Aplikace matematiky

Helena Růžičková; Alexander Ženíšek

Finite elements methods for solving viscoelastic thin plates

Aplikace matematiky, Vol. 29 (1984), No. 2, 81–103

Persistent URL: http://dml.cz/dmlcz/104073

Terms of use:

© Institute of Mathematics AS CR, 1984

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

FINITE ELEMENTS METHODS FOR SOLVING VISCOELASTIC THIN PLATES

HELENA RŮŽIČKOVÁ and ALEXANDER ŽENÍŠEK

(Received December 27, 1982)

1. INTRODUCTION

In [3] several mathematical models of viscoelastic bending of anisotropic thin plates are derived by generalizing one-dimensional rheological models, e.g. Voigt's model or Zener's model, to more dimensions. These problems are numerically solved by a combination of finite elements and Laplace transform in [4], [11], where further references can be found. The initial-boundary value problem (1)–(6), dealt with in the present paper, represents according to [3] and [4] the viscoelastic bending of a thin plate in the cases $c_2 = 0$, $c_1 = 1$ and $c_2 = c_1 = 1$. Moreover, our considerations cover the case $c_2 = 1$, $c_1 = 0$. Our numerical approach is more standard than in [4] and [11]: we combine finite elements with finite differences.

In Section 2 the corresponding variational problem is presented. In Section 3 the variational problem is discretized by finite elements in space and by finite differences in time. Triangular finite C^1-elements are used including curved triangular C^1-elements along the boundary if it is curved. When the problem is of the first order with respect to time (i.e. $c_2 = 0$) the discretization in time is done by means of one and two step A-stable methods. In the case $c_2 = 1$ we combine the formulas of two step A-stable methods with the second difference – this is nothing else than the Newmark method written for the linear equation. In this way we obtain the discrete problem (21)–(23). Further, applying numerical integration, the discrete problem (27)–(29) is derived. Existence and uniqueness of a solution of both discrete problems is proved.

In Section 4 the convergence of the approximate solution is proved and the rate of convergence is estimated. Theorems 3 and 4 are devoted to the cases $c_2 = 1$ and $c_2 = 0$, respectively, when numerical integration is not taken into account. The effect of numerical integration is studied in Theorems 5 and 6. In Section 5 numerical results are presented.

2. FORMULATION OF THE PROBLEM

We shall consider the following initial-boundary value problem

(1) $$c_2 D^{(2)}_{ijkl}\ddot{w}_{,ijkl} + c_1 D^{(1)}_{ijkl}\dot{w}_{,ijkl} + D^{(0)}_{ijkl}w_{,ijkl} = Q \quad \text{in} \quad \Omega \times (0, t^*] ,$$

(2) $$w(x_1, x_2, t)|_{\Gamma} = 0 ,$$

(3) $$\left.\frac{\partial w}{\partial v}\right|_{\Gamma_1} = 0 ,$$

(4) $$\left(c_2 D^{(2)}_{ijkl}\ddot{w}_{,ij} + c_1 D^{(1)}_{ijkl}\dot{w}_{,ij} + D^{(0)}_{ijkl}w_{,ij}\right) v_k v_l|_{\Gamma_2} = 0 ,$$

(5) $$w(x_1, x_2, 0) = f(x_1, x_2) ,$$

(6) $$c_2\dot{w}(x_1, x_2, 0) = c_2 g(x_1, x_2) ,$$

where at least one of the relations $c_1 = 1$, $c_2 = 1$ takes place and where $c_i = 0$ or 1 ($i = 1, 2$). The summation convention over repeated subscripts is adopted. A comma is employed to denote partial differentiation with respect to spatial coordinates x_1, x_2 and a dot denotes the derivative with respect to time t. $D^{(m)}_{ijkl}$ are constants with the properties

(7) $$D^{(m)}_{ijkl} = D^{(m)}_{jikl} = D^{(m)}_{klij} \quad (m = 0, 1, 2) ,$$

(8) $$D^{(m)}_{ijkl}\xi_{ij}\xi_{kl} \geqq \mu_m \xi_{ij}\xi_{ij} \quad \forall \xi_{ji} = \xi_{ij} \in R , \quad \mu_m = \text{const} > 0 \quad (m = 0, 1, 2) ,$$

and Q, f, g are sufficiently smooth given functions. The symbol v denotes the unit outward normal to the boundary Γ of the given domain Ω and v_1, v_2 are the components of the vector v. The symbols Γ_1, Γ_2 denote parts of Γ with the following properties: $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Before presenting a variational formulation of problem (1)–(6) let us introduce some notation. By $H^m(\Omega)$ we denote the Sobolev space of real functions which together with their generalized derivatives up to order m inclusive are square integrable over Ω. The inner product and the norm are denoted by $(\cdot, \cdot)_{m,\Omega}$ and $\|\cdot\|_{m,\Omega}$, respectively. $H^m_0(\Omega)$ is the closure in the H^m-norm of the set of infinitely differentiable functions having compact support in Ω. $C^m(H^k(\Omega))$ is the space of continuous functions $f: [0, t^*] \to H^k(\Omega)$ which have continuous derivatives up to order m on $[0, t^*]$. $L^2(H^k(\Omega))$ is the space of strongly measurable functions $f: (0, t^*) \to H^k(\Omega)$ such that

$$\int_0^{t^*} \|f(t)\|^2_{k,\Omega}\, dt < \infty .$$

Let us define the space V_0 by

(9) $$V_0 = \{v \in H^2(\Omega) \cap H^1_0(\Omega): \partial v/\partial v = 0 \text{ on } \Gamma_1 \text{ in the sense of traces}\} .$$

(Of course, if $\Gamma_1 = \Gamma$ then $V_0 = H^2_0(\Omega)$.) Multiplying equation (1) by $v \in V_0$, integrating over Ω and using (4) and Green's theorem we find

$$c_2 a_2(\ddot{w}, v) + c_1 a_1(\dot{w}, v) + a_0(w, v) = (Q, v)_{0,\Omega} \quad \forall v \in V_0 \,, \tag{10}$$

where

$$a_m(u, v) = \iint_\Omega D^{(m)}_{ijkl} u_{,ij} v_{,kl} \, \mathrm{d}x \,. \tag{11}$$

Thus the variational formulation of problem (1)–(6) reads: Find a function w which has the following properties:

(a) $w \in L^2(V_0)$, $\dot{w} \in L^2(V_0)$, $c_2 \ddot{w} \in L^2(V_0)$;

(b) relation (10) holds;

(c) the function w satisfies initial conditions (5), (6).

In what follows we shall suppose that problem (a)–(c) has a solution w. Then this solution is unique: It is sufficient to prove that the corresponding homogeneous problem has the trivial solution only. Thus let us assume $Q = f = g = 0$ and set $v = \dot{w}$ in (10). We obtain

$$c_2 a_2(\ddot{w}, \dot{w}) + c_1 a_1(\dot{w}, \dot{w}) + a_0(w, \dot{w}) = 0 \,. \tag{12}$$

Relations (7) and (11) imply

$$a_m(u, \dot{u}) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} a_m(u, u) \,, \quad a_m(u, v) = a_m(v, u) \quad (m = 0, 1, 2) \,, \tag{13}$$

and inequality (8) gives

$$a_m(u, u) \geqq K_m \|u\|^2_{2,\Omega} \,, \quad K_m = \text{const} > 0 \quad (m = 0, 1, 2) \quad \forall\, u \in V_0 \tag{14}$$

It follows from (12)–(14) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \{c_2 a_2(\dot{w}, \dot{w}) + a_0(w, w)\} = -2\, c_1 a_1(\dot{w}, \dot{w}) \leqq 0 \,. \tag{15}$$

As we assume initial conditions (5), (6) to be homogeneous the expression $c_2 a_2(\dot{w}, \dot{w}) + a_0(w, w)$ is initially zero. By (15) this expression either decreases and therefore becomes negative or remains equal to zero. Since (14) holds, however, only the latter alternative is possible. Then, according to (14), $\|w\|_{2,\Omega} = 0 \ \forall t \in [0, t^*]$ which was to be proved.

3. FINITE ELEMENT SOLUTION

We shall solve the variational problem (a)–(c) approximately, approximating the space V_0 by a finite dimensional space created by finite C^1-elements and discretizing the time variable by finite differences.

We shall use triangular elements. In the case of polygonal boundary Γ we shall consider both a triangular C^1-element with a full polynomial of the fifth degree [12] and Bell's C^1-element [1]. In the case of curved boundary Γ we shall consider Bell's elements on the interior triangles (whose sides are straight); along Γ_1 we shall consider curved C^1-elements with a cubic curved side (see [14]), along Γ_2 curved C^1-elements with a quintic curved side [7]. Details concerning the construction of these elements can be found in [14], [8]. For our considerations we shall need only the fact that each finite C^1-element of this kind corresponds to a polynomial $v^*(\xi, \eta)$ of degree N^*,

$$N^* = 4 + n\,, \tag{16}$$

where $n = 1$ for elements with straight sides, $n = 3$ for elements with a cubic curved side and $n = 5$ for elements with a quintic curved side. ((16) is a special form of the relation $N^* = 4m + 1 + (n - 1)\,m$ obtained for curved C^m-elements in [14].) The polynomial $v^*(\xi, \eta)$ is a C^1-element constructed on the unit triangle T_0 which lies in the ξ, η-plane and has the vertices $R_1(0, 0)$, $R_2(1, 0)$, $R_3(0, 1)$. The form of the correspondence is given by the mapping of T_0 on the element considered.

In the rest of this section we shall restrict ourselves to the case of curved boundary. The considerations in the case of a polygonal boundary are similar and simpler.

Let us construct a domain Ω_h (an approximation of Ω) and its triangulation τ_h in the same way as in [14], [7]. Using finite C^1-elements just mentioned we can construct on the triangulation τ_h a finite dimensional space V_h which is a subspace of $C^1(\bar{\Omega}_h)$. Every function $v \in V_h$ is uniquely determined by the parameters $D^\alpha v(P_i)$, $|\alpha| \leqq 2$, prescribed at the nodal points P_i of τ_h (P_i are the vertices of the triangles from which τ_h consists). Let V_{h0} be a subspace of V_h defined by

$$V_{h0} = \left\{ v \in V_h : v\Big|_{\Gamma_h} = 0\,,\ \frac{\partial v}{\partial x_1}\Big|_{\Gamma_{h1}} = \frac{\partial v}{\partial x_2}\Big|_{\Gamma_{h1}} = 0 \right\}, \tag{17}$$

where Γ_h is the boundary of Ω_h and Γ_{h1} is the part of Γ_h approximating Γ_1.

