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SOLUTIONS OF ABSTRACT HYPERBOLIC EQUATIONS BY ROTHE METHOD

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INTRODUCTION

In this paper we consider the abstract hyperbolic equation

$$\frac{d^2u}{dt^2} + A(t)\,u(t) = f(t)\,, \quad t \in \langle 0, T\rangle\,, \quad T < \infty \tag{1.1}$$

with the initial conditions

$$u(0) = u_0\,, \quad \frac{du}{dt}(0) = u_1\,, \tag{1.2}$$

where $A(t)$ is supposed to be a symmetric elliptic operator which depends on t.

The above mentioned equation is solved in the following way:

a) for a uniform partition of $\langle 0, T\rangle$, $h = T/n$ and ${}^nt_j = jh$ we construct an approximative solution

$$\begin{gathered} {}^nu(t) = {}^nz_{j-1} + (t - t_{j-1})\,h^{-1}({}^nz_j - {}^nz_{j-1}) \\ \text{for} \quad t_{j-1} \leqq t \leqq t_j\,, \quad j = 1, \ldots, n\,, \end{gathered} \tag{1.3}$$

where ${}^nz_j$, $j = 1, \ldots, n$ are the solutions of the system of equations

$$\frac{{}^nz_j - 2\,{}^nz_{j-1} + {}^nz_{j-2}}{h^2} + {}^nA_j\,{}^nz_j = {}^nf_j\,, \quad j = 1, \ldots, n\,, \tag{1.4}$$

$${}^nz_0 = u_0\,, \quad {}^nz_{-1} = u_0 - hu_1\,, \tag{1.5}$$

where ${}^nA_j = A({}^nt_j)\,, \quad {}^nf_j = f({}^nt_j)\,;$

b) under certain assumptions on $A(t)$, $f(t)$ and the initial conditions u_0 and u_1 we prove that the sequence ${}^nu(t)$ converges for $n \to \infty$ to the unique solution $u(t)$ of (1.1), (1.2);

c) in the next part some numeric aspects of the above mentioned method are dealt with, some estimates of the difference of the approximate solution $^n u$ and the exact solution u of the problem (1.1), (1.2) are presented here;

d) in the last part of this paper we present results concerning estimates of errors of the approximate solutions which are constructed by using only the approximate solutions of the equations of the system (1.4), (1.5).

The method described is called the Rothe method or the method of lines.

The Rothe method was introduced in [9] and later on has been used by many authors for the solution of parabolic equations – see [1]–[6]. Hyperbolic equations have been solved by the Rothe method first in [7] and [8]. In this papers the operator A was considered only independent of time t. Questions concerning hyperbolic equations have been solved also for instance in [10] by using other methods.

NOTATIONS AND DEFINITIONS

Let V be a reflexive Banach space which is contained in a Hilbert space H. We assume that V is dense in H and V is continuously imbedded in H. We identify the space H with its dual H' and denote by V' the dual space of V. In this way we can identify H with the subspace of V':

$$V \subset H \subset V' . \tag{2.1}$$

We denote by $|\cdot|$ and $\|\cdot\|$ the norms in H and V, respectively. The inner product in H as well as the pairing between V and V' are both denoted by $(\cdot,\cdot)$ (which is possible due to (2.1) and density V in H).

$((\cdot,\cdot))$ denotes a continuous symmetric bilinear form defined on $V \times V$. Such form can be identified with a symmetric operator $A \in \mathscr{L}(V, V')$, where for $u \in V$ in the following way we define $Au \in V'$

$$((u, v)) = (Au, v) \quad \text{for all} \quad v \in V . \tag{2.2}$$

The space of all such operators is denoted by $\mathscr{A}$.

In the whole article $((\cdot,\cdot))_t$, $t \in \langle 0, T \rangle = I$ denotes the set of continuous symmetric bilinear forms defined on $V \times V$ for which there exists $\alpha > 0$ such that

$$((u, u))_t \geqq \alpha \|u\|^2 , \quad u \in V , \quad t \in I . \tag{2.3}$$

A_t denotes the set of the corresponding continuous symmetric bilinear operators, i.e.

$$(A_t u, v) = ((u, v))_t , \quad v \in V , \quad t \in I . \tag{2.4}$$

For brevity we denote $((u, u))_t = \|u\|_t^2$.

Definition 2.1. *Let X be a Banach space with a norm $\|\cdot\|_X$. $L_p(I, X)$, $1 \leqq p \leqq \infty$, denotes the space of functions $f : I \to X$ for which*

$$\int_0^T \|f(t)\|_X^p < \infty .$$

Definition 2.2. *$W^{k,p}(I, X)$, $1 \leqq p \leqq \infty$, denotes the space of functions f for which*

$$f^{(i)} \in L_p(I, X), \quad 0 \leqq i \leqq k,$$

where $f^{(i)}$ is the derivative in the sense of distributions.

The next two lemmas are presented without proofs, which are evident. Their purpose is to introduce the notation of constants which are used in the same sense throughout the whole article.

Lemma 2.1. *Let $f \in W^{k,\infty}(I, H)$, $k = 0, 1, 2$. Then there exist constants L_i, $i = 0, \ldots, k$, independent of t and such that*

$$\begin{aligned} &|f(t)| \leqq L_0 , \\ &|f(t) - f(t - h)| \leqq L_1 h , \\ &|f(t) - 2f(t - h) + f(t - 2h)| \leqq L_2 h^2 , \end{aligned}$$

for $t, t - h, t - 2h \in I$.

