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A REMARK CONCERNING UNIQUENESS OF THE WOLD DECOMPOSION OF FINITE-DIMENSIONAL STATIONARY PROCESSES

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Summary. The uniqueness of the Wold decomposition of a finite-dimensional stationary process without assumption of time-containedness is proved. As a corollary the correspondence between the Wold decomposition of full rank stationary process and the Lebesgue decomposition of its spectral measure is easily obtained.

Key words: Stationary process, Wold decomposition, spectral measure.

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1. INTRODUCTION AND PRELIMINARIES

An orthogonal decomposition of a stationary process into the regular and singular parts was established for the first time by H. Wold [7]. A more abstract form which points out the operator-theoretical nature of the fact can be found in [1] (cf. also [5]). It may seem to be little surprising that the natural assumption of the so-called time containedness of the regular part is of no importance for the uniqueness of the decomposition in the one-dimensional case. In fact, the same argument applies to stationary processes generated by a set of elements for which the regular part is n-dimensional. The proof requires elementary Hilbert space geometry only.

As a consequence of the uniqueness theorem we obtain a new and more elementary proof of the correspondence between the Wold decomposition of a full rank stationary process and the Lebesgue decomposition of its spectral measure.

Let $\mathscr{H}$ be a Hilbert space. We shall denote by $P(\mathscr{L})$ the orthogonal projection of $\mathscr{H}$ onto a closed subspace $\mathscr{L}$ of $\mathscr{H}$. All projections are considered to be orthogonal.

A sequence $(f_n)_{n \in \mathbf{Z}}$ of vectors in $\mathscr{H}$ is called a (discrete time) stationary process if the scalar products (f_n, f_m) depend of the difference $n - m$ only, i.e.

$$(f_{n+k}, f_{m+k}) = (f_n, f_m) \quad \text{for all} \quad n, m, k \in \mathbf{Z} .$$

Since an analogous relation holds for linear combinations of vectors f_j it follows that there exists a unitary operator U acting on the whole space $\mathscr{H}$ which satisfies

$$Uf_n = f_{n+1} \quad \text{or equivalently} \quad U^n f_0 = f_n$$

for all $n \in \mathbf{Z}$, and U is uniquely determined on the reducing subspace containing $\bigvee_{j \in \mathbf{Z}} f_j$, the closed linear span of all f_j. Conversely, given a unitary operator $U \in B(\mathscr{H})$ and an $x \in \mathscr{H}$, the sequence $(f_n = U^n x)_{n \in \mathbf{Z}}$ is a stationary process. The above consideration allows us to introduce the following definition.

1.1 Definition. *A triplet $(\mathscr{H}, U, x)$, $\mathscr{H}$ a Hilbert space, $U \in B(\mathscr{H})$ a unitary operator and $x \in \mathscr{H}$, is called a stationary process.*

Similarly, a double sequence $(f_n^i)_{n \in \mathbf{Z}}$, $i = 1, 2, \ldots, N$, of vectors from $\mathscr{H}$ is called a finite dimensional stationary process if the Gram matrix $(f_n^i, f_m^j)_{i,j=1}^N$ depends on the difference $n - m$ only. Obviously, we can use the same reasoning as before so that the following definition describes the more general situation.

1.2 Definition. *Let $U \in B(\mathscr{H})$ be a unitary operator and $\mathscr{X}$ a subset of $\mathscr{H}$. Then $(\mathscr{H}, U, \mathscr{X})$ is called a stationary process.*

Consider now a stationary process $(\mathscr{H}, U, \mathscr{X})$, $\mathscr{X} \subset \mathscr{H}$. Denote by $E_{\mathscr{X}}(H_{\mathscr{X}})$ the smallest invariant (reducing, respectively) subspace of U^* containing $\mathscr{X}$, i.e.

$$E_{\mathscr{X}} = \bigvee_{k \leq 0} U^k \mathscr{X}, \quad H_{\mathscr{X}} = \bigvee_{k=-\infty}^{\infty} U^k \mathscr{X}.$$

The restriction $U^* \mid E_{\mathscr{X}}$ is an isometry so that the Wold decomposition applies. In other words, the space $E_{\mathscr{X}}$ can be decomposed into a direct sum of two subspaces reducing with respect to $U^* \mid E_{\mathscr{X}}$,

$$E_{\mathscr{X}} = \Big(\bigcap_{k \leq 0} U^k E_{\mathscr{X}} \Big) \oplus \big((E_{\mathscr{X}} \ominus U^* E_{\mathscr{X}}) \oplus U^*(E_{\mathscr{X}} \ominus U^* E_{\mathscr{X}}) \oplus \ldots \big),$$

so that the restriction of U^* to the first subspace is a unitary operator and the restriction to the second is a unilateral shift of multiplicity $\dim (E_{\mathscr{X}} \ominus U^* E_{\mathscr{X}}) \leq \dim$ span $\mathscr{X}$ (see [5], p. 4).