According to [14], [7] the spaces V_h, V_{h0} have the following property (P): Let $u \in C^2(\bar{\Omega})$ satisfy boundary conditions (2), (3) and let $u_I \in V_h$ be the interpolate of u (i.e. $D^\alpha u_I(P_i) = D^\alpha u(P_i)$, $|\alpha| \leqq 2$ at all nodal points P_i). Then $u_I \in V_{h0}$.

Let us choose an integer M, set $\Delta t = t^*/M$ and define

$$t_m = m\,\Delta t \quad (m = 0, 1, \ldots, M)\,. \tag{18}$$

If $f = f(x_1, x_2, t)$ then the symbol f^m will denote a function in two variables x_1, x_2 defined by the relation

$$f^m \equiv f^m(x_1, x_2) = f(x_1, x_2, m\,\Delta t)\,. \tag{19}$$

Setting finally

$$\tilde{a}_{rh}(u, v) = \iint_{\Omega_h} D^{(r)}_{ijkl} u_{,ij} v_{,kl}\, \mathrm{d}x \quad (r = 0, 1, 2) \tag{20}$$

we can define the discrete problem for the approximate solution of our variational problem (a)–(c):

Let $\mu = 1$ or 2. For each $m = 0, 1, \ldots, M - \mu$ find a function $w_d^{m+\mu} \in V_{h0}$ such that

$$(21)\qquad c_2\tilde{a}_{2h}(\Delta^2 w_d^m, v) + \Delta t c_1 \tilde{a}_{1h}\Big(\sum_{p=0}^{\mu} \alpha_p w_d^{m+p}, v\Big) + \\ + \Delta t^2 \tilde{a}_{0h}\Big(\sum_{p=0}^{\mu} \beta_p w_d^{m+p}, v\Big) = \Delta t^2\Big(\sum_{p=0}^{\mu} \beta_p \tilde{Q}^{m+p}, v\Big)_{0,\Omega_h} \quad \forall v \in V_{h0}$$

with initial condition (22) for $\mu = 1$ and initial conditions (22), (23) for $\mu = 2$:

$$(22)\qquad w_d^0 = f^{apr}(x_1, x_2)\,,$$

$$(23)\qquad w_d^1 = f^{apr} + \Delta t\, g^{apr}\,,$$

where $f^{apr} \in V_h$ and $g^{apr} \in V_h$ are approximations of the right-hand sides of (5) and (6), respectively. The relation (23) is motived by the Taylor expansion. In applications we usually define $f^{apr} = f^{int}$, $g^{apr} = g^{int}$, where f^{int} and g^{int} are interpolates of f and g in V_h, respectively. However, we shall see in Section 4 that there exist better approximations of f and g.

Remark. If $c_2 = 0$ we can choose both $\mu = 1$ and $\mu = 2$. However, the initial condition (6) is not prescribed for $c_2 = 0$ and therefore we cannot use (23). Instead of it we compute w_d^1 by means of a one step method. If $c_2 = 1$ then $\mu = 2$.

The operator Δ^2 is defined by

$$(24)\qquad \Delta^2 F^m = F^{m+2} - 2F^{m+1} + F^m\,.$$

For $\mu = 1$ we have

$$(25)\qquad \alpha_0 = -1\,,\quad \alpha_1 = 1\,,\quad \beta_0 = \vartheta\,,\quad \beta_1 = 1 - \vartheta \quad (\vartheta \leqq \tfrac{1}{2})$$

and for $\mu = 2$

$$(26)\qquad \alpha_0 = -1 + \vartheta\,,\quad \alpha_1 = 1 - 2\vartheta\,,\quad \alpha_2 = \vartheta\,,\quad \beta_0 = \tfrac{1}{2} - \tfrac{1}{2}\vartheta + \delta\,, \beta_1 = \tfrac{1}{2} - 2\delta\,, \\ \beta_2 = \tfrac{1}{2}\vartheta + \delta\,,\quad (\vartheta \geqq \tfrac{1}{2},\ \delta > 0 \ \text{ for } \ c_2 = 0,\ \delta \geqq 0 \ \text{ for } \ c_2 = 1)\,.$$

If $c_2 = 0$ then (25) and (26) define the coefficients of the μ-step A-stable method (see [10], [13]). If $c_2 = 1$ then (24), (26) define the Newmark method which is unconditionally stable for $\vartheta \geqq \frac{1}{2}$, $\delta \geqq 0$ [17].

The symbol $\tilde{Q}$ in (21) denotes a continuous extension of the function Q to a domain $\tilde{\Omega} \supset \Omega_h \cup \Omega$.

As usual, we approximate the integrals defining the forms $\tilde{a}_{rh}(\cdot, \cdot)$ and $(\cdot, \cdot)_{0,\Omega_h}$ in (21) by quadrature formulas with integration points lying in Ω. This is done in the same way as in [5] or [14], [10]. Thus we obtain the forms $a_{rh}(\cdot, \cdot)$ and $(\cdot, \cdot)_h$. Now we can formulate the fully discretized problem for the approximate solution of the variational problem (a)–(c) in the domain Ω_h:

Let $\mu = 1$ or 2. For each $m = 0, 1, \ldots, M - \mu$ find a function $w_h^{m+\mu} \in V_{h0}$ such that

$$c_2 a_{2h}(\Delta^2 w_h^m, v) + \Delta t\, c_1 a_{1h}(\sum_{p=0}^{\mu} \alpha_p w_h^{m+p}, v) + \tag{27}$$

$$+ \Delta t^2\, a_{0h}(\sum_{p=0}^{\mu} \beta_p w_h^{m+p}, v) = \Delta t^2(\sum_{p=0}^{\mu} \beta_p Q^{m+p}, v)_h \quad \forall v \in V_{h0}$$

with initial condition (28) for $\mu = 1$ and initial conditions (28), (29) for $\mu = 2$:

$$w_h^0 = f^{apr}(x_1, x_2)\,, \tag{28}$$

$$w_h^1 = f^{apr} + \Delta t\, g^{apr}\,, \tag{29}$$

where the meaning of f^{apr} and g^{apr} is the same as in (22), (23).

It should be noted that the computing experience shows that in the case of equations with constant coefficients (as equation (1)) it is advantageous to use quadrature formulas only on the curved triangles. On the triangles with straight sides it is better to use the technique using standard matrices [9]. This approach saves the computer time.

In Section 4 the error analysis is presented for problems (21)–(23) and (27)–(29). Now we prove existence and uniqueness of a solution of these two problems.

Lemma 1. *Let the boundary Γ be piecewise of class C^4. Let every triangulation τ_h satisfy the condition*

$$\bar{h}/h \geqq c_0 \;\; (c_0 = \text{const} > 0,\; \bar{h} = \min_{T \in \tau_h} h_T,\; h = \max_{T \in \tau_h} h_T)\,, \tag{30}$$

h_T being the greatest side of the triangle with straight sides having the same vertices as T. Then for $h < \tilde{h}$, where $\tilde{h}$ is sufficiently small, we have

$$C_1 \|v\|_{2,\Omega_h}^2 \leqq \tilde{a}_{rh}(v, v) \leqq C_2 \|v\|_{2,\Omega_h}^2 \quad \forall v \in V_{h0}\,, \tag{31}$$

where C_1, C_2 are constants independent of h and v.

Proof. The first inequality (31) follows from the results introduced in [15]. The second inequality (31) is obvious.

Lemma 2. *Let the assumptions of Lemma 1 be satisfied. Let the quadrature formulas on the unit triangle T_0 for the calculation of the forms $a_{rh}(u, v)$ be of the degree of precision $2n + 4$, where n depends on $T \in \tau_h$ and is the same as in (16). Then for $h < \tilde{h}$, where $\tilde{h}$ is sufficiently small, we have*

$$C_3 \|v\|_{2,\Omega_h}^2 \leqq a_{rh}(v, v) \leqq C_4 \|v\|_{2,\Omega_h}^2 \quad \forall v \in V_{h0}\,, \tag{32}$$

where C_3, C_4 are constants independent of h and v.

Proof. According to (16), $2N^* - 4 = 2n + 4$. Thus we can use Theorem 7 from [14] and Theorem 3.1 from [7] and prove the first inequality (32) similarly to [14, Corollary 1]. The second inequality (32) is a straightforward consequence of the second inequality (31) and [14, Theorem 7] and [7, Theorem 3.1].

Theorem 1. *a) Let the assumptions of Lemma* 1 *be satisfied. Then for sufficiently small h, problem* (21)–(23) *has one and only one solution.*

b) *Let the assumptions of Lemma* 2 *be satisfied. Then for sufficiently small h, problem* (27)–(29) *has one and only one solution.*

Proof. a) As relation (21) represents for each m a system of linear equations it suffices to prove that the following homogeneous problem has only the trivial solution: Find a function $w_d^{m+\mu} \in V_{h0}$ such that

$$c_2 \tilde{a}_{2h}(w_d^{m+\mu}, v) + \Delta t\, c_1 \alpha_\mu \tilde{a}_{1h}(w_d^{m+\mu}, v) + \\ + \Delta t^2\, \beta_\mu \tilde{a}_{0h}(w_d^{m+\mu}, v) = 0 \quad \forall v \in V_{h0}\,. \tag{33}$$

Let us set $v = w_d^{m+\mu}$ in (33) and use the first inequality (31). Then we obtain $w_d^{m+\mu} \equiv 0$.

b) Using the first inequality (32), part b) can be proved in the same way as part a). Theorem 1 is proved.

It should be noted that part a) of Theorem 1 can be proved without the restriction $h < \tilde{h}$, i.e. for every h. The only change is that in the proof we use the first inequality (31) with a constant C_1 dependent on h. However, in Section 4 we shall need inequalities (31) with constants independent of h.