Lemma 2.2. *Let $A_t \in W^{k,\infty}(I, \mathscr{A})$, $k = 1, 2, 3$. Then there exist constants K_i, $i = 0, \ldots, k$, independent of t and such that*

$$\begin{aligned} &|((u, v))_t| \leqq K_0 \|u\| \,.\, \|v\| , \\ &|((u, v))_t - ((u, v))_{t-h}| \leqq K_1 \alpha\, h \|u\| \,.\, \|v\| , \\ &|((u, v))_t - 2((u, v))_{t-h} + ((u, v))_{t-2h}| \leqq K_2 \alpha\, h^3 \|u\| \,.\, \|v\| , \\ &|((u, v)_t - 3((u, v))_{t-h} + 3((u, v))_{t-2h} - (u, v)_{t-3h}| \leqq K_3 \alpha\, h^2 \|u\| \,.\, \|v\| , \end{aligned}$$

for $t, t - h, t - 2h, t - 3h \in I$.

Now we can formulate the problem:

Solve the equation

$$u''(t) + A_t u(t) = f(t) , \tag{2.5}$$

$$u(0) = u_0 , \quad u'(0) = u_1 \tag{2.6}$$

with $A_t \in W^{k,\infty}(I, \mathscr{A})$, $f \in W^{l,\infty}(I, H)$, where k, l will be specified further and the properties of the solution depend on them. We denote by u' and u'' the strong derivatives in the space V, H or V', which will be specified.

Lemma 3.1. *Let the assumption* (2.3) *be satisfied. Then every equation of the system* (1.4), (1.5) *has a unique solution.*

This lemma is a simple consequence of the Lax Milgram theorem.

Before formulating a priority estimate theorems we add to the mesh points ${}^n t_j$, $j = -1, \ldots, n$, two auxliary points $t_{-2} = -2h$ and $t_{-3} = -3h$ and to the system of the equations (1.4) two further equations for $j = 0$ and $j = -1$. To this aim we must define ${}^n A_{-1}$ and ${}^n f_{-1}$. We do that in the following way:

$$
\begin{aligned}
{}^n A_{-1} &= 2\,{}^n A_0 - {}^n A_1 \,, \\
{}^n f_{-1} &= 2\,{}^n f_0 - {}^n f_1 \,.
\end{aligned}
$$

The reason for this definition will be clear from the proof of Theorem 3.3.

Further we shall use the following notation:

$$Z_j = \frac{z_j - z_{j-1}}{h}\,, \quad j = -2, \ldots, n\,,$$

$$s_j = \frac{Z_j - Z_{j-1}}{h}\,, \quad j = -1, \ldots, n\,,$$

$$S_j = \frac{s_j - s_{j-1}}{h}\,, \quad j = 0, \ldots, n\,.$$

We shall omit the upper index n if there is no danger of misunderstanding.

Theorem 3.1. *Let z_j be solutions of* (1.4), (1.5), $A_t \in W^{1,\infty}(I, \mathscr{A})$. *Then the estimate*

$$\|z_j\|_j^2 + |Z_j|^2 \leqq \frac{1}{1-h}\left[(1 + K_1 h)\,\|z_0\|_0^2 + |Z_0|^2 + T\max|f_i|^2\right] \mathrm{e}^{T\max(1,K_1)/(1-h)}$$

takes place for $j = 1, \ldots, n$.

Proof. z_j is a solution of the equation (1.4) if and only if

$$((z_j, v))_j + \left(\frac{z_j - 2z_{j-1} + z_{j-2}}{h^2}\,,\; v\right) = (f_j, v)$$

(where $((\cdot,\cdot))_j = ((\cdot,\cdot))_{t_j}$) holds for all $v \in V$. Putting $v = z_j - z_{j-1}$ we have

$$((z_j, z_j - z_{j-1}))_j + (s_j, z_j - z_{j-1}) = (f_j, z_j - z_{j-1})\,.$$

Hence we obtain

$$\tfrac{1}{2}\|z_j\|_j^2 + \tfrac{1}{2}\|z_j - z_{j-1}\|_j^2 - \tfrac{1}{2}\|z_{j-1}\|_j^2 + \tfrac{1}{2}|Z_j|^2 + \tfrac{1}{2}|Z_j - Z_{j-1}|^2 - {}$$
$$- \tfrac{1}{2}|Z_{j-1}|^2 = h(f_j, Z_j)\,.$$

By omitting the nonnegative expressions $\frac{1}{2}\|z_j - z_{j-1}\|_j^2$ and $\frac{1}{2}|Z_j - Z_{j-1}|^2$ we obtain

$$\|z_j\|_j^2 + |Z_j|^2 \leqq \|z_{j-1}\|_{j-1}^2 + |Z_{j-1}|^2 + 2h(f_j, Z_j) + \|z_{j-1}\|_j^2 - \|z_{j-1}\|_{j-1}^2 .$$

From Lemmas 2.1 and 2.2 we deduce

$$\|z_j\|_j^2 + |Z_j|^2 \leqq \|z_{j-1}\|_{j-1}^2 + |Z_{j-1}|^2 + h|Z_j|^2 + h \max |f_i|^2 + h K_1 \|z_{j-1}\|_{j-1}^2 .$$

This estimate is recurrent and enables us to obtain successively $(h < 1)$

$$\|z_j\|_j^2 + |Z_j|^2 \leqq \frac{1}{1-h}\left[(1 + hK_1)\|z_0\|_0^2 + |Z_1|^2 + T\max |f_i|^2\right] + \\ + \frac{h}{1-h}\max(1, K_1)\sum_{i=1}^{j-1}(\|z_i\|_i^2 + |Z_i|^2) .$$

Now, the assertion is a consequence of the following lemma:

Lemma 3.2. *Let a_j be a consequence of nonnegative numbers for which*

$$a_1 \leqq A ,$$
$$a_j \leqq A + hB\sum_{i=1}^{j-1} a_i ,$$

where A, B are nonnegative constants. Then

$$a_j \leqq Ae^{(j-1)hB} .$$

The proof of this lemma is well-known.

