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# CONVERGENCE OF APPROXIMATION METHODS FOR EIGENVALUE PROBLEM FOR TWO FORMS

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## INTRODUCTION

In [2] R. D. Brown investigated approximation methods for eigenvalues of a real quadratic form $b$ relative to a positive definite quadratic form $a$, where $a$ and $b$ are defined on a vector space $V$. He considered a general procedure for approximation, outlined by Aronszajn in [1]. His investigations were carried out on the basis of the theory of discrete convergence in Banach spaces in the form developed by Stummel in [6]. In this paper we prove a general convergence theorem in a different way. Namely, it is shown how the theory of external approximation of eigenvalue problems described in [5] can be adopted to the study of the methods considered by Brown. The convergence criteria obtained are somewhat weaker than those presented in [2].

## 1. EXTERNAL APPROXIMATION OF EIGENVALUE PROBLEMS

In this section we present a brief summary of the results contained in [5] concerning external approximation of eigenproblems.

Let $X$ be a Banach space and $T \in \mathscr{L}(X)$. Let $F$ be a normed space such that there exists an isomorphism $\omega : X \xrightarrow{\text{in}} F$. Next, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces with norms denoted by $\| \ \|_n$ and let $\{r_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences of linear maps from $X$ onto $X_n$ and $X_n$ into $F$ $(n = 1, 2, \ldots)$, respectively.

**Definition 1.** *An approximation $\{X_n, r_n, p_n\}$ of $X$ is said to be an external approximation convergent in $F$ if $r_n$ and $p_n$ are uniformly bounded and*

$$\forall u \in X \lim_{n \to \infty} \|\omega u - p_n r_n u\|_F = 0 .$$

Let us introduce a sequence $\{T_n\}_{n=1}^{\infty}$ of linear bounded operators $T_n \in \mathscr{L}(X_n)$, $n = 1, 2, \ldots$. As usual, $\sigma(T)$, $\varrho(T)$ and $\sigma(T_n)$, $\varrho(T_n)$ denote the spectrum and the resolvent set of $T$ and $T_n$, respectively.

**Definition 2.** *The approximation* $\{T_n\}_{n=1}^{\infty}$ *is stable at a point* $\lambda \in \varrho(T)$ iff $\exists N_\lambda$ *and* $\exists M_\lambda \, \forall n > N_\lambda \; \lambda \in \varrho(T_n)$ *and* $\|(\lambda - T_n)^{-1}\| \leqq M_\lambda < \infty$.

Let $N(r_n)$ denote the null space of $r_n$. We assume that for any $n$, $N(r_n)$ has a complementary subspace in $X$. So, we can introduce the set $\mathscr{F}$ of all sequences of complementary subspaces for $N(r_n)$:

$$\mathscr{F} = \{\{V_n\}_{n=1}^{\infty} : V_n \subset X,\; V_n \oplus N(r_n) = X\}\,.$$

**Theorem 1.** *If there exists* $\{V_n\} \in \mathscr{F}$ *such that*

$$\delta_n = \sup_{\substack{v \in V_n \\ \|v\| = 1}} \|\omega T v - p_n T_n r_n v\|_F \to 0\,, \tag{1.1}$$

$$\varepsilon_n = \sup_{\substack{v \in V_n \\ \|v\| = 1}} \|\omega v - p_n r_n v\|_F \to 0\,, \tag{1.2}$$

*then for any* $\lambda \in \varrho(T)$ *there exists a constant* $M_\lambda < \infty$ *such that*

$$\|(\lambda - T_n)^{-1}\| \leqq M_\lambda\,.$$

Remark 1. If the residual spectrum $\sigma_r(T_n)$ of $T_n$ $(\sigma_r(T_n) = \{\lambda \in \sigma(T_n) : (\lambda - T_n) x = 0 \equiv x = 0$, and $\overline{(\lambda - T_n) X_n} \neq X_n\})$ does not contain the points of $\varrho(T)$, then Theorem 1 implies that $\{T_n\}$ is stable at any $\lambda \in \varrho(T)$.