In the case of polygonal boundary Γ inequalities (31) hold for every h and without assumption (30). Thus we can state the following theorem:

Theorem 2. a) *Let the domain Ω have a polygonal boundary Γ. Then problem* (21)–(23) *has one and only one solution for every h.*

b) *Let the domain Ω have a polygonal boundary Γ and let the quadrature formulas on the unit triangle T_0 for the calculation of the forms $a_{rh}(u, v)$ be of the degree of precision* 6. *Then for sufficiently small h problem* (27)–(29) *has one and only one solution.*

4. ERROR ESTIMATES

We start with some definitions and a lemma. The symbol $\tilde{w}$ will denote the Calderon extension of the exact solution w to the domain $\tilde{\Omega}$, where $\tilde{\Omega} \supset \Omega \cup \Omega_h \;\; \forall h < \tilde{h}$.

The function $\eta \in V_{h0}$ satisfying

$$\tilde{a}_{0h}(\tilde{w} - \eta, v) = 0 \quad \forall v \in V_{h0} \tag{34}$$

is called the Ritz approximation of the function $\tilde{w}$.

Lemma 3. *Let in the case of a curved boundary Γ the assumption of Lemma 1 be satisfied. Let the solution $w(x_1, x_2, t)$ satisfy*

(35) $$w(x_1, x_2, t) \in H^{k+2}(\Omega) \quad \forall t \in (0, t^*]$$

where $k = 4$ when we use only triangles with straight sides and full polynomials of the fifth degree, and $k = 3$ in the remaining cases. Let $\tilde{w}(x_1, x_2, t) \in H^{k+2}(\tilde{\Omega})$ be the Calderon extension of $w(x_1, x_2, t)$ from Ω to $\tilde{\Omega}$, i.e.

(36) $$\|\tilde{w}\|_{k+2,\tilde{\Omega}} \leqq C_5 \|w\|_{k+2,\Omega},$$

where C_5 is a constant independent of w and t. Let $\eta(x_1, x_2, t) \in V_{h0}$ $(\forall t \in (0, t^*])$ be the Ritz approximation of the function $\tilde{w}(x_1, x_2, t)$. Then there exists a constant C independent of w, h and t such that

(37) $$\|\tilde{w} - \eta\|_{2,\Omega_h} \leqq C h^k \|w\|_{k+2,\Omega}.$$

Proof. Let $\tilde{w}_I \in V_{h0}$ be the interpolate of the function $\tilde{w} \in H^{k+2}(\tilde{\Omega})$. Then, according to corresponding interpolation theorems on classical and curved triangles (see [2, p. 812 and 819] and [14, p. 358]),

(38) $$\|\tilde{w} - \tilde{w}_I\|_{2,\Omega_h} \leqq C h^k \|\tilde{w}\|_{k+2,\tilde{\Omega}}.$$

The triangular inequality gives

(39) $$\|\tilde{w} - \eta\|_{2,\Omega_h} \leqq \|\tilde{w} - \tilde{w}_I\|_{2,\Omega_h} + \|\eta - \tilde{w}_I\|_{2,\Omega_h}.$$

As $\eta \in V_{h0}$, $\tilde{w}_I \in V_{h0}$ we have $\eta - \tilde{w}_I \in V_{h0}$. Using the first inequality (31) and relation (34) we obtain

(40) $$C_1 \|\eta - \tilde{w}_I\|^2_{2,\Omega_h} \leqq \tilde{a}_{0h}(\eta - \tilde{w}_I, \eta - \tilde{w}_I) =$$
$$= \tilde{a}_{0h}(\tilde{w} - \tilde{w}_I, \eta - \tilde{w}_I) \leqq K \|\tilde{w} - \tilde{w}_I\|_{2,\Omega_h} \|\eta - \tilde{w}_I\|_{2,\Omega_h},$$

where the constant K does not depend on h (because $D^{(0)}_{ijkl}$ are constants). If $\eta \not\equiv \tilde{w}_I$ then we obtain from (40)

(41) $$\|\eta - \tilde{w}_I\|_{2,\Omega_h} \leqq (K/C_1) \|\tilde{w} - \tilde{w}_I\|_{2,\Omega_h}.$$

Inequality (37) follows from (36), (38), (39) and (41). If $\eta = \tilde{w}_I$ then (37) follows from (36) and (38). Lemma 3 is proved.

First we prove convergence of the solution of problem (21)–(23). In order to estimate the expressions $\|\tilde{w}^m - w_d^m\|_{2,\Omega_h}$ $(m = \mu, \ldots, M)$ let us set

(42) $$\xi^m = \tilde{w}^m - \eta^m, \quad \varepsilon_d^m = \eta^m - w_d^m.$$

Then

(43) $$\|\tilde{w}^m - w_d^m\|_{2,\Omega_h} \leqq \|\xi^m\|_{2,\Omega_h} + \|\varepsilon_d^m\|_{2,\Omega_h}.$$

The first term on the right-hand side of (43) can be estimated by means of Lemma 3. It remains to estimate the second term. The complete result is contained in Theorems 3 and 4.

Theorem 3. *Let $c_2 = 1$ and let in the case of a curved boundary Γ the assumptions of Lemma* 1 *be satisfied. Let*

$$w \in C^{q+3}(H^{k+2}(\Omega)), \tag{44}$$

where k is defined in Lemma 3 *and $q = 2$ for $\vartheta = \frac{1}{2}$ and $q = 1$ for $\vartheta \neq \frac{1}{2}$ (cf* (25), (26)). *Then for $m = 2, \ldots, M$ and for sufficiently small h the following estimate holds:*

$$\|\tilde{w}^m - w_d^m\|_{2,\Omega_h} \leqq C\{\Delta t^q + h^k + \sum_{j=0}^{1} \|\varepsilon_d^j\|_{2,\Omega_h} + \Delta t^{-1}\|\varepsilon_d^1 - \varepsilon_d^0\|_{2,\Omega_h}\}, \tag{45}$$

where the constant C does not depend on Δt and h.

Theorem 4. *Let $c_2 = 0$, $c_1 = 1$ and let in the case of a curved boundary Γ the assumptions of Lemma* 1 *be satisfied. Let*

$$w \in C^{q+1}(H^{k+2}(\Omega)). \tag{46}$$

Then for $m = \mu, \ldots, M$ ($\mu = 1$ or 2) and for sufficiently small h the following estimate holds:

$$\|\tilde{w}^m - w_d^m\|_{2,\Omega_h} \leqq C\{\Delta t^q + h^k + \sum_{j=0}^{\mu-1} \|\varepsilon_d^j\|_{2,\Omega_h}\}, \tag{47}$$

where the meaning of k, q and C is the same as in Theorem 3.

Proof of Theorems 3 and 4. For the sake of simplicity we shall write ε^m instead of ε_d^m. First we prove Theorem 3. Some of our arguments are similar to considerations from [16].

Let $\tilde{w}$, $\tilde{\dot{w}}$ and $\tilde{\ddot{w}}$ denote the Calderon extensions of functions w, $\dot{w}$ and $\ddot{w}$, respectively, from Ω to $\tilde{\Omega}$. Let us define a function $\tilde{Q}$ by the relation

$$c_2 D_{ijkl}^{(2)} \tilde{\ddot{w}}_{,ijkl} + c_1 D_{ijkl}^{(1)} \tilde{\dot{w}}_{,ijkl} + D_{ijkl}^{(0)} \tilde{w}_{,ijkl} = \tilde{Q}. \tag{48}$$

Equation (48) is identical with equation (1) for $(x_1, x_2) \in \Omega$. Let us multiply (48) by an arbitrary function $v \in V_{h0}$ and integrate over Ω_h. We obtain

$$c_2 \tilde{a}_{2h}(\tilde{\ddot{w}}, v) + c_1 \tilde{a}_{1h}(\tilde{\dot{w}}, v) + \tilde{a}_{0h}(\tilde{w}, v) = (\tilde{Q}, v)_{0,\Omega_h}. \tag{49}$$

Relations (34) and (49) imply

$$\Delta t^2 \tilde{a}_{0h}(\sum_{p=0}^{2} \beta_p \eta^{m+p}, v) = \Delta t^2 (\sum_{p=0}^{2} \beta_p \tilde{Q}^{m+p}, v)_{0,\Omega_h} - \tag{50}$$

$$- c_2 \Delta t^2 \tilde{a}_{2h}(\sum_{p=0}^{2} \beta_p \tilde{\ddot{w}}^{m+p}, v) - c_1 \Delta t^2 \tilde{a}_{1h}(\sum_{p=0}^{2} \beta_p \tilde{\dot{w}}^{m+p}, v).$$

Let us add to both sides of (50) the expression

$$c_2 \tilde{a}_{2h}(\Delta^2 \eta^m, v) + c_1 \, \Delta t \, \tilde{a}_{1h}(\sum_{p=0}^{2} \alpha_p \eta^{m+p}, v) ;$$

we obtain

$$(51) \qquad c_2 \tilde{a}_{2h}(\Delta^2 \eta^m, v) + c_1 \, \Delta t \, \tilde{a}_{1h}(\sum_{p=0}^{2} \alpha_p \eta^{m+p}, v) + \Delta t^2 \, \tilde{a}_{0h}(\sum_{p=0}^{2} \beta_p \eta^{m+p}, v) =$$

$$= \Delta t^2 (\sum_{p=0}^{2} \beta_p \tilde{Q}^{m+p}, v)_{0,\Omega_h} + c_1 \, \Delta t \, \tilde{a}_{1h}(\sum_{p=0}^{2} (\alpha_p \eta^{m+p} - \Delta t \, \beta_p \tilde{\dot{w}}^{m+p}), v) +$$

$$+ c_2 \tilde{a}_{2h}(\Delta^2 \eta^m - \Delta t^2 \sum_{p=0}^{2} \beta_p \tilde{\ddot{w}}^{m+p}, v) .$$