Remark. From the previous theorem it is evident that there exist a constant $C_1 > 0$ independent of j and h (after restriction to "small h") which depends only on $\|z_0\|_0$, $|Z_0|$ and $\|f\|_{0,\infty}$, where $\|\cdot\|_{k,\infty}$ denotes the norm in the space $W^{k,\infty}(I, H)$, such that the following estimate holds:

$$\|z_j\|_j^2 + |Z_j|^2 \leqq C_1^2 .$$

Theorem 3.2. *Let $A_t \in W^{2,\infty}(I, \mathscr{A})$, $f \in W^{1,\infty}(I, H)$. Then there exists a continuous function*

$$C_2 = C_2(\|u_0\|_0, \|u_1\|_0, |f_0 - A_0 u_0|, \|f\|_{1,\infty})$$

independent of h and j, $C_2(0, 0, 0, 0) = 0$, such that

$$\|Z_j\|_j^2 + |s_j|^2 \leqq C_2 .$$

Proof. By subtracting two equations of (1.4) with the indices j and $j - 1$ and putting $v = s_j$ we obtain

$$((z_j, s_j))_j - ((z_{j-1}, s_j))_{j-1} + (s_j - s_{j-1}, s_j) = (f_j - f_{j-1}, s_j) , \quad j = 1, \ldots, n .$$

Hence we deduce

$$\|Z_j\|_j^2 + |s_j|^2 \leqq \|Z_{j-1}\|_{j-1}^2 + |s_{j-1}|^2 + 2h\left(\frac{f_j - f_{j-1}}{h}, s_j\right) +$$
$$+ \|Z_{j-1}\|_j^2 - \|Z_{j-1}\|_{j-1}^2 + 2((z_{j-1}, s_j))_{j-1} - 2((z_{j-1}, s_j))_j .$$

Now, we modify the last part of the preceding inequality

$$2((z_{j-1}, s_j))_{j-1} - 2((z_{j-1}, s_j))_j = (2/h)((z_j, Z_j))_{j-1} - (2/h)((z_{j-1}, Z_{j-1}))_{j-1} -$$
$$- (2/h)((z_j, Z_j))_j + (2/h)((z_{j-1}, Z_{j-1}))_j + 2\|Z_j\|_j^2 - 2\|Z_j\|_{j-1}^2 .$$

The remainder of the proof is based on the same principle as the proof of the previous theorem and therefore it is not presented here.

Theorem 3.3. *Let* $A_t \in W^{3,\infty}(I, \mathscr{A})$, $f \in W^{2,\infty}(I, H)$. *Let there exist constants* K_4 *and* K_5, $\delta > 0$ *such that*

$$\left|\frac{A_t - A_0}{t} u_0\right| \leqq K_4 , \quad |A_t u_1| \leqq K_5 ,$$

for $0 \leqq t \leqq \delta$. *Then there exists a continuous function*

$$C_3 = C_3(\|u_0\|_0, \|u_1\|_0, \|f_0 - A_0 u_0\|_0, \|f\|_{2,\infty}, K_4, K_5)$$

independent of j *and* h, $C_3(0, 0, 0, 0, 0, 0) = 0$, *such that*

$$\|s_j\|_j^2 + |S_j|^2 \leqq C_3^2 .$$

Proof. By multiplying three equations of (1.4) with the indices j, $j - 1$ and $j - 2$ successively by 1, -2 and 1 and then by their adding and putting $v = S_j$ we obtain

$$((z_j, S_j))_j - 2((z_{j-1}, S_j))_{j-1} + ((z_{j-2}, S_j))_{j-2} +$$
$$+ (s_j - 2s_{j-1} + s_{j-2}, S_j) = (f_j - 2f_{j-1} + f_{j-2}, S_j) ,$$

$j = 1, \ldots, n$. Hence we deduce

$$((s_j, s_j - s_{j-1}))_j + (S_j - S_{j-1}, S_j) =$$
$$= h\left(\frac{f_j - 2f_{j-1} + f_{j-2}}{h^2}, S_j\right) + \frac{2}{h}((z_{j-1}, S_j))_{j-1} - \frac{2}{h}((z_{j-1}, S_j))_j -$$
$$- \frac{1}{h}((z_{j-2}, S_j))_{j-2} + \frac{1}{h}((z_{j-2}, S_j))_j .$$

which yields similarly as in two previous proofs

$$\|s_j\|_j^2 + |S_j|^2 \leqq \|s_{j-1}\|_{j-1}^2 + |S_{j-1}|^2 +$$
$$+ 2h\left(\frac{f_j - 2f_{j-1} + f_{j-2}}{h^2}, S_j\right) + \|s_{j-1}\|_j^2 - \|s_{j-1}\|_{j-1}^2 +$$

$$+\frac{4}{h}((z_{j-1}, S_j))_{j-1} - \frac{4}{h}((z_{j-1}, S_j))_j - \frac{2}{h}((z_{j-2}, S_j))_{j-2} + \frac{2}{h}((z_{j-2}, S_j))_j.$$

The remainder of the proof is again similar to the proof of Theorem 3.1.