Let $\mathscr{R}_{\mathscr{X}} = \bigcap_{k \leq 0} U^k E_{\mathscr{X}}$, and denote by $\mathscr{F}_{\mathscr{X}}$ the wandering subspace, $\mathscr{F}_{\mathscr{X}} = E_{\mathscr{X}} \ominus U^* E_{\mathscr{X}}$. We shall also use the notation $M_+(\mathscr{F}_{\mathscr{X}}) = \bigoplus_{k \leq 0} U^k \mathscr{F}_{\mathscr{X}}$ and $M(\mathscr{F}_{\mathscr{X}}) = \bigoplus_{-\infty}^{\infty} U^k \mathscr{F}_{\mathscr{X}}$.

Moreover, this decomposition is unique in the following sense: if $E_{\mathscr{X}} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and $U^* \mid \mathscr{H}_1$ is unitary and $U^* \mid \mathscr{H}_2$ is a unilateral shift then $\mathscr{H}_1 = \mathscr{R}_{\mathscr{X}}$ and $\mathscr{H}_2 = M_+(\mathscr{F}_{\mathscr{X}})$.

Clearly,

$$E_{\mathscr{X}} = M_+(\mathscr{F}_{\mathscr{X}}) \oplus \mathscr{R}_{\mathscr{X}}, \quad H_{\mathscr{X}} = M(\mathscr{F}_{\mathscr{X}}) \oplus \mathscr{R}_{\mathscr{X}}.$$

1.3 Definition. *A stationary process* $(\mathcal{H}, U, \mathcal{X})$ *is called singular if* $E_{\mathcal{X}} = H_{\mathcal{X}}$. *It is called regular if* $\mathcal{R}_{\mathcal{X}} = \{0\}$.

If we denote $Q = 1 - P(\mathcal{R}_{\mathcal{X}})$ then $QU = UQ$ and $Q(\mathcal{X}) \subset M_+(\mathcal{F}_{\mathcal{X}}) \subset E_{\mathcal{X}}$. Since $QU = UQ$ the subspaces $H_{Q\mathcal{X}}$ and $H_{(1-Q)\mathcal{X}}$ are orthogonal and $x = Qx + (1 - Q)x$ for each $x \in \mathcal{X}$. Further, the process $(\mathcal{H}, U, Q\mathcal{X})$ is regular and $(\mathcal{H}, U, (1 - Q)\mathcal{X})$ is singular. Indeed,

$$E_{(1-Q)\mathcal{X}} = \bigvee_{n \leq 0} U^n P(\mathcal{R}_{\mathcal{X}}) \mathcal{X} = \operatorname{clos}\left(P(\mathcal{R}_{\mathcal{X}}) \bigvee_{n \leq 0} U^n \mathcal{X}\right) = \operatorname{clos}\left(P(\mathcal{R}_{\mathcal{X}}) E_{\mathcal{X}}\right) =$$

$$= \mathcal{R}_{\mathcal{X}} = \operatorname{clos}\left(P(\mathcal{R}_{\mathcal{X}}) H_{\mathcal{X}}\right) = \operatorname{clos}\left(P(\mathcal{R}_{\mathcal{X}}) \bigvee_{n \in \mathbf{Z}} U^n \mathcal{X}\right) = \bigvee_{n \in \mathbf{Z}} U^n P(\mathcal{R}_{\mathcal{X}}) \mathcal{X} = H_{(1-Q)\mathcal{X}}.$$

Since $Q\mathcal{X} \subset M_+(\mathcal{F}_{\mathcal{X}})$ we also have $E_{Q\mathcal{X}} \subset M_+(\mathcal{F}_{\mathcal{X}})$ and

$$\mathcal{R}_{Q\mathcal{X}} = \bigcap_{k \leq 0} U^k E_{Q\mathcal{X}} \subset \bigcap_{k \leq 0} U^k M_+(\mathcal{F}_{\mathcal{X}}) = \{0\}.$$

On the other hand, it follows from the uniqueness of the Wold decomposition that if P is any projection such that it commutes with U, maps $\mathcal{X}$ into $E_{\mathcal{X}}$, $(\mathcal{H}, U, P\mathcal{X})$ is regular and $(\mathcal{H}, U, (1 - P)\mathcal{X})$ singular, then $P \mid H_{\mathcal{X}} = Q \mid H_{\mathcal{X}}$. We can now sum up these facts in the following definition.