As $\tilde{w} = \xi + \eta$, we have

$$(52) \qquad \eta^p = \tilde{w}^p - \xi^p .$$

Let us insert (52) into (51) and use the notation

$$(53) \qquad \pi^m = \sum_{p=0}^{2} (\alpha_p \tilde{w}^{m+p} - \Delta t \, \beta_p \tilde{\dot{w}}^{m+p}) ,$$

$$(54) \qquad \omega^m = \sum_{p=0}^{2} \alpha_p \xi^{m+p} ,$$

$$(55) \qquad y^m = \Delta^2 \tilde{w}^m - \Delta t^2 \sum_{p=0}^{2} \beta_p \tilde{\ddot{w}}^{m+p} ;$$

we obtain

$$(56) \qquad c_2 \tilde{a}_{2h}(\Delta^2 \eta^m, v) + c_1 \, \Delta t \, \tilde{a}_{1h}(\sum_{p=0}^{2} \alpha_p \eta^{m+p}, v) +$$

$$+ \Delta t^2 \, \tilde{a}_{0h}(\sum_{p=0}^{2} \beta_p \eta^{m+p}, v) = \Delta t^2 (\sum_{p=0}^{2} \beta_p \tilde{Q}^{m+p}, v)_{0,\Omega_h} +$$

$$+ c_1 \, \Delta t \, \tilde{a}_{1h}(\pi^m - \omega^m, v) + c_2 \tilde{a}_{2h}(y^m, v) - c_2 \tilde{a}_{2h}(\Delta^2 \xi^m, v) .$$

Let us subtract (21) from (56) and use (42). We get

$$(57) \qquad c_2 \tilde{a}_{2h}(\Delta^2 \varepsilon^m, v) + c_1 \, \Delta t \, \tilde{a}_{1h}(\sum_{p=0}^{2} \alpha_p \varepsilon^{m+p}, v) +$$

$$+ \Delta t^2 \, \tilde{a}_{0h}(\sum_{p=0}^{2} \beta_p \varepsilon^{m+p}, v) = c_2 \tilde{a}_{2h}(y^m, v) - c_2 \tilde{a}_{2h}(\Delta^2 \xi^m, v) +$$

$$+ c_1 \, \Delta t \, \tilde{a}_{1h}(\pi^m - \omega^m, v) \quad \forall v \in V_{h0} .$$

Let us substitute into (57) the expression

$$(58) \qquad v = \sum_{p=0}^{2} \alpha_p \varepsilon^{m+p} \in V_{h0}$$

Adding up the obtained relation from $m = 0$ up to $m = s - 2$ we get

$$(59) \qquad \sum_{m=0}^{s-2} (c_2 \tilde{A}^m + c_1 \, \Delta t \, \tilde{B}^m + \Delta t^2 \, \tilde{C}^m) = \sum_{m=0}^{s-2} (c_2 \tilde{D}^m - c_2 \tilde{E}^m + c_1 \, \Delta t \, \tilde{F}^m),$$

where

$$(60) \qquad \tilde{A}^m = \tilde{a}_{2h}(\Delta^2 \varepsilon^m, \sum_{p=0}^{2} \alpha_p \varepsilon^{m+p}),$$

$$(61) \qquad \tilde{B}^m = \tilde{a}_{1h}(\sum_{p=0}^{2} \alpha_p \varepsilon^{m+p}, \sum_{p=0}^{2} \alpha_p \varepsilon^{m+p}),$$

$$(62) \qquad \tilde{C}^m = \tilde{a}_{0h}(\sum_{p=0}^{2} \beta_p \varepsilon^{m+p}, \sum_{p=0}^{2} \alpha_p \varepsilon^{m+p}),$$

$$(63) \qquad \tilde{D}^m = \tilde{a}_{2h}(y^m, \sum_{p=0}^{2} \alpha_p \varepsilon^{m+p}),$$

$$(64) \qquad \tilde{E}^m = \tilde{a}_{2h}(\Delta^2 \xi^m, \sum_{p=0}^{2} \alpha_p \varepsilon^{m+p}),$$

$$(65) \qquad \tilde{F}^m = \tilde{a}_{1h}(\pi^m - \omega^m, \sum_{p=0}^{2} \alpha_p \varepsilon^{m+p})$$

The left-hand side of (59) will be estimated from below and the right-hand side from above. Using (24), (26) and the notation $\Delta \varepsilon^m = \varepsilon^{m+1} - \varepsilon^m$ we obtain from (60)

$$\tilde{A}^m = \tilde{a}_{2h}(\Delta \varepsilon^{m+1} - \Delta \varepsilon^m, \, \vartheta \Delta \varepsilon^{m+1} + (1 - \vartheta) \, \Delta \varepsilon^m).$$

Taking into account the relation $\vartheta \geqq \frac{1}{2}$ we can find that

$$\tilde{A}^m \geqq \tfrac{1}{2}[\tilde{a}_{2h}(\Delta \varepsilon^{m+1}, \Delta \varepsilon^{m+1}) - \tilde{a}_{2h}(\Delta \varepsilon^m, \Delta \varepsilon^m)].$$

Similarly, using (26), (62) and $\vartheta \geqq \frac{1}{2}$ we obtain

$$\begin{aligned} \tilde{C}^m &= \tilde{a}_{0h}(\vartheta \varepsilon^{m+2} + (1 - \vartheta) \, \varepsilon^{m+1} - (\vartheta \varepsilon^{m+1} + (1 - \vartheta) \, \varepsilon^m), \\ &\tfrac{1}{2}(\vartheta \varepsilon^{m+2} + (1 - \vartheta) \, \varepsilon^{m+1} + \vartheta \varepsilon^{m+1} + (1 - \vartheta) \, \varepsilon^m + \delta \Delta^2 \varepsilon^m) \geqq \\ &\geqq \tfrac{1}{2}[\tilde{a}_{0h}(\vartheta \varepsilon^{m+2} + (1 - \vartheta) \, \varepsilon^{m+1}, \, \vartheta \varepsilon^{m+2} + (1 - \vartheta) \, \varepsilon^{m+1}) - \\ &- \tilde{a}_{0h}(\vartheta \varepsilon^{m+1} + (1 - \vartheta) \, \varepsilon^m, \vartheta \varepsilon^{m+1} + (1 - \vartheta) \, \varepsilon^m) + \\ &+ \delta \tilde{a}_{0h}(\Delta \varepsilon^{m+1}, \Delta \varepsilon^{m+1}) - \delta \tilde{a}_{0h}(\Delta \varepsilon^m, \Delta \varepsilon^m)]. \end{aligned}$$

It follows from (61), (31) that $\tilde{B}^m \geqq 0$. Thus using Lemma 1 we get for $\Delta t \leqq 1$

$$(66) \qquad \sum_{m=0}^{s-2} (\tilde{A}^m + \Delta t \, c_1 \tilde{B}^m + \Delta t^2 \, \tilde{C}^m) \geqq \tfrac{1}{2} C_1 \, \Delta t^2 \, P - \tfrac{1}{2} C_2 \|\Delta \varepsilon^0\|_{2,\Omega_h}^2 - \tfrac{1}{2} C_2 \, \Delta t^2 \, R,$$

where

$$(67) \qquad \begin{aligned} -R &= -\|\vartheta \varepsilon^1 + (1 - \vartheta) \, \varepsilon^0\|_{2,\Omega_h}^2 - \delta \|\varepsilon^1 - \varepsilon^0\|_{2,\Omega_h}^2 \geqq \\ &\geqq -K_1(\|\varepsilon^1\|_{2,\Omega_h}^2 + \|\varepsilon^0\|_{2,\Omega_h}^2) \end{aligned}$$

with

$$K_1 = \max\{\vartheta, \vartheta(2\vartheta - 1) + 2\delta\}$$

and

$$P = \|\vartheta\varepsilon^s + (1 - \vartheta)\varepsilon^{s-1}\|^2_{2,\Omega_h} + (1 + \delta)\|\varepsilon^s - \varepsilon^{s-1}\|^2_{2,\Omega_h} =$$
$$= (\vartheta^2 + 1 + \delta)\|\varepsilon^s\|^2_{2,\Omega_h} + [(1 - \vartheta)^2 + 1 + \delta]\|\varepsilon^{s-1}\|^2_{2,\Omega_h} +$$
$$+ 2[\vartheta(1 - \vartheta) - 1 - \delta](\varepsilon^s, \varepsilon^{s-1})_{2,\Omega_h}.$$

As $\vartheta(1 - \vartheta) - 1 - \delta < 0$ for $\vartheta \geqq \frac{1}{2}$, $\delta \geqq 0$, we have for any $\tau > 0$

$$2[\vartheta(1 - \vartheta) - 1 - \delta](\varepsilon^s, \varepsilon^{s-1})_{2,\Omega_h} \geqq$$
$$\geqq [\vartheta(1 - \vartheta) - 1 - \delta]\left(\tau\|\varepsilon^s\|^2_{2,\Omega_h} + \frac{1}{\tau}\|\varepsilon^{s-1}\|^2_{2,\Omega_h}\right).$$

We can set

$$\tau = \frac{[1 + \delta - \vartheta(1 - \vartheta)]^2 + [1 + \delta + (1 - \vartheta)^2](1 + \delta + \vartheta^2)}{2[1 + \delta + (1 - \vartheta)^2][1 + \delta - \vartheta(1 - \vartheta)]}$$

thus obtaining

(68) $$P \geqq K_2(\|\varepsilon^s\|^2_{2,\Omega_h} + \|\varepsilon^{s-1}\|^2_{2,\Omega_h}),$$

where

$$K_2 = \min\left\{\frac{1 + \delta}{2[1 + \delta + (1 - \vartheta)^2]},\right.$$
$$\left.\frac{(1 + \delta)[1 + \delta + (1 - \vartheta)^2]}{[1 + \delta - \vartheta(1 - \vartheta)]^2 + (1 + \delta + \vartheta^2)[1 + \delta + (1 - \vartheta)^2]}\right\}.$$

According to (26), we can write

(69) $$\sum_{p=0}^{2}\alpha_p\varepsilon^{m+p} = \Delta(\vartheta\,\Delta\varepsilon^m + \varepsilon^m).$$

Using (63) and (69) we obtain with respect to the boundedness of $\tilde{a}_{rh}(u, v)$