EXISTENCE OF SOLUTION

Theorem 4.1. *Let $A_0 u_0 \in H$, $u_1 \in V$, $A_t \in W^{2,\infty}(I, \mathscr{A})$, $f \in W^{1,\infty}(I, H)$. Then there exists a unique function u with the following properties:*

a) *$A_t u(t) \in H$ for almost all $t \in I$,*

b) *the function u is Lipschitz - continuous from I into V and thus $u' L_\infty(I, V)$,*

c) *the function u' is Lipschitz - continuous from I into H and thus $u'' \in L_\infty(I, H)$,*

d) *$A_t u(t) + u''(t) = f(t)$ holds for a.e. $t \in I$,*

e) *$u(0) = u_0$, $u'(0) = u_1$,*

f) *if two functions u and v have the above defined properties, where u is a solution corresponding to initial conditions u_0, u_1 and the right hand side f, v is a solution corresponding to the initial conditions v_0, v_1 and the right hand side g, then the estimate*

$$\|u(t) - v(t)\|_t^2 + |u'(t) - v'(t)|^2 \leqq$$
$$\leqq [K_0 \|u_0 - v_0\|_0^2 + |u_1 - v_1|^2 + \|f - g\|_{L_2(I,H)}^2] \, e^{T\max(1,K_1)}$$

holds for $t \in I$.

Proof. In addition to the function ${}^n u(t)$ defined by (1.3) we define the following auxiliary functions:

$$
\begin{aligned}
{}^n\bar{u}(t) &= {}^n z_{j+1}, \quad \text{for} \quad t \in (t_j, t_{j+1}\rangle\,,\\
{}^n\bar{u}(0) &= z_0\,,\\
{}^n U(t) &= {}^n Z_j + \frac{t - t_j}{h}({}^n Z_{j+1} - {}^n Z_j)\,, \quad \text{for} \quad t \in \langle t_j, t_{j+1}\rangle\,,\\
{}^n\mathscr{U}(t) &= {}^n s_j + \frac{t - t_j}{h}({}^n s_{j+1} - {}^n s_j)\,, \quad \text{for} \quad t \in \langle t_j, t_{j+1}\rangle\,,\\
{}^n f(t) &= f({}^n t_j) + \frac{t - t_j}{h}(f({}^n t_{j+1}) - f({}^n t_j))\,, \quad \text{for} \quad t \in \langle t_j, t_{j+1}\rangle\,,\\
{}^n\bar{f}(t) &= f({}^n t_{j+1})\,, \quad \text{for} \quad t \in (t_j, t_{j+1}\rangle\,,\\
{}^n\bar{f}(0) &= f(0)\,,
\end{aligned}
$$

$j = 0, \ldots, n - 1$.

The operator functions ${}^n A_t$ and ${}^n\bar{A}_t$ are defined analogously. The bilinear forms corresponding to ${}^n A_t$ and ${}^n\bar{A}_t$ are denoted by ${}^n((\cdot,\cdot))_t$ and ${}^n((\cdot,\cdot))_t^*$. $((\cdot,\cdot))_t'$ and ${}^n((\cdot,\cdot))_t'$

denote the derivatives of the forms $((\cdot,\cdot))_t$ and ${}^n((\cdot,\cdot))_t$ with respect to t. The whole proof of Theorem 4.1 consists of several lemmas:

Lemma 4.1. *There exist functions $u \in C(I, V)$ and $U \in C(I, H)$ such that ${}^n u \to u$ uniformly in V and ${}^n U \to U$ uniformly in H. Further there exists a constant M independent of h, t such that*

$$\alpha\|u(t) - {}^n u(t)\|^2 + |U(t) - {}^n U(t)|^2 \leqq TMh\, e^{K_1 T}.$$

Proof. First we assert that

$$(4.1) \qquad {}^n\bar{A}_t\, {}^n\bar{u}(t) + {}^n U'(t) = {}^n\bar{f}(t),$$

for a.e. $t \in I$, which is equivalent to

$${}^n(({}^n\bar{u}(t), v))_t^* + ({}^n U'(t), v) = ({}^n\bar{f}(t), v)$$

for all $v \in V$.

After subtracting this equations with different indices m and n and putting $v = {}^n u'(t) - {}^m u'(t)$ we obtain

$$(4.2) \qquad {}^n(({}^n\bar{u}(t), {}^n u'(t) - {}^m u'(t)))_t^* - {}^m(({}^m\bar{u}(t), {}^n u'(t) - {}^m u'(t)))_t^* + \\ + ({}^n U'(t) - {}^m U'(t), {}^n u'(t) - {}^m u'(t)) = ({}^n\bar{f}(t) - {}^m\bar{f}(t), {}^n u'(t) - {}^m u'(t)).$$

