}_T)$, $\varphi \geqq 0$ and $\varphi = 0$ on ∂D, $0 \leqq t \leqq T$. A solution on $[0, \infty)$ (global solution) means a solution on each $[0, T]$, a subsolution (supersolution) is defined by (i) *and* (ii) *with equality replaced by $\leqq$ ($\geqq$).*

Clearly, the strong solution from Theorem 1.2 is also a weak solution of Problem (1.1) in the sense of the above definition.

Theorem 2.4. (i) *Let u, v be solutions of Problem (2.1) on $[0, T]$ with initial data u_0, v_0, respectively. Let K be a Lipschitz constant for g on $[-M, M]$, where $M = \max(\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)})$. Then*

$$(2.5)\qquad \|\eta(u(t)) - \eta(v(t))\|_{L^1(D)} \leqq \|\eta(u_0) - \eta(v_0)\|_{L^1(D)} \exp(Kt)\,.$$

(ii) *Let u be a subsolution and v a supersolution of Problem (2.1) with initial data u_0 and v_0, respectively. Then $u_0 \leqq v_0$ implies that $u \leqq v$ a.e. on Q_T.*

Proof. We start with (ii). For u and v, (2.3) gives

$$\int_D (\eta(u(t)) - \eta(v(t)))\,\varphi(t) - \iint_{Q_t} (u - v)\,(a\varphi_s + \Delta\varphi) \leqq \tag{2.6}$$
$$\leqq \int_D (\eta(u_0) - \eta(v_0))\,\varphi(0) + \iint_{Q_t} (g(u) - g(v))\,\varphi$$

for any test function φ, where

$$a = \begin{cases} (\eta(u) - \eta(v))/(u - v) & \text{for} \quad u \neq v\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and under our assumptions it is easy to see that $a \in L^\infty(Q_T)$ and $a \geqq 0$.

Now let $a_n = R_\varepsilon a + n^{-1}$, where R_ε is a mollifier (see [5, page 72]) and ε is such that $\|a - R_\varepsilon a\|_{L^2(Q_T)} \leqq n^{-1}$. Then a_n are smooth and it is not difficult to see that

$$n^{-1} \leqq a_n \leqq \|a\|_{L^\infty(Q_T)} + n^{-1}\,, \quad (a_n - a)/\sqrt{a_n} \to 0 \quad \text{in} \quad L^2(Q_T) \tag{2.7}$$
$$\text{as} \quad n \to \infty\,.$$

From a_n we obtain functions φ_n as solutions of the backward problems

$$\begin{aligned} & a_n(x, s)\,(\varphi_n)_s + \Delta\varphi_n = \lambda a_n(x, s)\,\varphi_n && \text{for} \quad x \in D\,, \quad s \in [0, t)\,, \\ & \varphi_n(x, s) = 0 && \text{on} \quad \partial D\,, \quad 0 \leqq s < t\,, \\ & \varphi_n(x, t) = \chi(x) && \text{on} \quad D\,, \end{aligned} \tag{2.8}$$

where $0 < t \leqq T$ is now arbitrary but fixed $\chi(x) \in C_0^\infty(D)$, $0 \leqq \chi(x) \leqq 1$. The existence of $\varphi_n \in C^{2,1}(\bar{Q}_T)$ follows from the fact that (2.8) is a nondegenerate parabolic problem for each n, which may be rewritten into

$$\begin{aligned} & (\psi_n)_\tau - a_n^{-1}(x, t - \tau)\,\Delta\psi_n = -\lambda\psi_n && \text{in} \quad D \times (0, t]\,, \\ & \psi_n(x, \tau) = 0 && \text{on} \quad \partial D \times [0, t]\,, \\ & \psi_n(x, 0) = \chi(x) && \text{in} \quad D\,, \end{aligned} \tag{2.9}$$

and all its data are smooth (see e.g. [10, page 364]).