**Definition 3.** *We will say that* $\sigma(T_n)$ *approximates* $\sigma(T)$ *if the following three implications take place*:

i) *if* $\Omega \subset \mathbb{C}$ *is open and* $\Omega \cap \sigma(T) \neq \emptyset$, *then* $\Omega \cap \sigma(T_n) \neq \emptyset$ *for sufficiently large* $n$;

ii) *if* $\lambda \in \sigma(T)$ *and there is* $\delta_0 < 0$ *such that* $K(\lambda, \delta_0) \cap \sigma(T) = \{\lambda\}$, *where* $K(\lambda, \delta_0)$ *is a circle with radius* $\delta_0$ *and center* $\lambda$, *then for every* $\delta$ *such that* $0 < \delta < \delta_0$ : $\sigma(T_n) \cap K(\lambda, \delta_0) \subset K(\lambda, \delta)$ *for sufficiently large* $n$;

iii) *if* $\lambda_n \in \sigma(T_n)$ *and* $\lambda_n \to \lambda_0$ *as* $n \to \infty$, *then* $\lambda_0 \in \sigma(T)$.

In the sequel we quote four theorems concerning the convergence of an approximation.

**Theorem 2.** *Let* $\{X_n, r_n, p_n\}$ *be an external approximation of* $X$, *convergent in* $F$, *and let* $\{T_n\}$ *be stable in* $\varrho(T)$. *If for any* $u \in X$

$$\lim_{n \to \infty} \|r_n T u - T_n r_n u\|_n = 0\,, \tag{1.3}$$

*where* $\|\cdot\|_n$ *stands for the norm in* $X_n$, *then* $\sigma(T_n)$ *approximates* $\sigma(T)$ *in the sense of Definition* 3.

Let $\Gamma$ be a Jordan curve in the resolvent set $\varrho(T)$. If $\{T_n\}$ is stable for all $\lambda \in \Gamma$, then $\Gamma \subset \varrho(T_n)$ for $n > N_0$. So the spectral projectors associated with $\Gamma$, i.e. $E : X \to \to X$ and $E_n : X_n \to X_n$, are well defined and

$$E = \frac{1}{2\pi i} \int_\Gamma (\lambda - T)^{-1} \, d\lambda \,, \quad E_n = \frac{1}{2\pi i} \int_\Gamma (\lambda - T_n)^{-1} \, d\lambda \,.$$

**Theorem 3.** *If the assumptions of Theorem* 2 *are satisfied, then*

i) *if* $\dim EX = \infty$, *then* $\dim E_nX_n \to \infty$ *as* $n \to \infty$,

ii) *if* $\dim EX = n$, *then* $\dim p_nE_nX_n \geqq n$ *for* $n > n_0$.

The preservation of algebraic multiplicities of isolated eigenvalues can be obtained under a certain stronger assumption on $T_n$. Namely, we have

**Theorem 4.** *Let the assumptions of Theorem* 2 *be satisfied. If* $\dim EX < \infty$ *and*

$$\|(T_n r_n - r_n T)(\lambda - T)^{-1}|_{V_n}\| \to 0 \quad \text{for} \quad \lambda \in \Gamma \,, \tag{1.4}$$

*then* $\dim EX = \dim p_nE_nX_n$.

The eigensubspace $EX$ of $T$ is approximated by $E_nX_n$ in the following sense (cf. [5]):

**Theorem 5.** *If the assumptions of Theorem* 2 *are satisfied. then*

$$\forall v \in EX \quad \operatorname{dist}(\omega v, p_nE_nX_n) \to 0 \,.$$

*If, moreover* (1.4) *is satisfied, then*

$$\hat{\delta}(\omega EX, p_nE_nX_n) \to 0 \,,$$

*where* $\hat{\delta}(Y, Z)$ *is the gap between closed subspaces* $Y$ *and* $Z$ *of* $X$ ($\hat{\delta}(Y, Z) = \max(\delta(Y, Z), \delta(Z, Y))$ *where* $\delta(Y, Z) = \sup_{y \in Y, \|y\| = 1} \operatorname{dist}(y, Z)$).