By adding convenient expressions we have

$$\begin{aligned}
&{}^n(({}^n u(t) - {}^m u(t), {}^n u'(t) - {}^m u'(t)))_t + \tfrac{1}{2}\, {}^n(({}^n u(t) - {}^m u(t), {}^n u(t) - {}^m u(t)))_t' + \\
&+ ({}^n U'(t) - {}^m U'(t), {}^n U(t) - {}^m U(t)) = ({}^n\bar{f}(t) - {}^m\bar{f}(t), {}^n u'(t) - {}^m u'(t)) + \\
&+ {}^n(({}^n\bar{u}(t) - {}^m\bar{u}(t), {}^n u'(t) - {}^m u'(t)))_t - {}^n(({}^n\bar{u}(t) - {}^m\bar{u}(t), {}^n u'(t) - {}^m u'(t)))_t^* + \\
&+ {}^n(({}^n u(t) - {}^n\bar{u}(t) - ({}^m u(t) - {}^m\bar{u}(t)), {}^n u'(t) - {}^m u'(t)))_t + \\
&+ ({}^n U'(t) - {}^m U'(t), {}^n U(t) - {}^n u'(t) - {}^m U(t) + {}^m u'(t)) + \\
&+ {}^m(({}^m\bar{u}(t), {}^n u'(t) - {}^m u'(t))_t - {}^n(({}^m\bar{u}(t), {}^n u'(t) - {}^m u'(t)))_t^* + \\
&+ \tfrac{1}{2}\, {}^n(({}^n u(t) - {}^m u(t), {}^n u(t) - {}^m u(t)))_t'.
\end{aligned}$$

The left hand side is the derivative of

$$\tfrac{1}{2}\, {}^n\|{}^n u(t) - {}^m u(t)\|_t^2 + \tfrac{1}{2}|{}^n U(t) - {}^m U(t)|^2.$$

Applying the a priori estimates of Theorems 3.1 and 3.2 we obtain

$$\begin{aligned}
&({}^n\|{}^n u(t) - {}^m u(t)\|_t^2 + |{}^n U(t) - {}^m U(t)|^2)' \leqq \\
&\leqq 4\left(\frac{1}{n} + \frac{1}{m}\right)(L_1 C_1 + 3K_1 C_1 C_2 + 4C_2^2) + K_1\, {}^n\|{}^n u(t) - {}^m u(t)\|_t^2.
\end{aligned}$$

By the adding a nonnegative term $K_1|{}^nU(t) - {}^mU(t)|^2$ to the right hand side and integrating we obtain

$$(4.3) \qquad {}^n\|{}^nu(t) - {}^mu(t)\|_t^2 + |{}^nU(t) - {}^mU(t)|^2 \leqq$$

$$\leqq 4T\left(\frac{1}{n} + \frac{1}{m}\right)(L_1C_1 + 3K_1C_1C_2 + 4C_2^2) +$$

$$+ K_1\int_0^t ({}^n\|{}^nu(t) - {}^mu(t)\|_t^2 + |{}^nU(t) - {}^mU(t)|^2)\,\mathrm{d}t\,.$$

As a consequence of Gronwall's lemma and (2.3) we have

$$\|{}^nu(t) - {}^mu(t)\|_t^2 + |{}^nU(t) - {}^mU(t)|^2 \leqq T\left(\frac{1}{n} + \frac{1}{m}\right) M\,\mathrm{e}^{K_1T}\,,$$

where $M = 4(L_1C_1 + 3K_1C_1C_2 + 4C_2^2)$.

Hence, $\{{}^nu\}$ is a Cauchy sequence in $C(I, V)$ and $\{{}^nU\}$ is a Cauchy sequence in $C(I, H)$ and thus, as a consequence of the completeness of H and V, there exist limit functions $u \in C(I, V)$ and $U \in C(I, H)$.

The estimate from Lemma 4.1 can be obtained by the limiting process $m \to \infty$.

Lemma 4.2. *The function u is Lipschitz-continuous in V. The estimate*

$$\|u(t + h) - u(t)\| \leqq \frac{C_2H}{\sqrt{\alpha}}$$

holds for t, $t + h \in I$. The function U is Lipschitz-continuous in H. The estimate

$$|U(t + h) - U(t)| \leqq C_2h$$

holds for $t + h$, $t \in I$.

The proof of the first estimate can be obtain by the limiting process $n \to \infty$ in the following inequality

$$\|u(t + h) - u(t)\|_V \leqq \|u(t + h) - {}^nu(t + h)\|_V +$$

$$+ \|{}^nu(t + h) - {}^nu(t)\|_V + \|{}^nu(t) - u(t)\|_V \leqq 2T\frac{1}{n}\,\mathrm{e}^{K_1T} + \frac{C_2h}{\sqrt{\alpha}}\,.$$

The proof of the second estimate is similar.

Lemma 4.3. $u'(t) = U(t)$ holds for a.e. $t \in I$.

Proof. It is evident that

$${}^nu(t) = u_0 + \int_0^t {}^nu'(s)\,\mathrm{d}s$$

holds for every function ${}^n u$, $t \in I$, where the integral is considered in the space H. As a consequence of Lemma 3.2 and Lemma 4.1 the sequence $\{{}^n n'\}$ converges uniformly in H to the function U and thus we can pass to the limit with $n \to \infty$. From

$$u(t) = u_0 + \int_0^t U(s)\,\mathrm{d}s$$

we see that the functions u' and U coincide for all $t \in I$ (as a consequence of the continuity of the function U).

Lemma 4.4. *The sequence $\{{}^n U'\}$ converges weakly in $L_2(I, H)$ to the function $U'(=u'')$.*

Proof. Let w be an arbitrary function from $\mathscr{D}(I, H)$. Then using Lemma 4.1 we obtain

$$\int_0^T ({}^n U', w)\,\mathrm{d}t = -\int_0^T ({}^n U, w')\,\mathrm{d}t \xrightarrow{n\to\infty} -\int_0^T (U, w')\,\mathrm{d}t = \int_0^T (U', w)\,\mathrm{d}t\,.$$

The space $\mathscr{D}(I, H)$ is dense in $L_2(I, H)$ and the sequence $\{{}^n U'\}$ is bounded in $L_\infty(I, H)$ which enables us to prove that the above mentioned relation holds for every $w \in$ $\in L_2(I, H)$.