Moreover,

$$0 \leqq \varphi_n(x, s) \leqq \exp(-\lambda(t - s)) \text{ on } D\,,\ 0 \leqq s \leqq t \text{ and } \iint_{Q_t} a_n((\varphi_n)_s)^2 \leqq C\,, \tag{2.10}$$

where the constant C does not depend on n. The first assertion is a consequence of the maximum principle [12, page 173] and the second can be obtained from (2.9). Multiplying the equation by $a_n(\psi_n)_\tau$ and integrating over Q_t we find after integrating by parts and using Young's inequality

$$\iint_{Q_t} a_n((\psi_n)_\tau)^2 + \frac{1}{2}\int_D |\nabla\psi_n(t)|^2 \leqq \frac{1}{2}\int_D |\nabla\chi|^2 + \varepsilon\iint_{Q_t} a_n((\psi_n)_\tau)^2 + C(\varepsilon)\iint_{Q_t} a_n\lambda^2\psi_n^2\,,$$

hence the assertion, because

$$\iint_{Q_t} a_n(x, t - \tau)\,((\psi_n)_\tau)^2\,dx\,d\tau = \iint_{Q_t} a_n(x, s)\,((\varphi_n)_s)^2\,dx\,ds\,.$$

Now, if we put $\varphi = \varphi_n$ into (2.6) we obtain

$$(2.11)\qquad \int_D (\eta(u(t)) - \eta(v(t)))\,\chi - \iint_{Q_t} (u - v)\,\lambda a_n \varphi_n \leqq$$

$$\leqq \int_D (\eta(u_0) - \eta(v_0))\,\varphi_n(0) + \iint_{Q_t} (g(u) - g(v))\,\varphi_n + 2M \iint_{Q_t} \left|\frac{a - a_n}{\sqrt{a_n}}\sqrt{a_n}\,(\varphi_n)_s\right|,$$

where the last term tends to zero as $n \to \infty$ by (2.7) and $(2.10)_2$. Thus, letting $n \to \infty$ in (2.11) we have by $(2.10)_1$

$$(2.12)\qquad \begin{aligned} \int_D (\eta(u(t)) - \eta(v(t)))\,\chi &\leqq \int_D (\eta(u_0) - \eta(v_0))^+ \exp(-\lambda t) + \\ &+ \iint_{Q_t} (g(u) - g(v) + \lambda(\eta(u) - \eta(v)))^+ \exp(\lambda(s - t))\,, \end{aligned}$$

where $\zeta^+ = \max(\zeta, 0)$, because by (2.7), $a_n \to a$ in $L^2(Q_T)$ as $n \to \infty$. By the same argument as in [1] (we omit it) (2.12) implies

$$\int_D (\eta(u(t)) - \eta(v(t)))^+ \leqq \int_D (\eta(u_0) - \eta(v_0))^+ \exp(Kt)\,,$$

which proves Theorem 2.4 (ii), and (i) follows by adding the corresponding inequality for $(\eta(v(t)) - \eta(u(t)))^+$.

As a consequence of Theorem 2.4 the existence of a global weak solution of Problem (1.1) may be proved for $u_0 \in L^\infty(D)$.

Theorem 2.13. *Let f be as in Theorem* 1.2 *and $u_0 \geqq 0$, $u_0 \in L^\infty(D)$. Then Problem* (1.1) *has a nonnegative weak solution on* $[0, \infty)$ (*in the sense of Definition* 2.2) *and the maximum principle* (1.4) *holds again.*