## 2. APPROXIMATION OF THE EIGENVALUE PROBLEM FOR TWO FORMS AND THE CONVERGENCE RESULTS

The eigenvalue problem for a pair of sesquilinear forms $a$ and $b$ on a complex vector space $V$ is considered. It is assumed that $a$ is symmetric and positive definite and, moreover, $b$ is continuous with respect to $a$, i.e.: $\forall u, v \in V \, |b(u, v)| \leqq c \, a^{1/2}(u, u) \,.$ $. \, a^{1/2}(v, v)$, $c$ a positive constant. Assume also that $V$ is separable with respect to the norm $a^{1/2}$. Let $X$ be the closure of $V$ in the norm $a^{1/2}$. The form $b$ can be continuously extended to $X$. So, our eigenvalue problems takes the form

(2.1) find $\lambda \in \mathbb{C}$ and $0 \neq u \in X$ such that

$$b(u, v) = \lambda\, a(u, v) \quad \forall v \in V,$$

which is equivalent to the eigenproblem for an operator $T \in \mathscr{L}(X)$ defined by $a$ and $b$ as follows:

(2.2) $$\forall u \in X \quad b(u, v) = a(Tu, v) \quad \forall v \in V.$$

We will consider the approximate methods for the problem (2.1), which are generated by sequences of sesquilinear forms $a_n$ and $b_n$ defined on $V \times V$. It is assumed that $a_n$ ($n = 0, 1, \ldots$) are symmetric and positive definite and $b_n$ are bounded with respect to $a_n$.

Let $X_n$ be the closure of $V$ in the norm $a_n^{1/2}$, $n = 0, 1, \ldots$. The norms in $X$ and $X_n$ will be denoted by $\| \ \|$ and $\| \ \|_n$, respectively. The forms $b_n$ have continuous extensions on $X_n$. The $n$-th approximate eigenvalue problem takes the form

(2.3) find $\lambda \in \mathbb{C}$ and $0 \neq u \in X_n$ such that $b_n(u, v) = \lambda\, a_n(u, v)\ \forall v \in V$.

This problem is equivalent to the eigenproblem for an operator $T_n \in \mathscr{L}(X_n)$ which is defined by $a_n$ and $b_n$ as follows:

(2.4) $$\forall u \in X_n \quad b_n(u, v) = a_n(T_n u, v) \quad \forall v \in V.$$

It will be assumed that the following conditions are satisfied:

C 1 $$a_0 \leqq a_n \leqq a\,;$$

C 2 $a$ is quasi-bounded with respect to $a_0$, i.e.

$$\forall u \in V \quad \exists M_u < \infty \quad |a(u, v)| \leqq M_u \|v\|_0 \quad \forall v \in V.$$

($a$ is quasi-bounded with respect to $a_0$ iff there exists a symmetric operator $\hat{L}$ in $X_0$ such that $\forall u, v \in V\ a(u, v) = a_0(\hat{L}u, v)$). The forms $a_n$ generate a certain approximation of the space $X$. We will show that it is a special kind of the external approximation of $X$. We are going to construct suitable maps $r_n$ and $p_n$.

Let us first remark that the assumptions C 1 and C 2 imply that $a$ is quasi-bounded with respect to $a_n$, $n = 1, 2, \ldots$. In fact, $a(u, v) = a_n(A_n \hat{L}u, v)\ \forall v \in V$, where $A_n$ is a bounded operator defined by $a_0(u, v) = a_n(A_n u, v)\, \forall v \in V$. Denote $\hat{L}_n = A_n \hat{L}$. The operator $\hat{L}_n$ considered in $X_n$ is bounded from below ($a_n(\hat{L}_n u, u) \geqq a_n(u, u)$ $\forall u \in V$), so $\hat{L}_n$ is semi-bounded in $X_n$. Every semi-bounded symmetric operator with a dense domain has a semi-bounded selfadjoint extension with the same lower bound (cf. [3], XII. 5.1). Let $L_n$ be the selfadjoint extension of $\hat{L}_n$ on the space $X_n$. $L_n$ is positive definite. Thus, there is a unique positive definite and selfadjoint square root $L_n^{1/2}$ of $L_n$ and the domain $D(L_n)$ of $L_n$ is dense in $D(L_n^{1/2})$ (cf. [4], V. § 3.11).