(70) $$\sum_{m=0}^{s-2}\tilde{D}^m = \tilde{a}_{2h}(y^{s-2}, \vartheta\,\Delta\varepsilon^{s-1} + \varepsilon^{s-1}) - \tilde{a}_{2h}(y^0, \vartheta\,\Delta\varepsilon^0 + \varepsilon^0) -$$
$$- \sum_{m=1}^{s-2}\tilde{a}_{2h}(y^m - y^{m-1}, \vartheta\,\Delta\varepsilon^m + \varepsilon^m) \leqq \vartheta C_2\{\|y^{s-2}\|_{2,\Omega_h}(\|\varepsilon^s\|_{2,\Omega_h} +$$
$$+ \|\varepsilon^{s-1}\|_{2,\Omega_h}) + \|y^0\|_{2,\Omega_h}(\|\varepsilon^1\|_{2,\Omega_h} + \|\varepsilon^0\|_{2,\Omega_h}) +$$
$$+ \sum_{m=1}^{s-2}\|\Delta y^{m-1}\|_{2,\Omega_h}(\|\varepsilon^{m+1}\|_{2,\Omega_h} + \|\varepsilon^m\|_{2,\Omega_h})\}.$$

From the Calderon theorem and from (55) we obtain

$$\|y^m\|_{2,\Omega_h} \leqq C\|\Delta^2 w^m - \Delta t^2\sum_{p=0}^{2}\beta_p\ddot{w}^{m+p}\|_{k+2,\Omega}.$$

Let $q = 2$ (i.e. $\vartheta = \frac{1}{2}$). Then the Taylor theorem yields

$$\Delta^2 w^m - \Delta t^2 \sum_{p=0}^{2} \beta_p \ddot{w}^{m+p} = \frac{2}{3} \Delta t^4 \frac{\partial^4 w^{\varkappa_1}}{\partial t^4} - \frac{1}{12} \Delta t^4 \frac{\partial^4 w^{\varkappa_2}}{\partial t^4} - $$
$$- \left(\tfrac{1}{4} + \delta\right) \Delta t^4 \frac{\partial^4 w^{\varkappa_3}}{\partial t^4} - \left(\tfrac{1}{4} - \delta\right) \Delta t^4 \frac{\partial^4 w^{\varkappa_4}}{\partial t^4} .$$

Hence by means of (44)

(71) $$\|y^m\|_{2,\Omega_h} \leqq C \, \Delta t^4 .$$

Let $q = 1$ (i.e. $\vartheta > \frac{1}{2}$). Then the Taylor theorem yields

$$\Delta^2 w^m - \Delta t^2 \sum_{p=0}^{2} \beta_p \ddot{w}^{m+p} =$$
$$= \Delta t^3 \left[\frac{4}{3} \frac{\partial^3 w^{\varkappa_5}}{\partial t^3} - \frac{1}{3} \frac{\partial^3 w^{\varkappa_6}}{\partial t^3} - (\vartheta + 2\delta) \frac{\partial^3 w^{\varkappa_7}}{\partial t^3} - \left(\tfrac{1}{2} - 2\delta\right) \frac{\partial^3 w^{\varkappa_8}}{\partial t^3}\right].$$

Hence by means of (44)

(72) $$\|y^m\|_{2,\Omega_h} \leqq C \, \Delta t^3 .$$

From (71) and (72) we obtain

(73) $$\|y^m\|_{2,\Omega_h} \leqq C \, \Delta t^{q+2} .$$

Similarly we can derive

(74) $$\|\Delta y^m\|_{2,\Omega_h} \leqq C \, \Delta t^{q+3} .$$

Relations (70), (73) and (74) imply

(75) $$\Delta t^{-2} \sum_{m=0}^{s-2} \tilde{D}^m \leqq C \, \Delta t^q \Big\{ \sum_{i=0}^{1} \big[\|\varepsilon^{s-i}\|_{2,\Omega_h} + \|\varepsilon^i\|_{2,\Omega_h}\big] + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|_{2,\Omega_h} \Big\} .$$

Using (64) and (69) we obtain similarly as in the case of relation (70)

$$- \sum_{m=0}^{s-2} \tilde{E}^m \leqq \vartheta C_2 \sum_{i=0}^{1} \Big\{ \|\Delta^2 \xi^{s-2}\|_{2,\Omega_h} \|\varepsilon^{s-i}\|_{2,\Omega_h} + \|\Delta^2 \xi^0\|_{2,\Omega_h} \|\varepsilon^i\|_{2,\Omega_h} +$$
$$+ \sum_{m=1}^{s-2} \|\Delta^3 \xi^{m-1}\|_{2,\Omega_h} \|\varepsilon^{m+i}\|_{2,\Omega_h} \Big\} .$$

Relations (42_1), (44), Lemma 3 and the Taylor theorem imply

$$\|\Delta^j \xi^m\|_{2,\Omega_h} = \|\Delta^j \tilde{w} - \Delta^j \eta^m\|_{2,\Omega_h} \leqq C h^k \|\Delta^j w\|_{k+2,\Omega} \leqq C \, \Delta t^j h^k \quad (j = 2, 3) .$$

Thus

(76) $$- \Delta t^{-2} \sum_{m=0}^{s-2} \tilde{E}^m \leqq C h^k \Big[\Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|_{2,\Omega_h} + \sum_{i=0}^{1} \big(\|\varepsilon^{s-i}\|_{2,\Omega_h} + \|\varepsilon^i\|_{2,\Omega_h}\big)\Big] .$$

Using (65) and (69) we obtain similarly as in the case of relation (70):

$$\sum_{m=0}^{s-2} \tilde{F}^m \leqq \vartheta C_2 \sum_{i=0}^{1} \{(\|\pi^{s-2}\|_{2,\Omega_h} + \|\omega^{s-2}\|_{2,\Omega_h}) \|\varepsilon^{s-i}\|_{2,\Omega_h} + + (\|\pi^0\|_{2,\Omega_h} + \|\omega^0\|_{2,\Omega_h}) \|\varepsilon^i\|_{2,\Omega_h} + \sum_{m=1}^{s-2} (\|\Delta\pi^{m-1}\|_{2,\Omega_h} + + \|\Delta\omega^{m-1}\|_{2,\Omega_h}) \|\varepsilon^{m+i}\|_{2,\Omega_h}\} .$$

In the same way as in [10, p. 432] we obtain

$$\|\pi^m\|_{2,\Omega_h} \leqq C\,\Delta t^{q+1}, \quad \|\omega^m\|_{2,\Omega_h} \leqq Ch^k\,\Delta t .$$

Similarly

$$\|\Delta\pi^m\|_{2,\Omega_h} \leqq C\,\Delta t^{q+2}, \quad \|\Delta\omega^m\|_{2,\Omega_h} \leqq Ch^k\,\Delta t^2 .$$

Thus

(77)
$$\Delta t^{-1} \sum_{m=0}^{s-2} \tilde{F}^m \leqq C(h^k + \Delta t^q) \{\Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|_{2,\Omega_h} + + \sum_{i=0}^{1} (\|\varepsilon^{s-i}\|_{2,\Omega_h} + \|\varepsilon^i\|_{2,\Omega_h})\} .$$

Dividing (59) by Δt^2 and using (66)–(68), (75)–(77) we obtain

(78)
$$\sum_{i=0}^{1} \|\varepsilon^{s-i}\|^2_{2,\Omega_h} \leqq K_0\{\Delta t^{-2}\|\Delta\varepsilon^0\|^2_{2,\Omega_h} + \sum_{i=0}^{1} \|\varepsilon^i\|^2_{2,\Omega_h} + + (\Delta t^q + h^k) [\sum_{i=0}^{1} (\|\varepsilon^{s-i}\|_{2,\Omega_h} + \|\varepsilon^i\|_{2,\Omega_h}) + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|_{2,\Omega_h}]\} ,$$

where K_0 is a constant independent of h, Δt and s. Now we shall use five times the inequality

(79)
$$|ab| \leqq \frac{1}{2} \tau a^2 + \frac{1}{2\tau} b^2 .$$

In all five cases we set $a = \Delta t^q + h^k$. As to b and τ we define (denoting by K_i the absolute constant obtained after the i-th step):

1. $b = \|\varepsilon^s\|_{2,\Omega_h}$, $\quad \tau = K_0$,
2. $b = \|\varepsilon^{s-1}\|_{2,\Omega_h}$, $\quad \tau = K_1/2$,
3. $b = \|\varepsilon^1\|_{2,\Omega_h}$, $\quad \tau = 1$,
4. $b = \|\varepsilon^0\|_{2,\Omega_h}$, $\quad \tau = 1$,
5. $b = \|\varepsilon^m\|_{2,\Omega_h}$, $\quad \tau = K_4/2$.

The result is

$$\|\varepsilon^s\|^2_{2,\Omega_h} \leqq K\{\Delta t^{2q} + h^{2k} + \Delta t^{-2}\|\Delta\varepsilon^0\|^2_{2,\Omega_h} + \tag{80}$$
$$+ \sum_{i=0}^{1} \|\varepsilon^i\|^2_{2,\Omega_h}\} + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|^2_{2,\Omega_h} .$$

Using the discrete form of Gronwall's lemma (see e.g. [6, p. 176]) we obtain

$$\|\varepsilon^s\|^2_{2,\Omega_h} \leqq K\{\Delta t^{2q} + h^{2k} + \sum_{i=0}^{1} \|\varepsilon^i\|^2_{2,\Omega_h} + \Delta t^{-2}\|\Delta\varepsilon^0\|^2_{2,\Omega_h}\} \, e^{s\Delta t} .$$

Hence

$$\|\varepsilon^s\|_{2,\Omega_h} \leqq C\{\Delta t^q + h^k + \sum_{i=0}^{1} \|\varepsilon^i\|_{2,\Omega_h} + \Delta t^{-1}\|\Delta\varepsilon^0\|_{2,\Omega_h}\} . \tag{81}$$

The assertion of Theorem 3 follows from (42), (43), (81) and from Lemma 3.