1.4 Definition. *Let* $(\mathcal{H}, U, \mathcal{X})$ *be a stationary process. The only pair of stationary processes* $(\mathcal{H}, U, Q\mathcal{X})$ *and* $(\mathcal{H}, U, (1 - Q)\mathcal{X})$ *is called the Wold decomposition of* $(\mathcal{H}, U, \mathcal{X})$, *if*

1° Q *is a projection such that* $QU = UQ$ *and* $Q\mathcal{X} \subset E_{\mathcal{X}}$,

2° $(\mathcal{H}, U, Q\mathcal{X})$ *is regular and* $(\mathcal{H}, U, (1 - Q)\mathcal{X})$ *is singular.*

2. THE UNIQUENESS OF DECOMPOSITION

We shall use a slightly modified version of the Wold decomposition based on the fact that a bilateral shift of finite multiplicity cannot contain a bilateral shift of higher multiplicity (see [5], Proposition 2.1). Precisely, if W is a unitary operator and $\mathcal{L}_1$, $\mathcal{L}_2$ two wandering subspaces of W such that $M(\mathcal{L}_1) \subset M(\mathcal{L}_2)$ and $\dim \mathcal{L}_1 = \dim \mathcal{L}_2 < \infty$ then $M(\mathcal{L}_1) = M(\mathcal{L}_2)$.

The inclusion $Q\mathcal{X} \subset E_{\mathcal{X}}$ in condition 1° of 1.4 implies $E_{Q\mathcal{X}} \subset E_{\mathcal{X}}$ and has a natural meaning: "the past" of the regular part in the Wold decomposition depends on "the past" of the initial process only. Nevertheless, it may be replaced by a weaker one.

2.1 Proposition. *Let* $(\mathcal{H}, U, \mathcal{X})$ *be a stationary process. Then there exists an orthogonal projection* Q *such that*

1° $QU = UQ$, $Q\mathcal{X} \subset H_{\mathcal{X}}$,

2° $(\mathcal{H}, U, Q\mathcal{X})$ *is regular with* $\dim \mathcal{F}_{Q\mathcal{X}} = \dim \mathcal{F}_{\mathcal{X}}$ *and* $(\mathcal{H}, U, (1 - Q)\mathcal{X})$ *is singular.*

Conversely, if $\dim \mathscr{F}_{\mathfrak{X}} < \infty$ *and* Q *satisfis* 1° *and* 2° *then* $(\mathscr{H}, U, Q\mathfrak{X})$, $(\mathscr{H}, U, (1 - Q)\mathfrak{X})$ *is the Wold decomposition of* $(\mathscr{H}, U, \mathfrak{X})$.

Proof. It is easy to see that $Q = 1 - P(\mathscr{R}_{\mathfrak{X}})$ also satisfies $\dim \mathscr{F}_{Q\mathfrak{X}} = \dim \mathscr{F}_{\mathfrak{X}}$.

To prove the second part of the assertion let us consider a projection Q satisfying 1° and 2°. Condition 1° implies $H_{Q\mathfrak{X}} \subset H_{\mathfrak{X}}$. Using the singularity of $(\mathscr{H}, U, (1 - Q)\mathfrak{X})$ we have

$$E_{\mathfrak{X}} \subset E_{Q\mathfrak{X}} \oplus E_{(1-Q)\mathfrak{X}} = E_{Q\mathfrak{X}} \oplus H_{(1-Q)\mathfrak{X}}$$

and, for $n \in \mathbf{Z}$,

$$U^n E_{\mathfrak{X}} \subset U^n E_{Q\mathfrak{X}} \oplus U^n H_{(1-Q)\mathfrak{X}} = U^n E_{Q\mathfrak{X}} \oplus H_{(1-Q)\mathfrak{X}}\,.$$

Condition 1° and regularity of $(\mathscr{H}, U, Q\mathfrak{X})$ imply

$$\mathscr{R}_{\mathfrak{X}} = \bigcap_{n \leq 0} U^n E_{\mathfrak{X}} \subset \Big(\bigcap_{n \leq 0} U^n E_{Q\mathfrak{X}}\Big) \oplus H_{(1-Q)\mathfrak{X}} = H_{(1-Q)\mathfrak{X}} = H_{\mathfrak{X}} \ominus H_{Q\mathfrak{X}}\,,$$

hence $M(\mathscr{F}_{Q\mathfrak{X}}) = H_{Q\mathfrak{X}} \subset H_{\mathfrak{X}} \ominus \mathscr{R}_{\mathfrak{X}} = M(\mathscr{F}_{\mathfrak{X}})$. Both $\mathscr{F}_{Q\mathfrak{X}}$ and $\mathscr{F}_{\mathfrak{X}}$ are wandering subspaces of $U \mid H_{\mathfrak{X}}$ and, by 2°, $\dim \mathscr{F}_{Q\mathfrak{X}} = \dim \mathscr{F}_{\mathfrak{X}}$. If $\dim \mathscr{F}_{\mathfrak{X}} < \infty$ then $M(\mathscr{F}_{Q\mathfrak{X}}) = M(\mathscr{F}_{\mathfrak{X}})$ by Prop. 2.1 of [5].