Let $t_n : X \to X_n$ be the unique bounded linear transformation such that $t_n v = v$, $\forall v \in V$. We will show that $D(L_n^{1/2}) = t_n X$. To this end we apply the second representation theorem ([4], VI, § 2.6). The assumptions $x_k \in V$, $x_k \xrightarrow[k \to \infty]{} 0$

in $X_n$ and $\|x_k - x_1\| \underset{k,l\to\infty}{\longrightarrow} 0$ imply, by C 2, that for any $u \in V$, $|a(u, x_k)| \leqq \|L_n u\|_n \cdot \|x_k\|_n \to 0$. Thus the form $a$ is closable in $X_n$. So, let $\bar{a}^{(n)}$ be the closure of $a$ in $X_n$. For $u, v \in X$ we have $\bar{a}^{(n)}(t_n u, t_n v) = a(u, v)$, and the selfadjoint operator associated with $\bar{a}^{(n)}$ in $X_n$ is equal to $L_n$ defined above. The second representation theorem for the densely defined, closed symmetric, and positive definite form $\bar{a}^{(n)}$ yields that $D(L_n^{1/2}) = t_n X$ 1no $\forall u, v \in X$

$$a(u, v) = \bar{a}^{(n)}(t_n u, t_n v) = a_n(L_n^{1/2} t_n u, L_n^{1/2} t_n v)\,. \tag{2.5}$$

Finally, let us remark that the mapping $t_n$ of $X$ into $X_n$ is injective. In fact, if $x_k \in V$ and $x_k \underset{k\to\infty}{\longrightarrow} x$ in $X$ then $t_n x_k \underset{k\to\infty}{\longrightarrow} t_n x$ in $X_n$ and $\forall u \in V\, |a(u, x)| = \lim |a_n(L_n u, x_k)| \leqq \leqq \|L_n u\|_n \cdot \lim_{k\to\infty} \|x_k\|_n = \|L_n u\|_n \cdot \|t_n x\|_n$. So, if $\|t_n x\| = 0$ then $\forall u \in V\ a(u, x) = 0$, i.e. $x = 0$.

Let us define $r_n = L_n^{1/2} t_n$.

**Lemma 1.** *If* C 1 *and* C 2 *are satisfied, then* $r_n \in \mathscr{L}(X, X_n)$ *and* $r_n^{-1} \in \mathscr{L}(X_n, X)$ *for* $n = 0, 1, \ldots$ *Moreover,* $\|r_n\|_{\mathscr{L}(X,X_n)} = \|r_n^{-1}\|_{\mathscr{L}(X_n,X)} = 1$.

Proof. From (2.5) it follows that $\forall u \in X\ \|r_n u\|_n^2 = a_n(L_n^{1/2} t_n u, L_n^{1/2} t_n u) = \|u\|$. Next, let us remark that $\forall w \in D(L_n)\ a_n(L_n w, w) = \bar{a}^{(n)}(w, w) \geqq a_n(w, w)$. In [4] (V, § 3.11) it is proved that under that condition $L_n^{-1/2}$ is a bounded operator on $X_n$. So, $\forall v \in X_n\ r_n^{-1}$ is well defined since $t_n$ is injective, as has been shown above. Moreover, by (2.5) $\forall v \in X_n\ \|r_n^{-1} v\|^2 = \|t_n^{-1} L_n^{-1/2} v\|^2 = \bar{a}^{(n)}(L_n^{-1/2} v, L_n^{-1/2} v) = a_n(v, v)$ which completes the proof of Lemma 1.

So, we can put $p_n = r_n^{-1}$. We have $p_n r_n x = x$ for any $x \in X$. Thus we have

**Corollary 1.** $\{X_n, r_n, p_n\}$ *is an external approximation of* $X$, *convergent in* $X$ *in the sense of Definition* 1.

**Lemma 2.** *If* C 1 *and* C 2 *are satisfied together with*

$$\text{C 3} \qquad \forall u \in V \sup_{\substack{v \in V \\ \|v\| = 1}} |a_n(u, v) - a(u, v)| \to 0\,,$$

*then* $\forall u \in V\ \|r_n u - u\|_n \to 0$.