Now we prove Theorem 4. In this case $c_2 = 0$. Let us divide (57) by Δt and set

$$v = \sum_{p=0}^{\mu} \beta_p \varepsilon^{m+p} \in V_{h0} . \tag{82}$$

Adding up the obtained relation from $m = 0$ up to $m = s - \mu$ we get

$$\sum_{m=0}^{s-\mu} (\tilde{G}^m + \Delta t \, \tilde{H}^m) = \sum_{m=0}^{s-\mu} \tilde{J}^m , \tag{83}$$

where

$$\tilde{G}^m = \tilde{a}_{1h}\Big(\sum_{p=0}^{\mu} \alpha_p \varepsilon^{m+p}, \; \sum_{p=0}^{\mu} \beta_p \varepsilon^{m+p}\Big) , \tag{84}$$

$$\tilde{H}^m = \tilde{a}_{0h}\Big(\sum_{p=0}^{\mu} \beta_p \varepsilon^{m+p}, \; \sum_{p=0}^{\mu} \beta_p \varepsilon^{m+p}\Big) , \tag{85}$$

$$\tilde{J}^m = \tilde{a}_{1h}\Big(\pi^m - \omega^m, \; \sum_{p=0}^{\mu} \beta_p \varepsilon^{m+p}\Big) . \tag{86}$$

Similarly as in [10, pp. 430–431] we obtain

$$\sum_{m=0}^{s-\mu} \tilde{G}^m \geqq K_1 \|\varepsilon^s\|^2_{2,\Omega_h} - K_2 \sum_{i=0}^{\mu-1} \|\varepsilon^i\|^2_{2,\Omega_h} . \tag{87}$$

Lemma 1 and relation (85) give

$$\sum_{m=0}^{s-\mu} \tilde{H}^m \geqq C_1 \sum_{m=0}^{s-\mu} \Big\| \sum_{p=0}^{\mu} \beta_p \varepsilon^{m+p} \Big\|^2_{2,\Omega_h} . \tag{88}$$

Similarly as in [10, pp. 432–433] we obtain

$$\sum_{m=0}^{s-\mu} \tilde{J}^m \leqq C \, \Delta t(\Delta t^q + h^k) \sum_{m=0}^{s-\mu} \Big\| \sum_{p=0}^{\mu} \beta_p \varepsilon^{m+p} \Big\|^2_{2,\Omega_h} . \tag{89}$$

The inequality which follows from (83), (87), (88) and (89) can be estimated in the same way as [10, (171)]. The result is

$$\|\varepsilon^s\|_{2,\Omega_h} \leqq C(\Delta t^q + h^k + \sum_{i=0}^{\mu-1} \|\varepsilon^i\|_{2,\Omega_h}) . \tag{90}$$

The assertion of Theorem 4 follows from (42), (43), (90) and Lemma 3. Theorems 3 and 4 are proved.

At the end of this section we shall briefly analyze the effect of numerical integration. Let us define the error functionals by

$$E(uv) = (u, v)_{0,\Omega_h} - (u, v)_h , \tag{91}$$

$$E(D_{ijkl}^{(m)} u_{,ij} v_{,kl}) = \tilde{a}_{mh}(u, v) - a_{mh}(u, v) . \tag{92}$$

Let us express $(\cdot, \cdot)_{0,\Omega_h}$ and $\tilde{a}_{mh}(\cdot, \cdot)$ in (56) by means of (91) and (92), respectively, subtract (27) from (56) and use the relation

$$\varepsilon_h^j = \eta^j - w_h^j . \tag{93}$$

We obtain

$$\begin{aligned} c_2 a_{2h}(\Delta^2 \varepsilon_h^m, v) + c_1 \, \Delta t \, a_{1h}(\sum_{p=0}^{2} \alpha_p \varepsilon_h^{m+p}, v) + \\ + \Delta t^2 \, a_{0h}(\sum_{p=0}^{2} \beta_p \varepsilon^{m+p}, v) = c_2 \tilde{a}_{2h}(y^m, v) - c_2 \tilde{a}_{2h}(\Delta^2 \xi^m, v) + \\ + c_1 \, \Delta t \, \tilde{a}_{1h}(\pi^m - \omega^m, v) + \Delta t^2 \, E(\sum_{p=0}^{2} \beta_p \tilde{Q}^{m+p} v) - \\ - c_2 E(D_{ijkl}^{(2)} \Delta^2 \eta_{,ij}^m v_{,kl}) - c_1 \, \Delta t \, E(D_{ijkl}^{(1)} \sum_{p=0}^{2} \alpha_p \eta_{,ij}^{m+p} v_{,kl}) - \\ - \Delta t^2 \, E(D_{ijkl}^{(0)} \sum_{p=0}^{2} \beta_p \eta_{,ij}^{m+p} v_{,kl}) . \end{aligned} \tag{94}$$

We shall use (94) for estimating $\|\varepsilon_h^m\|_{2,\Omega_h}$. Let us assume that $c_2 = 1$ and set $v = \sum_{p=0}^{2} \alpha_p \varepsilon_h^{m+p}$ in (94). Adding up the obtained relation from $m = 0$ up to $m = s - 2$ and dividing the result by Δt^2 we obtain

$$\begin{aligned} \sum_{m=0}^{s-2} (\Delta t^{-2} A^m + c_1 \, \Delta t^{-1} B^m + C^m) = \sum_{m=0}^{s-2} (\Delta t^{-2} \tilde{D}^m - \Delta t^{-2} \tilde{E}^m + \\ + c_1 \, \Delta t^{-1} \tilde{F}^m + K^m - \Delta t^{-2} L^m - c_1 \, \Delta t^{-1} M^m - N^m) , \end{aligned} \tag{95}$$

where $\tilde{D}^m$, $\tilde{E}^m$ and $\tilde{F}^m$ are given by (63), (64) and (65), respectively, A^m, B^m and C^m by the relations which we obtain from relations (60), (61) and (62), respectively, by omitting tildas, and where

$$K^m = E(\sum_{p=0}^{2} \beta_p \tilde{Q}^{m+p} \sum_{q=0}^{2} \alpha_q \varepsilon_h^{m+q}), \tag{96}$$

$$L^m = E(D^{(2)}_{ijkl}(\Delta^2 \eta^m)_{,ij} (\sum_{q=0}^{2} \alpha_q \varepsilon_h^{m+q})_{,kl}), \tag{97}$$

$$M^m = E(D^{(1)}_{ijkl}(\sum_{p=0}^{2} \alpha_p \eta^{m+p})_{,ij} (\sum_{q=0}^{2} \alpha_q \varepsilon_h^{m+q})_{,kl}), \tag{98}$$

$$N^m = E(D^{(0)}_{ijkl}(\sum_{p=0}^{2} \beta_p \eta^{m+p})_{,ij} (\sum_{q=0}^{2} \alpha_q \varepsilon_h^{m+q})_{,kl}). \tag{99}$$

In estimating the terms (96)–(99) we shall use the following lemmas:

Lemma 4. *Let $\tilde{Q}(x_1, x_2, t) \in H^k(\tilde{\Omega})$ $\forall t \in [0, t^*]$ and let $v \in V_{0h}$. Let the quadrature formulas on the unit triangle $\bar{T}_0$ for the calculation of the form $(u, v)_h$ be of the degree of precision*

$$d = n + k + 1. \tag{100}$$

Then

$$|E(\tilde{Q}^m v)| \leqq Ch^k \|\tilde{Q}^m\|_{k,\tilde{\Omega}} \|v\|_{2,\Omega_h}, \tag{101}$$

where C is a constant independent of h, $\tilde{Q}$ and v.

It should be noted that n appearing in (16) and (100) depends on $T \in \tau_h$. We have $n = 1$ for triangles with straight sides and $n > 1$ for curved triangles ($n = 3$ if the curved side lies on Γ_{h1} and $n = 5$ if the curved side lies on Γ_{h2}). As $E(f) = \sum_T E_T(f)$ the proof of Lemma 4 is a consequence of [14, Theorem 9] and Cauchy's inequality. Relation (100) is relation [14, (157)] written for $r = k$, $N^* = n + 4$ (see (16)) and $m = 1$.

Lemma 5. *Let the assumptions of Lemma 1 be satisfied. Let the degrees of precision of the quadrature formulas on the unit triangle $\bar{T}_0$ for the calculation of the forms $a_{rh}(u, v)$ $(r = 0, 1, 2)$ be given by (100). Then*

$$|E(D^{(r)}_{ijkl} \eta_{,ij} v_{,kl})| \leqq Ch^k \|w\|_{k+2,\Omega} \|v\|_{2,\Omega_h}, \tag{102}$$

where C is a constant independent of h, w and v.

Proof. The proof is a consequence of the proof of [14, Theorem 8]. As in [14] it is sufficient to estimate terms of the form

$$E^*(c^* D^\gamma \eta^* D^\delta v^*) \tag{103}$$

with $|\gamma| = |\delta| = 2$ for interior elements and $1 \leqq |\gamma|, |\delta| \leqq 2$ for boundary elements.

The function c^* has the property $D^\mu c^* = O(h^{2-|\gamma|-|\delta|+|\mu|})$. It follows from [14, (174)] that

$$(104) \qquad |E^*(c^* D^\gamma \eta^* D^\delta v^*)| \leqq C h^{s+3-|\gamma|-|\delta|} \|v\|_{2,T} \sum_{j=0}^{s} h_T^{-j} |\eta^*|_{j+|\gamma|,T_0},$$

where

$$(105) \qquad s = k + |\delta| - 2\,.$$

It remains to estimate the term depending on η^*. We can write

$$(106) \qquad \sum_{j=0}^{s} h_T^{-j} |\eta^*|_{j+|\gamma|,T_0} \leqq \sum_{j=0}^{s} h_T^{-j} \{|\eta^* - (\pi_T \tilde{w})^*|_{j+|\gamma|,T_0} + $$
$$+ |(\pi_T \tilde{w})^* - \tilde{w}^*|_{j+|\gamma|,T_0} + |\tilde{w}^*|_{j+|\gamma|,T_0}\},$$

where $\pi_T \tilde{w}$ is the interpolate of $\tilde{w}$ on T. We have (using [14, (153), (106), (73)])

$$(107) \qquad \sum_{j=0}^{s} h_T^{-j} |\eta^* - (\pi_T \tilde{w})^*|_{j+|\gamma|,T_0} \leqq C h_T^{-s} |\eta^* - (\pi_T \tilde{w})^*|_{|\gamma|,T_0} \leqq$$
$$\leqq C h_T^{-s+|\gamma|-1} (\|\eta - \tilde{w}\|_{|\gamma|,T} + \|\tilde{w} - \pi_T \tilde{w}\|_{|\gamma|,T}) \leqq$$
$$\leqq C h_T^{-s+|\gamma|-1} (\|\eta - \tilde{w}\|_{2,T} + h^k \|\tilde{w}\|_{k+2,T})\,.$$