Proof. We choose $\{u_{0n}\} \subset H_0^1(D) \cap L^\infty(D)$ such that $\|u_{0n} - u_0\|_{L^1(D)} \to 0$ as $n \to \infty$ and $\|u_{0n}\|_{L^\infty(D)} \leqq \|u_0\|_{L^\infty(D)}$. Let u_n be strong solutions of Problem (1.1) with initial data u_{0n}. Then by (1.4) we have

$$\|u_n(t)\|_{L^\infty(D)} \leqq (\|u_0\|_{L^\infty(D)} + (f(0)/\varepsilon)^m) \exp((K + \varepsilon)\,mt)$$

and by (2.5),

$$\begin{aligned} \|\beta(u_n(t)) - \beta(u_k(t))\|_{L^1(D)} &\leqq \|\beta(u_{0n}) - \beta(u_{0k})\|_{L^1(D)} \exp(Kt) \leqq \\ &\leqq \alpha \|u_0\|_{L^\infty(D)}^{\alpha - 1} \|u_{0n} - u_{0k}\|_{L^1(D)} \exp(Kt)\,, \end{aligned}$$

which gives the existence of a function $w \in C([0, T]; L^1(D))$ such that $\beta(u_n) \to w$ strongly in $C([0, T]; L^1(D))$ as $n \to \infty$. Now by Lebesgue's dominated convergence theorem we have $u_n \to \beta^{-1}(w)$ strongly in $L^p(Q_T)$, $1 \leqq p < \infty$ (through a subsequence). Denoting $u = \beta^{-1}(w)$ it is easy to see that u is a weak solution of Problem (1.1) and satisfies (1.4). Moreover, due to Theorem 2.4, this solution is unique.

3. LOCAL EXISTENCE AND A BLOW UP RESULT

We start this section by stating its main results.

Theorem 3.1. *Let f be locally Lipschitz continuous, $f(0) \geqq 0$ and $u_0 \geqq 0$, $u_0 \in L^\infty(D)$. Then there exists $T_{\max}$, $0 < T_{\max} \leqq \infty$ such that Problem (1.1) has a unique weak solution on any $[0, T]$, $T < T_{\max}$.*

If in addition $u_0 \in H_0^1(D)$ then this solution is strong and satisfies (1.5) for any $0 \leqq t < T_{\max}$. In the case $T_{\max} < \infty$ we have

$$\lim_{t \to T_{\max}^-} \|u(t)\|_{L^\infty(D)} = +\infty . \tag{3.2}$$

Moreover, if f satisfies

$$f(r) \leqq Kr + f(0) \quad \text{for all} \quad 0 \leqq r < \infty , \tag{3.3}$$

where K is a constant, then $T_{\max} = \infty$, i.e. there exists a global solution of Problem (1.1).

Theorem 3.4. *Let f be locally Lipschitz continuous, $f(0) \geqq 0$ and*

$$f(r) \geqq cr^\gamma \quad \text{for some} \quad c > 0 , \quad \gamma > 1 \quad \text{and for all} \quad 0 \leqq r < \infty . \tag{3.5}$$

Suppose that

$$u_0 \geqq 0 , \quad u_0 \not\equiv 0 , \quad u_0 \in H_0^1(D) \cap L^\infty(D) , \quad 2^{-1} \int_D |\nabla u_0|^2 - c(\alpha\gamma + 1)^{-1} \int_D u_0^{\alpha\gamma+1} \leqq 0$$

and let $u(t)$ be a strong solution of (1.1) on $[0, T]$, $T > 0$. Then T satisfies

$$T < T_0 \equiv \left(\int_D u_0^{\alpha+1}\right)^{-(\gamma-1)} (\alpha\gamma + 1)/c(\gamma - 1)(\alpha + 1)(\alpha\gamma - 1) \tag{3.6}$$

and

$$\|u(t)\|_{L^{\alpha+1}(D)} \geqq \text{const}\,(T_0 - t)^{-1/(\gamma-1)} . \tag{3.7}$$

Proof of Theorem 3.1. Let $M = \|u_0\|_{L^\infty(D)} + (f(0))^m$ and define

$$f_M(\beta(r)) = \begin{cases} f(\beta(r)) & \text{for } |\beta(r)| \leqq M + 1 , \\ f(M + 1) & \text{otherwise} . \end{cases}$$