Proof. Let us apply the integral expression for $L_n^{1/2}$ (cf. [4], V, § 3.11):

$$L_n^{1/2} u = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (L_n + \lambda)^{-1} L_n u \,\mathrm{d}\lambda \quad \text{for} \quad u \in D(L_n) \subset X_n.$$

Similarly, we can express the identity operator on $X_n$:

$$I u = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (I + \lambda)^{-1} u \,\mathrm{d}\lambda\,.$$

Since $(L_n + \lambda)^{-1} L_n u = u - \lambda(L_n + \lambda)^{-1} u$ for $u \in D(L_n)$ and $(I + \lambda)^{-1} u = u - \lambda(I + \lambda)^{-1} u$; we have

$$L_n^{1/2} u - u = \frac{1}{\pi} \int_0^\infty \lambda^{1/2} [(I + \lambda)^{-1} - (L_n + \lambda)^{-1}] u \, d\lambda =$$
$$= \frac{1}{\pi} \int_0^\infty \lambda^{1/2} (L_n + \lambda)^{-1} (L_n - I)(I + \lambda)^{-1} u \, d\lambda \text{ for } u \in D(L_n) .$$

The last term is obtained from the resolvent equation

$$(I + \lambda)^{-1} - (L_n + \lambda)^{-1} = (L_n + \lambda)^{-1} (L_n - I)(I + \lambda)^{-1} .$$

From the above it follows that

$$\|L_n^{1/2} u - u\|_n \leqq \frac{1}{\pi} \int_0^\infty \lambda^{1/2} (1 + \lambda)^{-2} \, d\lambda \|L_n u - u\|_n ,$$

since $(I + \lambda)^{-1} u = (1 + \lambda)^{-1} u$ and $\|(L_n + \lambda)^{-1}\| \leqq [\text{dist}(-\lambda, \sigma(L_n)]^{-1} \leqq (1 + \lambda)^{-1}$. Thus, for any $u \in V$

$$\|L_n^{1/2} u - u\|_n \leqq c \|L_n u - u\|_n = c \sup_{\substack{v \in V \\ \|v\| = 1}} |a_n(L_n u - u, v)| =$$
$$= \sup_{\substack{v \in V \\ \|v\| = 1}} |a(u, v) - a_n(u, v)| \to 0$$

according to the assumption C 3.

**Theorem 6.** *If* C 1, C 2 *and* C 3 *are satisfied together with*

C 4 $$\sup_{\substack{u, v \in V \\ \|u\| = \|v\| = 1}} |b_n(u, v) - b(u, v)| \to 0 \quad as \quad n \to \infty ;$$

C 5 *if sequences* $\{u_n\}$ *and* $\{v_n\}$ *of elements of* $V$ *satisfy* $a_n(u_n, w) \to a(u, w)$ *and* $a_n(v_n, w) \to a(v, w)$ $\forall w \in V$ *and the norms* $\|u_n\|_n$, $\|v_n\|_n$ *are uniformly bounded then* $b_n(u_n, v_n) \to b(u, v)$,

*then the family* $\{T_n\}$ *defined by* (2.4) *is stable.*

Proof. We have to show that $\delta_n$ ($\delta_n = \|T - r_n^{-1} T_n r_n\|$) converges to zero as $n \to \infty$. Let us take $u$ and $v$ from the space $V$. Then

$$a(r_n^{-1} T_n r_n u, v) = a(t_n^{-1} L_n^{-1/2} T_n L_n^{1/2} u, v) = \bar{a}^{(n)}(v, L_n^{-1/2} T_n L_n^{1/2} u) =$$
$$= a_n(L_n v, L_n^{-1/2} T_n L_n^{1/2} u) .$$

Since $L_n^{1/2}$ is selfadjaont in $X_n$, by the definition of $T_n$

$$a_n(L_n v, L_n^{-1/2} T_n L_n^{1/2} u) = a_n(T_n L_n^{1/2} u, L_n^{1/2} v) = b_n(L_n^{1/2} u, L_n^{1/2} v) .$$