Clearly, $Q \mid H_{\mathfrak{X}}$ is an orthogonal projection and $QH_{\mathfrak{X}} = H_{Q\mathfrak{X}} = M(\mathscr{F}_{Q\mathfrak{X}})$. On the other hand, $M(\mathscr{F}_{\mathfrak{X}}) = (1 - P(\mathscr{R}_{\mathfrak{X}}))\, H_{\mathfrak{X}}$, thus $Q \mid H_{\mathfrak{X}} = (1 - P(\mathscr{R}_{\mathfrak{X}})) \mid H_{\mathfrak{X}}$. The proof is complete.

The following example shows that if $\dim \mathscr{F}_{\mathfrak{X}} = \infty$, conditions 1° and 2° do not imply the uniqueness of the decomposition.

2.2 Example. Consider the following double sequence of orthonormal vectors in a Hilbert space $\mathscr{H}$,

$$\begin{array}{c} \ldots\; e_{0,-2}\; e_{0,-1}\; e_{00}\; e_{01}\; e_{02}\; \ldots \\ \ldots\; e_{1,-1}\; e_{10}\; e_{11}\; \ldots \\ \ldots\; e_{20}\; \ldots \\ \ldots \end{array}$$

and define a unitary operator $U \in B(\mathscr{H})$ satisfying

$$Ue_{ij} = e_{i,j-1} \quad \text{for} \quad i \geqq 0\,, \quad j \in \mathbf{Z}\,.$$

If $\mathfrak{X} = \{e_{k0} : k \geqq 0\}$ then $(\mathscr{H}, U, \mathfrak{X})$ is clearly a regular stationary process and $\dim \mathscr{F}_{\mathfrak{X}} = \infty$. Let us define

$$m = \sum_{k=0}^{\infty} 2^{-k} e_{kk}\,, \quad \mathscr{M} = H_m\,.$$

The projection $Q = 1 - P(\mathscr{M})$ clearly satisfies condition 1° and we shall show that it also satisfies condition 2° of Proposition 2.1. By easy computation we have, for $k \geqq 0$,

$$P(\mathscr{M})\, e_{k0} = P(\mathscr{M})\, U^k e_{kk} = U^k P(\mathscr{M})\, e_{kk} = \\ = 2^{-k} U^k P(\mathscr{M})\, 2^k e_{kk} = 2^{-k} U^k P(\mathscr{M})\, e_{00}$$

because

$$2^k e_{kk} - e_{00} \perp \mathscr{M} .$$

Since

$$H_{P(\mathscr{M})\mathscr{X}} = \bigvee_{\substack{n\in\mathbf{Z}\\ k\geqq 0}} U^{*n} P(\mathscr{M})\, e_{k0} = \bigvee_{\substack{n\in\mathbf{Z}\\ k\geqq 0}} U^{*n}U^k P(\mathscr{M})\, e_{00} =$$

$$= \operatorname{clos}\big(P(\mathscr{M}) \bigvee_{\substack{n\in\mathbf{Z}\\ k\geqq 0}} U^{*n-k} e_{00}\big) = \operatorname{clos}\big(P(\mathscr{M}) \bigvee_{\substack{n\geqq 0\\ k\geqq 0}} U^{*n-k} e_{00}\big) =$$

$$= \bigvee_{\substack{n\geqq 0\\ k\geqq 0}} U^{*n}U^k P(\mathscr{M})\, e_{00} = \bigvee_{\substack{k\geqq 0\\ n\geqq 0}} U^{*n} P(\mathscr{M})\, e_{k0} = E_{P(\mathscr{M})\mathscr{X}} ,$$

the process $(\mathscr{H}, U, P(\mathscr{M})\, \mathscr{X})$ is singular.

Now, we shall show that $(\mathscr{H}, U, (1 - P(\mathscr{M}))\, \mathscr{X})$ is regular. If we denote by $\mathscr{L} = \bigvee_{k\geqq 0} e_{kk}$ then

$$H_{\mathscr{X}} = \bigoplus_{k\in\mathbf{Z}} U^k \mathscr{L} .$$

To compute $\mathscr{R}_{P(\mathscr{M}^\perp)}$ we shall use the inclusion

$$U^n E_{P(\mathscr{M}^\perp)\mathscr{X}} = U^n \operatorname{clos}\big(P(\mathscr{M}^\perp)\, E_{\mathscr{X}}\big) = \operatorname{clos}\big(P(\mathscr{M}^\perp)\, U^n E_{\mathscr{X}}\big) \subset U^n E_{\mathscr{X}} \vee \mathscr{M}$$

and the decomposition

$$U^{*n} E_{\mathscr{X}} \vee \mathscr{M} = \bigoplus_{k\in\mathbf{Z}} \big(U^{*n} E_{\mathscr{X}} \vee \mathscr{M}\big) \cap U^{*k} \mathscr{L} .$$