Lemma 4.5.

$$A_t\, u(t) + u''(t) = f(t),\ u(0) = u_0,\ u'(0) = u_1$$

takes place for almost all $t \in I$.

Proof. From (2.3) we deduce

$$\int_0^T {}^n(({}^n\bar{u}(t) - v(t), {}^n\bar{u}(t) - v(t)))_t\,\mathrm{d}t \geqq 0$$

for all $v \in L_2(I, V)$. From (4.1) we have

$$\int_0^T ({}^n\bar{f}(t) - {}^n U'(t), {}^n\bar{u}(t) - v(t))\,\mathrm{d}t - \int_0^T ((v(t), {}^n\bar{u}(t) - v(t)))_t\,\mathrm{d}t = 0\,.$$

Using the fact that ${}^n\bar{f} - {}^n U' \to f - u''$ weakly in $L_2(I, H)$ (Lemma 4.4) and ${}^n\bar{u} \to u$ in $L_2(I, V)$ we obtain

$$\int_0^T (f(t) - u''(t), u(t) - v(t))\,\mathrm{d}t - \int_0^T ((v(t), u(t) - v(t)))_t\,\mathrm{d}t = 0\,.$$

Now, we put $v = u - rw$ where w is an arbitrary function from $L_2(I, V)$ and r is a nonnegative constant:

$$\int_0^T (f(t) - u''(t), w(t))\,\mathrm{d}t - \int_0^T ((u(t) + r\,w(t), w(t)))_t \geqq 0\,.$$

After passing to the limit with $r \to 0$ we have (the inequality holds for both w and $-w$)

$$\int_0^T (f(t) - u''(t), w(t))\,\mathrm{d}t = \int_0^T ((u(t), w(t)))_t\,\mathrm{d}t$$

for all $w \in L_2(I, V)$ which is equivalent to

$$\int_0^T (f(t) - u''(t) - A_t\,u(t), w(t))\,\mathrm{d}t = 0$$

for all $w \in L_2(I, V)$ which yields

$$A_t\,u(t) + u''(t) = f(t)$$

for almost all $t \in I$.

Assertion a) follows from the fact that $u''(t) \in H$ and $f(t) \in H$. Now we shall prove assertion f). Since the equation considered is linear it suffices to prove that only for $v = 0$ and $g = 0$. In the equation

$$(u''(t), v) + ((u(t), v))_t = (f(t), v)\,,$$

which holds for all $v \in V$, we put $v = u'$ and by integration we obtain

$$\int_0^t [((u(s), u'(s)))_s + (u''(s), u'(s))]\,\mathrm{d}s = \int_0^t (f(s), u'(s))\,\mathrm{d}s\,, \tag{4.5}$$

which implies

$$\|u(t)\|_t^2 + |u'(t)|^2 = \|u(0)\|_0^2 + |u'(0)|^2 + \\ + \int_0^t ((u(s), u(s)))'_s\,\mathrm{d}s + 2\int_0^t (f(s), u'(s))\,\mathrm{d}s\,.$$

Then, owing to the assumption of the operator A, we have

$$\|u(t)\|_t^2 + |u'(t)|^2 \leqq \|u(0)\|_0^2 + |u'(0)|^2 + \|f\|_{L_2(I,H)}^2 + \\ + \max(1, K_1) \int_0^t [\|u(s)\|_s^2 + |u'(s)|^2]\,\mathrm{d}s$$

and the estimate f) follows from Gronwall's lemma.

PROPERTIES OF SOLUTIONS UNDER STRONGER ASSUMPTIONS

Theorem 5.1. *Let* $A_t \in W^{3,\infty}(I, \mathscr{A}), f \in W^{2,\infty}(I, H)$, $f(0) - A_0 u_0 \in V$. *Let there exist a derivative in* H *of the function* $A_t u_0$ *at the point* $t = 0$ *and let there exist*

$\delta > 0$ *such that the function* $A_t u_1$ *is bounded in* H *for* $0 \leqq t \leqq \delta$. *Then, in addition to the assertions of Theorem* 4.1, *the following assertions take place*:

a) *the function* u' *is Lipschitz-continuous from* I *into* V *and thus* $u'' \in L_\infty(I, V)$,

b) *the function* u'' *is Lipschitz-continuous from* I *into* H *and thus* $u''' \in L_\infty(I, H)$,

c) $(A_t u(t))' \in L_\infty(I, H)$,

d) *if in addition* $f(t) \in V$ *for all* $t \in I$ *then* $A_t u(t) \in V$ *for a.e.* $t \in I$,

e) *let* u *be a solution for the initial conditions* u_0, u_1 *and the right hand side* f *and let* v *be solution for the initial conditions* v_0, v_1 *and the right hand side* g; *then*

$$\|u'(t) - v'(t)\|_t^2 + |u''(t) - v''(t)|^2 \leqq$$
$$\leqq 2[(1 + K_1)\|u_1 - v_1\|_0^2 + K_1\|u_0 - v_0\|_0^2 +$$
$$+ |f(0) - g(0) - A_0 u_0 + A_0 v_0|^2 + \|f' - g'\|_{L_2(I,H)}^2 + TK_2 C_1^2] \, e^{T\max(1,3K_1+K_2)} +$$
$$+ 4K_1^2 C_1^2 \, e^{2T\max(1,3K_1+K_2)} .$$

The proof is similar to that of Theorem 4.1.