Thus

$$\delta_n = \sup_{\substack{u,v \in V \\ \|u\| = \|v\| = 1}} a((T - r_n^{-1} T_n r_n)\, u,\, v) = \sup_{\substack{u,v \in V \\ \|u\| = \|v\| = 1}} \left| b(u, v) - b_n(L_n^{1/2} u, L_n^{1/2} v) \right| \leqq$$

$$\leqq \sup_{\substack{u,v \in V \\ \|u\| = \|v\| = 1}} \left| b(u, v) - b_n(u, v) \right| + \sup_{\substack{u,v \in V \\ \|u\| = \|v\| = 1}} \left| b_n(u, v) - b_n(L_n^{1/2} u, L_n^{1/2} v) \right| .$$

The first term tends to zero according to the assumption C 4. Suppose that the second term does not converge to zero. Thus, there exist $\varepsilon < 0$ and sequences $\{u_n\}$ and $\{v_n\}$ from the unit sphere in $V \cap X$ such that

$$\left| b_n(u_n, v_n) - b_n(L_n^{1/2} u_n, L_n^{1/2} v_n) \right| \geqq \varepsilon . \tag{2.6}$$

From these sequences we can choose subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ weakly convergent in $X$. Let their weak limits be denoted by $u$ and $v$, respectively. Thus $\forall u \in V$

$$\left| a_{n_k}(u_{n_k}, w) - a(u, w) \right| \leqq \sup_{\substack{z \in V \\ \|z\| = 1}} \left| a_{n_k}(z, v) - a(z, v) \right| + a(u_{n_k}, w) - a(u, w) \right| ,$$

and the left-hand side converges to zero by the assumption C 3. So, C 5 implies that

$$b_{n_k}(u_{n_k}, v_{n_k}) \to b(u, v) .$$

We have to show that the sequence $\{b_{n_k}(L_{n_k}^{1/2} u_{n_k}, L_{n_k}^{1/2} v_{n_k})\}$ has the same limit. Let us notice that $a_n(L_n^{1/2} u_n, w) = a_n(u_n, w) + a_n(u_n, L_n^{1/2} w - w)$ for any $w \in V$ since $L_n^{1/2}$ is selfadjoint in $X_n$. Thus, by Lemma 2,

$$\lim_{k \to \infty} a_{n_k}(L_{n_k}^{1/2} u_{n_k}, w) = \lim_{k \to \infty} a_{n_k}(u_{n_k}, w) = a(u, w) .$$

Applying now C 5 to the sequences $\{L_{n_k}^{1/2} u_{n_k}\}$ and $\{L_{n_k}^{1/2} v_{n_k}\}$ we get $|b_{n_k}(u_{n_k}, v_{n_k}) - b_{n_k}(L_{n_k}^{1/2} u_{n_k}, L_{n_k}^{1/2} v_{n_k})| \to 0$ contrary to (2.6). Thus $\delta_n \to 0$. It is easy to show that if $\delta_n \to 0$, then $\varrho(T) \cap \sigma_r(T_n) = \emptyset$ for $n > n_0$. Thus $\{T_n\}$ is stable according to Remark 1.

Now, let us notice that, in our special case, the condition (1.1) of Theorem 1 implies the condition (1.3). Moreover, (1.1) implies the condition (1.4). Thus according to Corollary 1 and Theorem 6, all the assumptions of Theorems 2–5 are satisfied. Therefore, the final result concerning the convergence of the methods considered can be formulated in the form of the following theorem:

**Theorem 7.** *Let the conditions* C 1–C 5 *be satisfied. Then*

i) $\sigma(T_n)$ *approximates* $\sigma(T)$ *in the sense of Definition* 3;

ii) *if* $\Gamma$ *is a Jordan curve in* $\varrho(T)$ *and* $E$ *and* $E_n$ *are the spectral projectors associated with* $\Gamma$ *and* $T$ *and* $T_n$, *respectively, then*
*if* $\dim EX = \infty$, *then* $\dim E_n X_n \to \infty$,
*if* $\dim EX = n$, *then* $\dim E_n X_n = n$ *for sufficiently large* $n$;

iii) $\hat{\delta}(EX, p_n E_n X_n) \to 0$.