For any $n \geqq 0$, we have also

$$\big(U^{*n} E_{\mathscr{X}} \vee \mathscr{M}\big) \cap \mathscr{L} = m \vee \bigvee_{j\geqq n} e_{jj}$$

so that

$$U^{*n} E_{\mathscr{X}} \vee \mathscr{M} = \bigoplus_{k\in\mathbf{Z}} \big(U^{*n} E_{\mathscr{X}} \vee \mathscr{M}\big) \cap U^{*k}\mathscr{L} = \bigoplus_{k\in\mathbf{Z}} U^{*k}\big[\big(U^{*n-k} E_{\mathscr{X}} \vee \mathscr{M}\big) \cap \mathscr{L}\big] =$$

$$= \bigoplus_{k<n} U^{*k}\big(m \vee \bigvee_{j\geqq n-k} e_{jj}\big) \oplus \bigoplus_{k\geqq n} U^{*k} \mathscr{L} .$$

Denoting

$$A_{nk} = \begin{cases} U^{*k}\big(m \vee \bigvee_{j\geqq n-k} e_{jj}\big), & k < n ,\\ U^{*k}\mathscr{L} , & k \geqq n ,\end{cases}$$

for any $n \geqq 0$, we clearly have $A_{n+1,k} \subset A_{nk}$ and $\bigcap_{n\geqq 0} A_{nk} = \mathscr{M} \cap U^{*k}\mathscr{L}$. The equality

$$U^{*n} E_{\mathscr{X}} \vee \mathscr{M} = \bigoplus_{k\in\mathbf{Z}} A_{nk}$$

now implies

$$\mathscr{R}_{P(\mathscr{M}^\perp)\mathscr{X}} = \bigcap_{n\geqq 0} U^{*n} E_{P(\mathscr{M}^\perp)\mathscr{X}} \subset \bigcap_{n\geqq 0} \big(U^{*n} E_{\mathscr{X}} \vee \mathscr{M}\big) = \bigcap_{n\geqq 0} \bigoplus_{k\in\mathbf{Z}} A_{nk} \subset \mathscr{M} .$$

On the other hand, $\mathscr{R}_{P(\mathscr{M}^\perp)\mathscr{X}} \subset \mathscr{M}^\perp$ so that $\mathscr{R}_{P(\mathscr{M}^\perp)\mathscr{X}} = \{0\}$ and the regularity of $(\mathscr{H}, U, P(\mathscr{M}^\perp)\mathscr{X})$ is proved.

3. STATIONARY PROCESSES WITH THE SPECTRAL MEASURE ABSOLUTELY CONTINUOUS WITH RESPECT TO THE LEBESGUE MEASURE

Let us now consider the Hilbert space $L^2 = L^2(\mathbf{T})$ with the norm $|f|_2^2 = \int_{\mathbf{T}} |f|^2 \, dm$ where $\mathbf{T}$ is the unit circle and m the normalized Lebesgue measure on $\mathbf{T}$. As usual, denote by S the unitary operator of multiplication by e^{it} on L^2. Given a natural number n, we shall denote by $L^2(n)$ the Hilbert space of all n-tuples $f = (f_1, \ldots, f_n)$ with $f_i \in L^2$ $(i = 1, 2, \ldots, n)$ equipped with the scalar product $(f, g) = \sum_{i=1}^n (f_i, g_i)$. Let $S_n \in B(L^2(n))$ be the bilateral shift operator, $S_n f = (Sf_1, \ldots, Sf_n)$, $f \in L^2(n)$. Obviously $L^2(n) = M(\mathscr{F}_{\mathscr{M}})$ where $\mathscr{M} = \{e_j : e_{jk} = \delta_{jk}, j, k = 1, 2, \ldots, n\}$.

3.1 Definition. *Let $(\mathscr{H}, U, \mathscr{X})$ be a stationary process. Denote by E the spectral measure of U. The set of Borel measures*

$$\mu_{\mathscr{X}} = \{\mu_{x,y} = (E(\cdot)\, x, y) : x, y \in \mathscr{X}\}$$

will be called the spectral measure of $(\mathscr{H}, U, \mathscr{X})$. We shall say that $\mu_{\mathscr{X}} \ll m$ $(\mu_{\mathscr{X}} \perp m)$ iff $\mu_{x,y} \ll m$ $(\mu_{x,y} \perp m$, respectively$)$ for all $x, y \in \mathscr{X}$.

If $\mathscr{X}$ consists of a single element x then the spectral measure of $(\mathscr{H}, U, \mathscr{X})$ is nonnegative, $\mu_x = |E(\cdot)\, x|^2$.

If $\mathscr{X}$ is finite, $\mathscr{X} = \{x_1, \ldots, x_n\}$, then the spectral measure of $(\mathscr{H}, U, \mathscr{X})$ can be considered as a matrix $\mu_{\mathscr{X}} = (\mu_{ij})_1^n$ with nonnegative diagonal entries.