First, the existence of the functions U and $\mathscr{U}$ such that ${}^nU \to U$ uniformly in V and ${}^n\mathscr{U} \to \mathscr{U}$ uniformly in H and the existence of a constant N such that

$$\alpha\|U(t) - {}^nU(t)\|^2 + |\mathscr{U}(t) - {}^n\mathscr{U}(t)|^2 \leqq TNh \, e^{K_1 T} \tag{5.4}$$

can be proved similarly as in Lemma 4.1. From Theorems 3.2 and 3.3 we obtain $U' \in L_\infty(I, V)$ and $\mathscr{U}' \in L_\infty(I, H)$. $U' = \mathscr{U}$ follows from (5.4) analogously as in the proof of Lemma 4.4. If in addition we use Lemma 4.4, we can prove assertions a), b), c) d). The proof of assertion e) is similar to that of assertion f) of Theorem 4.1.

WEAK SOLUTION

Definition. $u \in C(I, V) \cap C^1(I, H)$ is *a weak solution of the equation* (2.5), (2.6) *if*

$$\int_0^T [((u, v))_t - (u', v')] \, dt = \int_0^T (f(t), v(t)) \, dt + (u_1, v(0)) , \tag{6.1}$$

$$U(0) = u_0 \tag{6.2}$$

takes place for all $v \in L_2(I, V) \cap W^{1,2}(I, H)$.

Theorem 6.1. *Let* $A_t \in W^{2,\infty}(I, \mathscr{A})$, $f \in L_2(I, H)$, $u_0 \in V$, $u_1 \in H$. *Then there exists a unique solution of the equation* (2.5), (2.6) *such that* $u'' \in L_2(I, V')$.
If in addition $f \in C(I, V')$ *then* $u'' \in C(I, V')$.
Estimate f) *from Theorem* 4.1 *takes place for two weak solutions.*

Proof. There exist sequences $\{u_1^n\} \subset V$, $\{f^n\} \subset W^{1,2}(I, H)$ and a sequence $\{u_0^n\}$, $A_0 u_0^n \in H$, such that $u_1^n \to u_1$ in H, $u_0^n \to u_0$ in V, $f^n \to f$ in $L_2(I, H)$. For every triplet u_0^n, u_1^n, f^n there exists a solution u^n (see Theorem 4.1). The equation (6.1), (6.2) holds for all such solutions u^n. Due to the estimate f) from Theorem 4.1 there exists a limit function $u \in C(I, V) \cap C^1(I, H)$ for which the equation (6.1), (6.2) holds.

Since $u \in C(I, V)$ we have $A_t u(t) \in C(I, V')$. $f \in L_2(I, H)$ implies $f \in L_2(I, V')$ and the assertion can be proved by passing to the limit in the formula

$$u^{n\prime}(t) = u_1^n + \int_0^t (f^n(s) - A_s u^n(s)) \, ds$$

in the space V', for $f^n \to f$ in $L_2(I, H)$, and thus also in $L_2(I, V')$, and it can be proved that $A_t u^n(t) \to A_t u(t)$ in $C(I, V')$.

The two remaining assertions of the theorem are evident. The proof of the uniqueness can by find for instance in [10].

Further we shall deal with the question of convergence of the Rothe sequence to a weak solution. Here we have some trouble with that we defined $f_j = f(t_j)$ which is not possible for a function only from $L_2(I, H)$. The definition of the value f_j must be therefore changed.

In the case $f \in C(I, H)$ we keep the original identity $f_j = f(t_j)$ while in the opposite case we define

$$f_j = h^{-1} \int_{(j-1)h}^{jh} f(t) \, dt \,, \quad j = 1, \ldots, n \,. \tag{6.3}$$

Now we can formulate a theorem:

Theorem 6.2. *Let the conditions of Theorem* 6.1. *be fulfilled. Then the sequence of Rothe approximations* $\{{}^n u\}$ *converges to the weak solution* u *of the equation* (2.5), (2.6) *in* $C(I, V)$ *and the sequence* $\{{}^n U\}$ *converges to* u' *in* $C(I, H)$.

Proof. Let $\{u_0^m, u_1^m, f^m\}$ be the sequence defined in the proof of Theorem 6.1 and let $\{u^m\}$ be the sequence of the corresponding solutions. In order to prove that ${}^n u \to u$ in $C(I, V)$ we use the following inequality:

$$\|u(t) - {}^n u(t)\| \leqq \|u(t) - u^m(t)\| + \|u^m(t) - {}^n u^m(t)\| + \|{}^n u^m(t) - {}^n u(t)\| \,. \tag{6.4}$$

We choose $\varepsilon > 0$. There exists $m_1 \in N$ such that

$$\|u - u^m\|_{C(I,V)} \leqq \tfrac{1}{3}\varepsilon \tag{6.5}$$

holds for $m \geqq m_1$. From Theorem 3.1 we deduce (f_j defined by (6.3)) the inequality

$$\alpha \|{}^n u^m(t) - {}^n u(t)\|^2 \leqq \frac{1}{1 - h} \left[(1 + K_1 h) \|u_0^m - u_0\|_0^2 + |u_1^m - u_1|^2 + \right.$$
$$\left. + \|f^m - f\|_{L_2(I,H)}^2 \right] e^{T \max(1, K_1)/(1-h)} \,.$$

Further there exists m_2 such that

$$(6.6) \qquad \|{}^n u^m - {}^n u\|_{C(I,V)} \leqq \tfrac{1}{3}\varepsilon$$

for all $m \geqq m_2$ and $n \in N$. Now, we choose a fixed $m \geqq \max(m_1, m_2)$. Owing to Theorem 4.1 there exists n_0 such that

$$(6.7) \qquad \|u^m - {}^n u^m\|_{C(I,V)} \leqq \tfrac{1}{3}\varepsilon$$

holds for all $n \geqq n_0$ (m fixed).