Then Problem (1.1) with f replaced by f_M has a unique global weak solution u_M (Theorem 2.13), and it satisfies

$$\|u_M(t)\|_{L^\infty(D)} \leqq M \exp((K + 1)\, mt) \quad ((1.4) \text{ for } \varepsilon = 1),$$

where K is a Lipschitz constant of the function f_M. Now we take δ so small as to have $M \exp((K + 1)\, m\delta) \leqq M + 1$. So $u_M(t)$ is a solution of the original Problem (1.1) on $[0, \delta]$ and by using the standard continuation procedure we obtain T_{max}, $0 < T_{max} \leqq \infty$, so that Problem (1.1) has a unique weak solution on $[0, T]$, $T < T_{max}$. The arguments for a strong solution are the same.

Now let $T_{max} < \infty$. We first show that

$$\overline{\lim_{t \to T_{max}^-}} \|u(t)\|_{L^\infty(D)} = +\infty . \tag{3.8}$$

If (3.8) does not hold then $\|u(t)\|_{L^\infty(D)} \leqq C$ for all $0 \leqq t < T_{max}$, and (1.5) yields

$$\int_0^{T_{max}} \|(u^{(\alpha+1)/2})_t\|^2_{L^2(D)} \leqq \text{const}.$$

So we have $\|u(t) - u(s)\|_{L^2(D)} \leqq C|t - s|^{1/(\alpha+1)}$ for all $0 \leqq t, s < T_{max}$, which implies that $\lim_{t \to T_{max}^-} u(t)$ exists in $L^2(D)$. Let us denote it by v. Then (1.5) gives that $u(t) \to v$ weakly in $H_0^1(D)$ as $t \to T_{max}^-$ and we have $v \in H_0^1(D) \cap L^\infty(D)$, contradicting the maximality of T_{max}.

Now suppose that (3.2) does not hold. Then there exists a sequence $t_n \to T_{max}^-$ as $n \to \infty$ with $\|u(t_n)\|_{L^\infty(D)} \leqq C$. Let K be a Lipschitz constant of f on $[0, C^* + 1]$, where $C^* = C + (f(0))^m$. Then for all n we have by (1.4)

$$\|u(t_n + t)\|_{L^\infty(D)} \leqq (\|u(t_n)\|_{L^\infty(D)} + (f(0))^m) \exp((K + 1)\, mt)$$

for $0 \leqq t \leqq t^*$, where $C^* \exp((K + 1)\, mt^*) = C^* + 1$. So

$$\|u(t_n + t)\|_{L^\infty(D)} \leqq C^* + 1 \quad \text{for all } n, \quad 0 \leqq t \leqq t^* .$$

But for sufficiently large n we have $T_{max} - t_n < t^*$, therefore

$$\|u(t)\|_{L^\infty(D)} \leqq C^* + 1 \quad \text{for} \quad t_n \leqq t < T_{max},$$

which contradicts (3.8).

In the end, let f satisfy (3.3), then the solution v of Problem (1.1) with $f(r) = Kr + f(0)$ is a supersolution of Problem (1.1) (with the original f) which exists globally (Theorem 1.2) and so does u by Theorem 2.4 (ii).

Proof of Theorem 3.4. Let v be a solution of Problem (1.1) with $f(r) = cr^\gamma$ and $v_0 = u_0$. Then v is a subsolution of (1.1) and by the comparison result stated in Theorem 2.4 (ii) it is sufficient to prove the assertion for v. The proof proceeds in a standard way (see e.g. [11]).