Similarly

$$(108) \qquad \sum_{j=0}^{s} h_T^{-j} |(\pi_T \tilde{w})^* - \tilde{w}^*|_{j+|\gamma|,T_0} \leqq C \sum_{j=0}^{s} h_T^{|\gamma|-1} \|(\pi_T \tilde{w}) - \tilde{w}\|_{j+|\gamma|,T} \leqq$$
$$\leqq C h^{|\gamma|-1} \|\pi_T \tilde{w} - \tilde{w}\|_{s+|\gamma|,T} \leqq C h_T^{|\gamma|-1} \|\tilde{w}\|_{k+2,T},$$

$$(109) \qquad \sum_{j=0}^{s} h_T^{-j} |\tilde{w}^*|_{j+|\gamma|,T_0} \leqq C h_T^{|\gamma|-1} \|\tilde{w}\|_{k+2,T}\,.$$

Substituting (106)–(108) into (104) and using (105) we obtain

$$(110) \qquad |E^*(c^* D^\gamma \eta^* D^\delta v^*)| \leqq C \|v\|_{2,T} (\|\eta - \tilde{w}\|_{2,T} + h^k \|\tilde{w}\|_{k+2,T})\,.$$

We have

$$(111) \qquad E(D_{ijkl} \eta_{,ij} v_{,kl}) = \sum_T E_T(D_{ijkl} \eta_{,ij} v_{,kl})\,.$$

Each term on the right-hand side of (111) can be written as a sum of terms (103). Thus, using Cauchy's inequality and Lemma 3 we obtain from (110), (111) the estimate (102). Lemma 5 is proved.

Theorem 5. *Let the assumptions of Theorem* 3, *Lemma* 2 *and Lemma* 4 *be satisfied. Let* $\partial Q/\partial t \in C(H^k(\Omega))$. *Then for* $m = 2, \ldots, M$ *and for sufficiently small* h *the following estimate holds*:

$$(112) \qquad \|\tilde{w}^m - w_h^m\|_{2,\Omega_h} \leqq$$
$$\leqq C\{\Delta t^q + h^k + \sum_{j=0}^{1} \|\varepsilon_h^j\|_{2,\Omega_h} + \Delta t^{-1} \|\varepsilon_h^1 - \varepsilon_h^0\|_{2,\Omega_h}\},$$

where the constant C *does not depend on* Δt *and* h.

Proof. We have

$$\|\tilde{w}^m - w_h^m\|_{2,\Omega_h} \leqq \|\zeta^m\|_{2,\Omega_h} + \|\varepsilon_h^m\|_{2,\Omega_h}. \tag{113}$$

The first term on the right-hand side of (113) can be estimated by means of Lemma 3. The estimate of the second term will be obtained in the same way as in the proof of Theorem 3: by estimating the left-hand side of (95) from below and the right-hand side from above.

First we estimate the last four terms on the right-hand side of (95). Using (96), summation by parts (see (69), (70)), Lemma 4 and the Taylor theorem we obtain

$$\begin{aligned}\sum_{m=0}^{s-2} K^m = {} & E\Big(\sum_{p=0}^{2} \beta_p \tilde{Q}^{s-2+p}(\vartheta\Delta\varepsilon_h^{s-1} + \varepsilon_h^{s-1})\Big) - \\ & - \sum_{m=1}^{s-1} E\Big(\sum_{p=0}^{2} \beta_p(\tilde{Q}^{m+p} - \tilde{Q}^{m-1+p})(\vartheta\Delta\varepsilon_h^m + \varepsilon_h^m)\Big) - \\ & - E\Big(\sum_{p=0}^{2} \beta_p \tilde{Q}^p(\vartheta\Delta\varepsilon_h^0 + \varepsilon_h^0)\Big) \leqq \\ \leqq {} & \vartheta C h^k\Big\{\Big\|\sum_{p=0}^{2} \beta_p \tilde{Q}^{s-2+p}\Big\|_{k,\tilde{\Omega}} \sum_{i=0}^{1} \|\varepsilon^{s-i}\|_{2,\Omega_h} + \\ & + \sum_{m=1}^{s-1} \Big\|\sum_{p=0}^{2} \beta_p \Delta\tilde{Q}^{m-1+p}\Big\|_{k,\tilde{\Omega}} \sum_{i=0}^{1} \|\varepsilon_h^{m+i}\|_{2,\Omega_h} + \\ & + \Big\|\sum_{p=0}^{2} \beta_p \tilde{Q}^p\Big\|_{k,\tilde{\Omega}} \sum_{i=0}^{1} \|\varepsilon_h^i\|_{2,\Omega_h}\Big\} \leqq \\ \leqq {} & C h^k\Big\{\sum_{i=0}^{1}\big[\|\varepsilon_h^{s-i}\|_{2,\Omega_h} + \|\varepsilon_h^i\|_{2,\Omega_h}\big] + \Delta t \sum_{m=1}^{s-1} \|\varepsilon_h^m\|_{2,\Omega_h}\Big\}.\end{aligned} \tag{114}$$

Similarly, using (97)–(99), summation by parts, Lemma 5 and the Taylor theorem, we can estimate

$$\begin{aligned}& \sum_{m=0}^{s-2} \big(-\Delta t^{-2} L^m - c_1\, \Delta t^{-1} M^m - N^m\big) \leqq \\ & \leqq C h^k\Big\{\Delta t \sum_{m=1}^{s-1} \|\varepsilon_h^m\|_{2,\Omega_h} + \sum_{i=0}^{1}\big[\|\varepsilon_h^{s-i}\|_{2,\Omega_h} + \|\varepsilon_h^i\|_{2,\Omega_h}\big]\Big\}.\end{aligned} \tag{115}$$

The remaining terms in (95), which contain A^m, B^m, C^m, $\tilde{D}^m$, $\tilde{E}^m$ and $\tilde{F}^m$, can be estimated similarly as in the proof of Theorem 3 (instead of Lemma 1 we use Lemma 2). We obtain an inequality of the same form as inequality (78). The rest of the proof is the same as the corresponding part of the proof of Theorem 3.

Similarly, generalizing the proof of Theorem 4 we can prove:

Theorem 6. *Let the assumptions of Theorem 4, Lemma 2 and Lemma 4 be satisfied. Then for $m = \mu, \ldots, M$ ($\mu = 1$ or 2) and for sufficiently small h the following estimate holds:*

$$(116) \qquad \|\tilde{w}^m - w_h^m\|_{2,\Omega_h} \leqq C\{\Delta t^q + h^k + \sum_{j=0}^{\mu-1} \|\varepsilon_h^j\|_{2,\Omega_h}\} ,$$

where the meaning of k, q and C is the same as in Theorem 3.

Remark. The assumptions of Theorem 5 can be weakened. If we define the forms $a_{rh}(u, v)$ $(r = 1, 2)$ by means of quadrature formulas of an arbitrary degree but with positive weights then $a_{rh}(v, v) \geqq 0$ (this follows from (8)). Thus $B^m \geqq 0$ and we can change Theorem 5 in the following way: Let the degrees of precision of the quadrature formulas on T_0 for the calculation of $a_{0h}(u, v)$ and $a_{rh}(u, v)$ $(r = 1, 2)$ be $2n + 4$ and $n + k + 1$, respectively, and let the formulas of the degree $n + k + 1$ have positive weights. Let the remaining assumptions of Theorem 5 hold. Then for $m = 2, \ldots, M$ and for sufficiently small h the following estimate holds:

$$(117) \qquad \|\tilde{w}^m - w_h^m\|_{2,\Omega_h} \leqq C\{\Delta t^q + h^k + \sum_{j=0}^{1} \|\varepsilon_h^j\|_{2,\Omega_h} + \Delta t^{-1}\|\varepsilon_h^1 - \varepsilon_h^0\|_*\} ,$$

where $\|v\|_*^2 = a_{2h}(v, v)$.

5. NUMERICAL RESULTS

Example 1. We solved problem (1)–(6) with Ω being a rectangle with sides $a, b, \Gamma_1 = \Gamma, \Gamma_2 = \emptyset, c_1 = 1$ or 0, $c_2 = 1$ or 0. The functions Q, f, g were chosen in such a way that the problem has the solution

$$(118) \qquad w(x_1, x_2, t) = \varphi(t)\, u(x_1, x_2) ,$$

where $\varphi(t)$ is a function of time variable t and

$$(119) \qquad u(x_1, x_2) = [x_1(a - x_1)\, x_2(b - x_2)]^2 .$$

As the problem is symmetrical the space discretization was done for one quarter of Ω only. It was divided into four equal rectangles and each of them by diagonals into four triangles. The number of unknowns is thus 65 for the full polynomial of the fifth degree and 45 for Bell's element. Numerical integration was not used. In the case $c_2 = 1$ the initial condition (6) was approximated either by the Taylor theorem

$$(120) \qquad w_d(x_1, x_2, \Delta t) = (\varphi(0) + \Delta t\, \dot{\varphi}(0))\, u(x_1, x_2) ,$$

or by using the exact value

$$(121) \qquad w_d(x_1, x_2, \Delta t) = \varphi(\Delta t)\, u(x_1, x_2) .$$

The results are expressed by the absolute value of maximum relative error in percent, i.e. by

$$(122) \qquad \max |(D^\gamma w^m - D^\gamma w_d^m)/D^\gamma w^m| \,.\, 100 ,$$

where in each time step maximum is taken over all nodal points and over $|\gamma| = 0, 1, 2$. Some results are presented in Tables 1 and 2 for the case $a = b$, $D_{ijkl}^{(m)} w_{,ijkl} = \Delta^2 w$,

$(m = 0, 1, 2)$, (Δ^2 denotes now the biharmonic operator), $\varphi(t) = 1 - \exp(-t)$ and Bell's element. We make the following remarks:

1. Using a one step method for $c_2 = 0$, $\varphi(t) = t$ we obtain the maximum relative error 1·93% (constant in all time steps) which may be considered as the error of the space discretization. This error is achieved at $\partial^2 w/\partial x_1^2$ at the point $(3a/8, b/8)$ whereas the relative error at $w(\frac{1}{2}a, \frac{1}{2}b)$ is only 0·005%. The great difference between the maximum error and error at $w(\frac{1}{2}a, \frac{1}{2}b)$ appears always when the error of the space discretization prevails. However, if the error of the time discretization is dominant

Table 1. Maximum relative error in % for one step methods.