3.2 Lemma. *Let $\mathscr{H}_1, \mathscr{H}_2$ be two Hilberts spaces, $U_1 \in B(\mathscr{H}_1)$, $U_2 \in B(\mathscr{H}_2)$ unitary operators and $\mathscr{X} \subset \mathscr{H}$. If $\Phi \in B(\mathscr{H}_1, \mathscr{H}_2)$ is an isometry such that $\Phi U_1 = U_2 \Phi$ then*

1° $E_{\Phi\mathscr{X}} = \Phi E_{\mathscr{X}}$ *and* $\mathscr{F}_{\Phi\mathscr{X}} = \Phi \mathscr{F}_{\mathscr{X}}$,
2° $H_{\Phi\mathscr{X}} = \Phi H_{\mathscr{X}}$,
3° $\mathscr{R}_{\Phi\mathscr{X}} = \Phi \mathscr{R}_{\mathscr{X}}$.

Proof.

$$E_{\Phi\mathscr{X}} = \bigvee_{k \leq 0} U_2^k \Phi \mathscr{X} = \bigvee_{k \leq 0} \Phi U_1^k \mathscr{X} = \Phi \bigvee_{k \leq 0} U_1^k \mathscr{X} = \Phi E_{\mathscr{X}}$$

and

$$\mathscr{F}_{\Phi\mathscr{X}} = E_{\Phi\mathscr{X}} \ominus U_2 E_{\Phi\mathscr{X}} = \Phi E_{\mathscr{X}} \ominus U_2 \Phi E_{\mathscr{X}} = \Phi E_{\mathscr{X}} \ominus \Phi U_1 E_{\mathscr{X}} = = \Phi(E_{\mathscr{X}} \ominus U_1 E_{\mathscr{X}}) = \Phi \mathscr{F}_{\mathscr{X}}\,.$$

Similarly,

$$H_{\Phi\mathscr{X}} = \Phi H_{\mathscr{X}}\,.$$

Further,

$$\mathscr{R}_{\Phi\mathscr{X}} = \bigcap_{k \leq 0} U_2^k E_{\Phi\mathscr{X}} = \bigcap_{k \leq 0} U_2^k \Phi E_{\mathscr{X}} = \bigcap_{k \leq 0} \Phi U_1^k E_{\mathscr{X}} = \Phi \mathscr{R}_{\mathscr{X}}\,.$$

3.3 Proposition. *Let $\mathscr{X} = \{x_1, \ldots, x_n\}$ be a subset of $\mathscr{H}$ such that the stationary process $(\mathscr{H}, U, \mathscr{X})$* satisfies *$\dim \mathscr{F}_{\mathscr{X}} = n$ and $\mu_{\mathscr{X}} \ll m$. Then $(\mathscr{H}, U, \mathscr{X})$ is regular.*

Proof. Since $\mu_{ij} \ll m$ there exist functions $f_{ij} \in L^1(\mathbf{T})$ such that $f_{ij} = \mathrm{d}\mu_{ij}/\mathrm{d}m$ $(i, j = 1, \ldots, n)$. Given $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ we have

$$\sum_{i,j} \lambda_i \lambda_j^* \, \mu_{ij}(\cdot) = |E(\cdot) \sum_i \lambda_i x_i|^2 \geqq 0$$

so that $\sum \lambda_i \lambda_j^* \, \mu_{ij}(\cdot)$ is a nonnegative Borel measure on $\mathbf{T}$ which is absolutely continuous with respect to m. Consequently, its density $\Sigma \lambda_i \lambda_j^* f_{ij}$ is nonnegative a.e. This implies that there exists a Borel subset σ_0 of $\mathbf{T}$ such that $m(\sigma_0) = 1$, all functions f_{ij} are defined on σ_0 and $\sum \lambda_i \lambda_j^* f_{ij}(t) \geqq 0$ for $t \in \sigma_0$, $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$.

In other words, matrices $(f_{ij}(t))$ are positive semidefinite so that there exist functions φ_{ij} defined on σ_0 such that

$$(f_{ij}(t)) = (\varphi_{ij}(t)) \, (\varphi_{ij}(t))^* \quad \text{for} \quad t \in \sigma_0 \, .$$

Since

$$\sum_{k=1}^{n} |\varphi_{ik}(t)|^2 = f_{ii}(t)$$

for $t \in \sigma_0$ we have $\varphi_{ij} \in L^2(\mathbf{T})$.