The assumptions on u_0 imply by (1.5) that

$$\int_D |\nabla v(t)|^2 \leqq \frac{2c}{\alpha\gamma + 1} \int_D (v(t))^{\alpha\gamma+1} \quad \text{for} \quad 0 \leqq t \leqq T. \tag{3.9}$$

Now putting $v(t)$ into (1.3) and integrating we obtain

$$\int_D v^{\alpha+1}(t) - \int_D u_0^{\alpha+1} = \frac{\alpha+1}{\alpha} \int_0^t \int_D \left(-|\nabla v|^2 + cv^{\alpha\gamma+1}\right). \tag{3.10}$$

Since $y(t) \stackrel{\text{def}}{=} \int_D v^{\alpha+1}(t)$ is absolutely continuous on $[0, T]$, (3.10) yields

$$y'(t) = \frac{\alpha+1}{\alpha} \int_D \left(-|\nabla v(t)|^2 + c(v(t))^{\alpha\gamma+1}\right),$$

which by (3.9) and the Hölder inequality gives the differential inequality

$$y'(t) - \frac{(\alpha+1)\, c(\alpha\gamma-1)}{(\alpha\gamma+1)\,\alpha} (y(t))^{(\alpha\gamma+1)/(\alpha+1)} \geqq 0$$

for a.e. $t \in [0, T]$, assuming for convenience that meas $(D) = 1$. As $(\alpha\gamma + 1)\,.\,(\alpha+1)^{-1} > 1$, by the standard comparison theorem for ordinary differential equations we have

$$(y(t))^{\gamma-1} \geqq \left[\left(\int_D u_0^{\alpha+1}\right)^{-(\gamma-1)} - \frac{(\gamma-1)(\alpha+1)\, c(\alpha\gamma-1)}{(\alpha\gamma+1)\,\alpha}\, t\right]^{-1},$$

hence (3.6) and (3.7). This completes the proof.

4. A FINITE EXTINCTION TIME

Let us begin with the simple problem

$$\begin{aligned} &(\beta(u))_t - \Delta u = 0 && \text{in} \quad D \times (0, \infty), \\ &u(x, t) = 0 && \text{on} \quad \partial D \times (0, \infty), \\ &u(x, 0) = u_0(x) && \text{in} \quad D, \end{aligned} \tag{4.1}$$

where $u_0 \in H_0^1(D) \cap L^\infty(D)$ for a while. It is known that the solution of (4.1) has a finite extinction time T^*, i.e. $u \equiv 0$ for $t \geqq T^*$ (see [13], [2]).

To estimate T^* let us sketch the proof of its existence. By Theorem 1.2, Problem (4.1) has a global strong solution and (1.3) for $w = u(t)$ yields

$$\int_D u^{\alpha+1}(t) - \int_D u_0^{\alpha+1} + \frac{\alpha+1}{\alpha} \int_0^t \int_D |\nabla u|^2 = 0,$$

which implies that $\int_D u^{\alpha+1}(t)$ is absolutely continuous in t. So we have

$$(4.2) \qquad \frac{d}{dt}\int_D u^{\alpha+1}(t) + \frac{\alpha+1}{\alpha}\int_D |\nabla u(t)|^2 = 0 \quad \text{for a.e.} \quad t\,.$$

If either $N = 1, 2$ and $\alpha > 1$ is arbitrary, or $N \geqq 3$ and $1 < \alpha \leqq (N+2)/(N-2)$, we obtain from (4.2) that

$$\frac{d}{dt}\int_D u^{\alpha+1}(t) + C\frac{\alpha+1}{\alpha}\left(\int_D u^{\alpha+1}(t)\right)^{2/(\alpha+1)} \leqq 0\,,$$

because of the continuous imbedding $H_0^1(D)$ into $L^{\alpha+1}(D)$. This inequality implies the existence of $T^*(=T^*(u_0)) > 0$ such that $u(t) \equiv 0$ for $t \geqq T^*$ (see [2]) and

$$(4.3) \qquad T^* = \alpha\|u_0\|^{\alpha-1}_{L^{\alpha+1}(D)}/C(\alpha-1)\,.$$

In the case $N \geqq 3$ and $\alpha > (N+2)/(N-2)$ we get, using the Nirenberg-Gagliardo inequality (see e.g. [5, page 27]), the estimate

$$(4.4) \qquad \|\xi\|_{L^{\alpha+1}(D)} \leqq C\left(\int_D |\nabla\xi|^2\right)^{a/2} (\|\xi\|_{L^\infty(D)})^{1-a}\,,$$