ϑ	1/2	1/2	1/2	1/2	1/3	1/4
Δt	1/8	1/4	1/2	1	1/2	1/2
t						
0·5	2·04	2·35	3·63		5·47	8·34
1·0	2·01	2·24	3·20	7·50	4·39	6·64
1·5	1·99	2·16	2·87		3·50	5·23
2·0	1·97	2·10	2·61	4·79	2·80	4·10
2·5	1·96	2·05	2·41		2·25	3·22
3·0	1·95	2·02	2·27	3·30	1·84	2·55
3·5	1·95	1·99	2·16		1·54	2·05
4·0	1·94	1·97	2·09	2·55	1·42	1·68
4·5	1·94	1·96	2·04		1·58	1·42
5·0	1·94	1·95	2·00	2·20	1·69	1·51

Table 2. Maximum relative error in % for two step methods.

c_2	0	1	1	1	1	1	1	1	1	1
c_1	1	1	1	0	0	0	1	1	0	0
ϑ	1/2	1/2	1/2	1/2	0·6	0·6	1/2	1/2	1/2	1/2
δ	1/12	1/12	1/12	1/12	0·3	0·3	1/12	1/12	1/12	1/12
Δt	1/8	1/8	1/8	1/8	1/8	1/8	1/4	1/4	1/4	1/4
t										
Δt	2·06	6·38	0	6·38	6·38	0	13·02	0	13·02	0
0·5	2·14	6·19	0·05	7·37	7·53	0·21	12·64	0·08	14·08	0·36
1·0	2·09	5·66	0·25	8·05	8·40	0·43	11·30	0·26	15·12	0·96
1·5	2·05	4·90	0·56	7·65	8·19	0·51	9·39	0·45	14·27	1·23
2·0	2·02	4·05	0·93	6·05	6·73	0·44	7·23	0·75	11·37	1·15
2·5	2·00	3·22	1·30	4·38	5·07	1·15	5·15	1·14	7·45	1·46
3·0	1·98	2·52	1·62	2·34	2·92	1·97	3·39	1·49	2·69	1·63
3·5	1·96	2·01	1·86	3·36	2·97	2·74	2·09	1·77	5·57	1·93
4·0	1·95	1·69	2·01	6·02	5·87	3·30	1·60	1·96	10·26	3·14
4·5	1·95	1·55	2·09	7·40	7·50	3·52	2·06	2·07	12·65	4·08
5·0	1·94	1·53	2·11	7·18	7·51	3·37	2·10	2·11	12·21	4·42

then the relative error is almost of the same magnitude for all $D^\gamma w$, $|\gamma| = 0, 1, 2$, at all nodal points.

2. In the case $c_2 = 1$ the sign of the error changes repeatedly. This can explain the fact that in some time steps greater error is obtained for $\varepsilon^1 = 0$ than in the case $\varepsilon^1 > 0$. (In our example $\varepsilon^0 = 0$ always holds.)

3. In the case $c_2 = 0$ the best results are obtained for the one step method with $\vartheta = \frac{1}{2}$. This corresponds to the properties of the one and two step A-stable methods for solving a single ordinary differential equation of the first order. In the case $c_2 = 1$ we got better results for $c_1 = 1$ than for $c_1 = 0$ not only for $\varphi(t) = 1 - \exp(-t)$ but also for some other functions $\varphi(t)$ and even for the solution of a single differential equation of the second order.

Example 2. We reproduced an example solved in [11] by a combination of finite elements and the Laplace transform. The data are given so that problem (1)–(6) should describe the viscoelastic bending of an orthotropic simply supported plate loaded by the uniform load. In this example $c_2 = 0$ and the initial condition (5) is given by

$$D^{(1)}_{ijkl} w_{,ijkl}(x_1, x_2, 0) = \text{const} \neq 0 . \tag{123}$$

Other details concerning the data can be found in [11]. In [11], the exact values of $w(x_1, x_2, t)$ at the centre of the plate, computed by means of an infinite series, are also given. If we divide the quarter of the rectangular plate into 16 triangles in the same way as in Example 1 (the problem is again symmetrical in space), use the full polynomial of the fifth degree and the one step method with $\vartheta = 0{\cdot}5$, $\Delta t = 0{\cdot}2$ in $(0, 1]$, $\Delta t = 0{\cdot}5$ in $(1, 2]$, $\Delta t = 1$ in $(2, 10]$ and $\Delta t = 5$ in $(10, 100]$, we get results which coincide with the exact solution up to 4 significant digits. However, we get surprisingly good results even if we divide a quarter of Ω only into two triangles, use Bell's element and make only three time steps with $\Delta t = 10$, $\Delta t = 20$ and $\Delta t = 70$, successively. The relative errors then are 1·13% for $t = 10$, 2·12% for $t = 30$ and 1·46% for $t = 100$. It should be noted that the function w at the centre of the plate is approximated better than w elsewhere except the boundary and much better than the first and second derivatives of w.

All computations have been performed in the Computing centre of Technical university of Brno on the computer DATASAAB D21.

References

[1] *K. Bell:* A refined triangular plate bending finite element. Int. J. Numer. Meth. Engng. 1 (1969), 101–122.

[2] *J. H. Bramble, M. Zlámal:* Triangular elements in the finite element method. Math. Comp. 24 (1970), 809–820.

[3] *J. Brilla:* Visco-elastic bending of anisotropic plates (in Slovak), Stav. čas. 17 (1969), 153–175.

[4] *J. Brilla:* Finite element method for quasiparabolic equations, in Proc. of the 4th symposium on basic problems of numer. math., Plzeň (1978), 25–36.

[5] *P. G. Ciarlet:* The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam, 1978.
[6] *V. Girault, P.-A. Raviart:* Finite Element Approximation of the Navier-Stokes Equations. Springer-Verlag, Berlin—Heidelberg—New York, 1979.
[7] *J. Hřebíček:* Numerical analysis of the general biharmonic problem by the finite element method. Apl. mat. 27 (1982), 352—374.
[8] *V. Kolář, J. Kratochvíl, F. Leitner, A. Ženíšek:* Calculation of plane and Space Constructions by the Finite Element Method (Czech). SNTL, Praha, 1979.
[9] *J. Kratochvíl, A. Ženíšek, M. Zlámal:* A simple algorithm for the stiffness matrix of triangular plate bending finite elements. Int. J. Numer. Meth. Engng. 3 (1971), 553—563.
[10] *J. Nedoma:* The finite element solution of parabolic equations. Apl. mat. 23 (1978), 408—438.
[11] *S. Turčok:* Solution of quasiparabolic differential equations by finite element method (in Slovak), Thesis, Komenský University Bratislava, (1978).
[12] *M. Zlámal:* On the finite element method. Numer. Math. 12 (1968), 394—409.
[13] *M. Zlámal:* Finite element methods for nonlinear parabolic equations. R.A.I.R.O. Numer. Anal. 11 (1977), 93—107.
[14] *A. Ženíšek:* Curved triangular finite C^m-elements. Apl. Mat. 23 (1978), 346—377.
[15] *A. Ženíšek:* Discrete forms of Friedrichs' inequalities in the finite element method. R.A.I. R.O. Numer. Anal. 15 (1981), 265—286.
[16] *A. Ženíšek:* Finite element methods for coupled thermoelasticity and coupled consolidation of clay. (To appear in R.A.I.R.O. Numer. Anal. 18 (1984).)
[17] *E. Godlewski, A. Puech-Raoult:* Équations d'évolution linéaires du second ordre et méthodes multipas. R.A.I.R.O. Numer. Anal. 13 (1979), 329—353.

Souhrn

ŘEŠENÍ VISKOELASTICKÉ TENKÉ DESKY METODOU KONEČNÝCH PRVKŮ

HELENA RŮŽIČKOVÁ, ALEXANDER ŽENÍŠEK

Řešení w problému $(1)-(6)$ představuje pro $c_2 = c_1 = 1$ nebo pro $c_2 = 0$, $c_1 = 1$ viskoelastický průhyb tenké desky (viz [3], [4]). Variační formulaci (a)—(c) problému $(1)-(6)$ diskretizujeme vzhledem k prostorovým proměnným metodou konečných prvků při použití trojúhelníkových C^1-prvků. Diskretizace vzhledem k časové proměnné je provedena jedno- nebo dvoukrokovou diferenční metodou s krokem Δt. V souvislosti s metodou konečných prvků analyzujeme také vliv numerické integrace a křivé hranice. Plně diskretizovaný problém je definován vztahy $(27)-(29)$. Hlavní výsledek práce je odhad chyby formulovaný ve větách 5 a 6 ve tvaru

$$\|\tilde{w}^m - w_h^m\|_{2,\Omega_h} \leqq C\{\Delta t^q + h^k + \|\varepsilon^0\| + \|\varepsilon^1\| + c_2\, \Delta t^{-1}\|\varepsilon^1 - \varepsilon^0\|_{2,\Omega_h}\}$$

kde $q = 1$ nebo 2 v závislosti na použité diferenční metodě, $k = 3$ nebo 4 v závislosti na konečných prvcích, ε^0 a ε^1 jsou chyby způsobené aproximací počátečních podmínek, $\tilde{w}^m$ je přesné a w_h^m přibližné řešení v čase $m\,\Delta t$. Přitom předpokládáme dostatečně vysoký řád přesnosti použitých integračních formulí a dostatečnou hladkost dat. V závěru práce jsou uvedeny numerické výsledky.

Authors' address: RNDr. *Helena Růžičková,* CSc., Doc. RNDr. *Alexander Ženíšek,* DrSc., Oblastní výpočetní centrum VUT, Obránců míru 21, 602 00 Brno.