Let us now set

$$\Phi x_j = \varphi_j = (\varphi_{j1}, \ldots, \varphi_{jn}) \in L^2(n) \, .$$

The relations $(x_i, x_j) = (\varphi_i, \varphi_j)$ $(i, j = 1, \ldots, n)$ make it possible to define an isometry $\tilde{\Phi}$ on $H_{\mathscr{X}}$ with values in $L^2(n)$ which satisfies

$$\tilde{\Phi} x_j = \Phi x_j \quad \text{and} \quad \tilde{\Phi} U = S_n \tilde{\Phi} \, .$$

According to Lemma 3.2 the process $(L^2(n), S_n, \Phi\mathscr{X})$ satisfies $\dim \mathscr{F}_{\Phi\mathscr{X}} = n$. Now, using Proposition 2.1 of [5] we deduce that $(L^2(n), S_n, \Phi\mathscr{X})$ is regular and, consequently, $(\mathscr{H}, U, \mathscr{X})$ is regular as well. The proof is complete.

If $n = 1$ then there are only two possibilities: either $(\mathscr{H}, U, x)$ is singular or $\dim \mathscr{F}_x = 1$. So we have

3.4 Corollary. *Let $(\mathscr{H}, U, x)$ be a stationary process satisfying $\mu_x \ll m$. Then it is either regular or singular.*

4. THE LEBESGUE DECOMPOSITION OF THE SPECTRAL MEASURE

Let $(\mathscr{H}, U, \mathscr{X})$ be a stationary process with the spectral measure $\mu_{\mathscr{X}}$. If P is a projection which commutes with U then P also commutes with $E(\cdot)$ and, for $x, y \in \mathscr{X}$,

$$\mu_{x,y} = (E(\cdot)\, x, y) = (E(\cdot)\, Px, Py) + (E(\cdot)\,(1 - P)\, x, (1 - P)\, y) =$$
$$= \mu_{Px,Py} + \mu_{(1-P)x,(1-P)y} \, ,$$

or shortly,

$$\mu_{\mathscr{X}} = \mu_{P\mathscr{X}} + \mu_{(1-P)\mathscr{X}} \, .$$

Clearly $\mu_{P\mathscr{X}} \ll \mu_{\mathscr{X}}$ and $\mu_{(1-P)\mathscr{X}} \ll \mu_{\mathscr{X}}$.

The spectral measure of a regular process $(\mathscr{H}, U, \mathscr{X})$ is absolutely continuous with respect to m. Indeed, the unitary operator $U \mid H_{\mathscr{X}}$ is a bilateral shift so that its spectral measure is equivalent to m ([5]). It follows that the spectral measure of a non-singular process $(\mathscr{H}, U, \mathscr{X})$ cannot be orthogonal to m. In other words, if $\mu_{\mathscr{X}} \perp m$ then $(\mathscr{H}, U, \mathscr{X})$ is singular.

On the other hand, if U is a bilateral shift and $\mathscr{X} \subset \mathscr{H}$ such that $H_{\mathscr{X}}$ is reducing to U then $(\mathscr{H}, U, \mathscr{X})$ is singular and $\mu_{\mathscr{X}} \ll m$. In view of these considerations it is not unnatural to ask what is the connection between the above decomposition and the Lebesgue decomposition of measures $\mu_{x,y}$ $(x, y \in \mathscr{X})$ into absolutely continuous and orthogonal parts with respect to m.

Let $\mathscr{X}$ be a subset of $\mathscr{H}$, $y \in H_{\mathscr{X}}$, and let us consider the nonnegative Borel measure $\mu_y = |E(\cdot)\, y|^2$. Let the Lebesgue decomposition of μ_y have the form

$$\mu_y = \mu^a + \mu^s\,, \quad \mu^a \ll m\,, \quad \mu^s \perp m\,.$$

If μ_y is concentrated on B then $y = E(B)\, y + E(B^c)\, y$ and the measure $\mu_{E(B)y} = |E(\cdot)\, E(B)\, y|^2 = |E(B \cap \cdot)\, y|^2$ is absolutely continuous while $\mu_{E(B^c)y}$ is orthogonal to m so that $\mu^a = \mu_{E(B)y}$ and $\mu^s = \mu_{E(B^c)y}$. Since subspaces reducing U are invariant to $E(\cdot)$, elements $E(B)y$ and $E(B^c)\, y$ are in $H_{\mathscr{X}}$ as well.

Now let us define subspaces

$$\mathscr{H}^a = \{y \in H_{\mathscr{X}}\colon \mu_y \ll m\}\,,$$

$$\mathscr{H}^s = \{y \in H_{\mathscr{X}}\colon \mu_y \perp m\}\,.$$

Both subspaces are closed, mutually orthogonal and $H_{\mathscr{X}} = \mathscr{H}^a \oplus \mathscr{H}^s$. The relation $\mu_{Uy} = \mu_y$ implies that they are also reducing to U.