where $\xi \in H_0^1(D) \cap L^\infty(D)$ and $a = 2N/(N-2)(\alpha+1) < 1$. Now by (4.4), (1.4) and (4.2) we obtain

$$\frac{d}{dt}\int_D u^{\alpha+1}(t) + \frac{\alpha+1}{\alpha}\, C'(\|u_0\|_{L^\infty(D)})^{2(a-1)/a}\left(\int_D u^{\alpha+1}(t)\right)^{(N-2)/N} \leqq 0\,.$$

Since $(N-2)/N < 1$, we again get $T^*(=T^*(u_0)) > 0$ such that $u(t) \equiv 0$ for $t \geqq T^*$ and

$$(4.5) \qquad T^* = \alpha N(\|u_0\|_{L^\infty(D)})^{2(1-a)/a}(\|u_0\|_{L^{\alpha+1}(D)})^{2(\alpha+1)/N}(2(\alpha+1)\,C')^{-1}\,.$$

So we have

Theorem 4.6. *For any $u_0 \in L^\infty(D)$, $u_0 \geqq 0$ there exists $T^*(u_0) \geqq 0$ such that the solution of Problem* (4.1) *vanishes for $t \geqq T^*$, i.e., $u(t) \equiv 0$ for $t \geqq T^*$ and $T^*(u_0) = 0$ only for $u_0 \equiv 0$.*

It remains to remove only the restriction $u_0 \in H_0^1(D)$ from the beginning. To this end, let $u_0 \in L^\infty(D)$ and choose a sequence $\{u_{0n}\} \subset H_0^1(D) \cap L^\infty(D)$ such that

$$\|u_0 - u_{0n}\|_{L^1(D)} \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \|u_{0n}\|_{L^\infty(D)} \leqq \|u_0\|_{L^\infty(D)}\,.$$

Let u_n be the solutions of Problem (4.1) with initial data u_{0n}. Then $u_n(t) \equiv 0$ for $t \geqq T^*(u_{0n})$ and it is easy to see that $\overline{\lim}_{n\to\infty} T^*(u_{0n}) \leqq T^*(u_0)$, where T^* is given by (4.3) or (4.5) with regard to N and α. Now due to (2.5) we have

$$\|\beta(u(t))\|_{L^1(D)} \leqq \|\beta(u_0) - \beta(u_{0n})\|_{L^1(D)} \leqq \alpha\|u_0\|^{\alpha-1}_{L^\infty(D)}\|u_0 - u_{0n}\|_{L^1(D)}$$

for $t \geqq T^*(u_{0n})$ and letting $n \to \infty$ we have the assertion.

Now we consider the problem

$$(4.7) \qquad \begin{aligned} &(\beta(u))_t - \Delta u = \lambda\, \beta(u) && \text{in} \quad D \times (0, \infty),\ \lambda > 0\,, \\ &u(x, t) = 0 && \text{on} \quad \partial D \times (0, \infty)\,, \\ &u(x, 0) = u_0(x) && \text{in} \quad D\,. \end{aligned}$$

Putting $u = v \exp(\lambda t/\alpha)$ the equation in (4.7) gives

$$(\beta(v))_t \exp(\lambda t(\alpha - 1)/\alpha) = \Delta v$$

and after a simple transformation of time, $t = -c^{-1} \ln(1 - c\tau)$ where $c = \lambda(\alpha - 1)/\alpha$, Problem (4.7) can be rewritten into

$$(4.8) \qquad \begin{aligned} &(\beta(w))_\tau - \Delta w = 0 && \text{in} \quad D \times (0, T_c)\,, \\ &w(x, \tau) = 0 && \text{on} \quad \partial D \times (0, T_c)\,, \\ &w(x, 0) = u_0(x) && \text{in} \quad D\,, \end{aligned}$$

where $T_c = c^{-1}$ $(\tau = c^{-1}(1 - \exp(-ct))$ (see [14]).