The theorem on convergence of eigenelements presented in [2] (cf. Th. 1.2) is proved under the additional assumptions on $b$ and $b_n$. Namely, it is assumed that $b$ and $b_n$ are symmetric forms on $V$ completely continuous with respect to $a$ and $a_n$, respectively ($n = 0, 1, \ldots$).

### 3. APPLICATION TO ARONSZAJN'S METHOD

Aronszajn's method is a special case of the approximation (2.3) considered in Section 2. Aronszajn's method is defined for the selfadjoint problem, i.e. $b$ is also symmetric (cf. [1], [2], [7]). Since our theorem admits nonselfadjoint case we will assume that $b$ is nonsymmetric, but $b_n = b$, $n = 0, 1, \ldots$.

The initial approximate eigenproblem is chosen so as to be easily solvable and $a_0 \leqq a$. To construct the intermediate forms $a_n$ one defines $a' = a - a_0$ and a sequence $\{\varphi_j\}$ in $V$ whose elements are linearly independent modulo the null space $N$ of $a'$ in $V$. Let $\pi_n$ be the projection, orthogonal with respect to $a'$, of $V$ onto span $(\varphi_1, \ldots, \varphi_n)$. Define

$$a_n(u) = a_0(u) + a'(\pi_n u) \quad n = 1, 2, \ldots$$

Then $a_0 \leqq a_1 \leqq \ldots \leqq a$. So $a_n$ is a finite dimensional perturbation of $a$. In [2] Brown proved the following theorem (cf. Prop. 2.1 and Th. 5.1).

**Theorem 8.** *If*

i) *$a$ is quasi-bounded with respect to $a_0$ (thus there exists a symmetric operator $\hat{L}$, $D(\hat{L}) = V$, such that $a(u, v) = a_0(\hat{L}u, v)\ \forall u, v \in V$),*

ii) *$b$ is completely continuous with respect to $a_0$,*

iii) *$V' := N +$ span $(\varphi_j)$ is dense in $X$,*

iv) *$\hat{L}(V')$ is dense in $X_0$,*

*then the condition* C 5 *is satisfied.*

It is easy to see that the assumption C 3 is also satisfied. In fact, since $a(u, v) - a_n(u, v) = a'(u - \Pi_n u, v)$, for $u \in N$ we have $a(u, v) - a_n(u, v) = 0\ \ \forall v \in V$. Moreover, for $u \in span\ (\varphi_j)$ $\|\Pi_n u - u\|_X \to 0$. Thus, since $\|v\|_X \geqq \|v\|_0$, we have

$$\sup_{\substack{v \in V \\ \|v\|_X = 1}} a'(u - \Pi_n u, v) \leqq \sup_{\substack{v \in V \\ \|v\|_X = 1}} a(u - \Pi_n u, v) + \sup_{\substack{v \in V \\ \|v\|_0 = 1}} a_0(u - \Pi_n u, v) =$$

$$= \|u - \Pi_n u\|_X + \|u - \Pi_n u\|_{X_0} \leqq 2\|u - \Pi_n u\|_X \to 0\,.$$

So, Theorem 7 yields.

**Corollary 2.** *If the assumptions* (i–iv) *of Theorem* 8 *are satisfied, then the eigenelements of the intermediate problems*

$$\textit{find } \lambda \in \mathbb{C} \textit{ and } 0 \neq u \in X_n \textit{ such that}$$

$$b(u, v) = \lambda\, a_n(u, v)\ \forall v \in V$$

*approximate suitable eigenelements of* (2.1) *in the sense of the points* i)–iii) *of Theorem* 7.

Similar results for Aronszajn's method are obtained in [2] in another way.

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## Souhrn

## KONVERGENCE APROXIMAČNÍ METODY PRO PROBLÉM VLASTNÍCH HODNOT DVOU FOREM


TERESA REGIŃSKA



Článek je věnován aproximaci problému vlastních hodnot dvou forem v Hilbertově prostoru $X$. Zkoumají se aproximační metody generované posloupnostmi forem $a_n$ a $b_n$ definovaných na hustém podprostoru $X$. Důkaz konvergence těchto metod je založen na teorii vnější aproximace problému vlastních hodnot. Obecné výsledky jsou aplikovány na Aronszajnovu metodu.



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