4.1 Proposition. *Let $\mathscr{X} = \{x_1, \dots, x_n\}$ be a finite subset of $\mathscr{H}$ and let $(\mathscr{H}, U, \mathscr{X})$ be a stationary process with* $\dim \mathscr{F}_{\mathscr{X}} = n$. *If $(\mathscr{H}, U, Q\mathscr{X})$, $(\mathscr{H}, U, (1 - Q)\, \mathscr{X})$ is the Wold decomposition of $(\mathscr{H}, U, \mathscr{X})$ then*

$$\mu_{\mathscr{X}} = \mu_{Q\mathscr{X}} + \mu_{(1-Q)\mathscr{X}}$$

is the Lebesgue decomposition of the spectral measure of $(\mathscr{H}, U, \mathscr{X})$ into absolutely continuous and orthogonal parts with respect to m, i.e.

$$\mu_{x_i,x_j} = \mu_{Qx_i,x_j} + \mu_{(1-Q)x_i,x_j}$$

is the Lebesgue decomposition of μ_{x_i,x_j}, $i, j = 1, 2, \dots, n$.

Proof. According to what has been said above both $\mathscr{H}^a$ and $\mathscr{H}^s$ are reducing subspaces to U, $\mathscr{H}^a \perp \mathscr{H}^s$ and $x = P(\mathscr{H}^a)\, x + P(\mathscr{H}^s)\, x$ for $x \in \mathscr{X}$.

Obviously, $H_{P(\mathscr{H}^a)\mathscr{X}} \subset \mathscr{H}^a$, $H_{P(\mathscr{H}^s)\mathscr{X}} \subset \mathscr{H}^s$ and thus $H_{P(\mathscr{H}^a)\mathscr{X}} \perp H_{P(\mathscr{H}^s)\mathscr{X}}$. Since $\mu_{P(\mathscr{H}^s)\mathscr{X}} \perp m$ the process $(\mathscr{H}, U, P(\mathscr{H}^s)\, \mathscr{X})$ is singular.

We shall show that $(\mathscr{H}, U, P(\mathscr{H}^a)\, \mathscr{X})$ is regular. Regularity of $(\mathscr{H}, U, Q\mathscr{X})$ implies $H_{Q\mathscr{X}} \subset \mathscr{H}^a$, and consequently, $\mathscr{F}_{\mathscr{X}} \subset M(\mathscr{F}_{\mathscr{X}}) = H_{Q\mathscr{X}} \subset \mathscr{H}^a$. Thus we have

$$\mathscr{F}_{\mathscr{X}} \subset P(\mathscr{H}^a)\, E_{\mathscr{X}} \ominus U^*E_{\mathscr{X}} = P(\mathscr{H}^a)\, E_{\mathscr{X}} \ominus P(\mathscr{H}^a)\, U^*E_{\mathscr{X}} \subset$$

$$\subset \operatorname{clos}\,(P(\mathscr{H}^a)\, E_{\mathscr{X}}) \ominus \operatorname{clos}\,(P(\mathscr{H}^a)\, U^*E_{\mathscr{X}}) = E_{P(\mathscr{H}^a)\mathscr{X}} \ominus U^*E_{P(\mathscr{H}^a)\mathscr{X}} = \mathscr{F}_{P(\mathscr{H}^a)\mathscr{X}}\,.$$

It follows that dim $\mathscr{F}_{P(\mathscr{H}^a)\mathscr{X}} \geqq$ dim $\mathscr{F}_{\mathscr{X}} = n$. Moreover, $\mu_{P(\mathscr{H}^a)\mathscr{X}} \ll m$ and, according to Proposition 3.3, $(\mathscr{H}, U, P(\mathscr{H}^a)\, \mathscr{X})$ is regular. The decomposition $(\mathscr{H}, U, P(\mathscr{H}^a)\, \mathscr{X})$ and $(\mathscr{H}, U, P(\mathscr{H}^s)\, \mathscr{X})$ satisfies condition 1° and 2° of 2.1 so that $P(\mathscr{H}^a)\, \mathscr{X} = Q\mathscr{X}$ and $P(\mathscr{H}^s)\, \mathscr{X} = (1 - Q)\, \mathscr{X}$. The proof is complete.

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Souhrn

POZNÁMKA O JEDNOZNAČNOSTI WOLDOVA ROZKLADU KONEČNĚROZMĚRNÝCH STACIONÁRNÍCH PROCESŮ

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V práci je dokázána jednoznačnost Woldova rozkladu konečněrozměrného stacionárního procesu bez předpokladu časové podřízenosti. Důsledkem je jednoduchý důkaz korespondence mezi Woldovým rozkladem stacionárního procesu plné hodnosti a Lebesgueovým rozkladem odpovídající spektrální míry.

Резюме

ЗАМЕЧАНИЕ ОБ ЕДИНСТВЕННОСТИ РАЗЛОЖЕНИЯ ВОЛЬДА КОНЕЧНОМЕРНЫХ СТАЦИОНАРНЫХ ПРОЦЕССОВ

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Доказывается единственность разложения Вольда конечномерного стационарного процесса без предположения подчиненности исходному процессу. Как следствие получается элементарное доказательство соответствия разложения Волда стационарного процесса максимального ранга и разложения Лебега его спектральной меры.

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