As a simple corollary of Theorem 4.6 and of the comparison principle stated in Theorem 2.4 we have

Theorem 4.9. *Let u be a weak solution of Problem* (1.1) *with a locally Lipschitz continuous function f satisfying*

$$f(r) \leqq \lambda r \quad \textit{for some} \quad \lambda > 0\,.$$

If $u_0(\in L^\infty(D))$ is such that $T^*(u_0) < T_c$ for T^* given by (4.3) *or* (4.5) (*corresponding to the cases mentioned above*) *then*

$$u(t) \equiv 0 \quad \textit{for} \quad t \geqq -c^{-1} \ln(1 - cT^*)\,.$$

Acknowledgment. I would like to thank Dr. J. Kačur for valuable discussions.

References

[1] *D. G. Aronson, M. G. Crandall, L. A. Peletier:* Stabilization of solutions of a degenerate nonlinear diffusion problem. Nonlinear Analysis, 6 (1982), 1001—1022.

[2] *P. Benilan, M. K. Crandall:* The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$. Indiana Univ. Math. J., Vol. 30, 2 (1981), 161—177.

[3] *J. G. Berryman, C. J. Holland:* Stability of the separable solution for fast diffusion. Arch. Ratl. Mech. Anal., 74 (1980), 379—388.

[4] *J. Filo:* A nonlinear diffusion equation with nonlinear boundary conditions: method of lines (to appear).

[5] *A. Friedman:* Partial differential equations. Holt, Rinehart and Winston, New York, 1969.
[6] *S. Fučík, A. Kufner:* Nonlinear differential equations. SNTL, Praha 1978 (Czech).
[7] *J. W. Jerome:* Horizontal line analysis of the multidimensional porous medium equation: existence, rate of convergence and maximum principles. Lecture Notes in Math. 506, Springer 1976.
[8] *J. Kačur:* Method of Rothe in Evolution Equations. Teubner-Texte zur Mathematik, 80, Leipzig, 1985.
[9] *A. Kufner, S. Fučík, O. John:* Function Spaces. Academia, Praha 1977.
[10] *O. A. Ladyženskaja, V. A. Solonikov, N. N. Ural'ceva:* Linear and quasilinear equations of parabolic type, Nauka, Moscow 1967 (Russian).
[11] *M. Nakao:* Existence, nonexistence and some asymptotic behaviour of global solutions of a nonlinear degenerate parabolic equation. Math. Rep. College of Gen. Edc., Kyushu Univ. 14 (1983), 1—21.
[12] *M. H. Protter, H. F. Weinberger:* Maximum principles. Prentice-Hall, 1967.
[13] *E. S. Sabinina:* On a class of quasilinear parabolic equations not solvable for the time derivative. Sibirsk. Mat. Z., 6 (1965), 1074—1100 (Russian).
[14] *M. E. Gurtin, R. C. MacCamy:* On the diffusion of biological populations. Math. Biosciences, 33 (1977), 35—49.

Súhrn

O RIEŠENIACH PERTURBOVANEJ ROVNICI RÝCHLEJ DIFÚZIE

JÁN FILO

Práca je venovaná otázkam existencie (lokálnej a globálnej), neexistencie, jednoznačnosti, porovnávania a niektorým vlastnostiam riešení počiatočno-okrajovej úlohy pre perturbovanú rovnicu rýchlej difúzie s homogénnymi Dirichletovými okrajovými podmienkami.

Резюме

О РЕШЕНИЯХ ВОЗМУЩЁННОГО УРАВНЕНИЯ БЫСТРОЙ ДИФФУЗИИ

JÁN FILO

Работа посвящена доказательству существования и некоторых особенностей решения возмущённой проблемы быстрой дифузии с нулевыми краевыми условиями.

Author's address: Dr. Ján Filo, CSc., ÚAMaVT UK, Mlynská dolina, 842 15 Bratislava.