Owing to (6.5), (6.6), (6.7) we have

$$\|u(t) - {}^n u(t)\| \leqq \varepsilon$$

which proves ${}^n u \to u$ in $C(I, V)$.

The proof of the second part of the theorem is similar.

ESTIMATE OF ERROR OF ROTHE APPROXIMATION

Theorem 7.1. *Let the assumptions of Theorem* 5.1 *be fulfilled. Then there exist constants M_1, M_2, M_3 independent of n and t such that*

$$\alpha\|{}^n U(t) - u'(t)\|^2 + |{}^n \mathscr{U}(t) - u''(t)|^2 \leqq M_1 h ,$$
$$\|{}^n u(t) - u(t)\| \leqq M_2 h ,$$
$$|{}^n U(t) - u'(t)| \leqq M_3 h$$

takes place for $t \in I$.

Proof. The first estimate immediately follows from (5.4) and the proof of Theorem 5.1. For the proof of the two remaining estimates we consider two partitions: with n and $2n$ mesh points. We denote

$$p_j = {}^{2n}z_{2j} - {}^n z_j , \quad j = -1, \ldots, n ,$$

$$P_j = \frac{p_j - p_{j-1}}{h} , \quad j = 0, \ldots, n ,$$

$$q_j = \frac{P_j - P_{j-1}}{h} , \quad j = 1, \ldots, n .$$

From the equation

$$A_j p_j + \frac{1}{h^2}\left(-{}^n z_j + 2\,{}^n z_{j-1} - {}^n z_{j-2} + 4\,{}^{2n}z_{2j} - 8\,{}^{2n}z_{2j-1} + 4\,{}^{2n}z_{j-2}\right) = 0$$

we obtain

$$A_j p_j + q_j = g_j ,$$

where

$$g_j = -\frac{h}{8}(3\,{}^{2n}S_{2j} + {}^{2n}S_{2j-1}).$$

From Theorem 3.1 we obtain

(7.1)
$$\alpha\|p_j\|^2 + |P_j|^2 \leqq$$
$$\leqq \frac{(1 + hK_1)\,\|p_0\|_0^2 + |P_0|^2 + T\max|g_j|^2}{1 - h}\,\mathrm{e}^{T\max(1,K_1)/(1-h)}$$

for all $j = 1, \ldots, n$.

Owing to Theoıem 3.3 and the fact that $p_0 = 0$, $P_0 = -(h/4)\,(f(0) - A_0u_0)$ there exists a constant M_2 such that

$$\|p_j\| \leqq M_2^*\frac{h}{2}$$

holds for $j = 1, \ldots, n$, $h < h_0$. By repeating this process for two partitions with $2n$ and $4n$ mesh points we obtain

$$\|{}^{4n}z_{4j} - {}^{2n}z_{2j}\| \leqq M_2^*\frac{h}{4}$$

for $j = 1, \ldots, n$. By another repeatition we obtain

$$\|{}^{2^kn}z_{2^kj} - {}^{n}z_j\| \leqq M_2^*h$$

and passing to the limit with $k \to \infty$ we obtain

$$\|u(t_j) - {}^nu(t_j)\| = \|u(t_j) - {}^nz_j\| \leqq M_2^*h$$

for $j = 1, \ldots, n$. In order to estimate the error not only at the mesh points we use Lemma 4.2:

$$\|u(t) - {}^nu(t)\| \leqq \frac{C_2}{\sqrt{\alpha}}\,h + M_2^*h$$

for all $t \in I$. If we put $M_2 = C_2/\sqrt{\alpha} + M_2^*$ the proof is complete.

The proof of the third estimate is similar.

The following theorem is a direct consequence of Lemmas 4.1 and 4.3.

Theorem 7.2. *Let the assumptions of Theorem* 4.1 *be fulfilled. Then the estimate*

$$\alpha\|u(t) - {}^nu(t)\|^2 + |u'(t) - {}^nU(t)|^2 \leqq TMh\,\mathrm{e}^{K_1T}$$

holds for $t \in I$.

Remark. We can obtain a certain improvement of the Rothe approximation ${}^nu$ if we put

$$z_0 = u_0\,, \quad z_{-1} = u_0 - hu_1 + \frac{h^2}{2}\left(f(0) - A_0u_0\right)$$

because then the estimate (7.1) reduces to

$$\alpha\|p_j\|^2 + |P_j|^2 \leqq \frac{T}{1-h}\max|g_j|^2\, \mathrm{e}^{T(\max(1,K_1)/1-h)}\,.$$

A certain disadvantage of this consists in the fact that the assumptions on $f(0) - A_0u_0$ must be stronger (the same as on u_1).

In the end we shall deal with the case when the equations (1.4) are solved only approximately, i.e. we consider the system of equations

$$A_j\bar{z}_j + \frac{\bar{z}_j - 2\bar{z}_{j-1} + \bar{z}_{j-2}}{h^2} = f_j + R_j\,, \quad j = 1, \ldots, n\,, \tag{7.2}$$

$$\bar{z}_0 = z_0\,, \quad \bar{z}_{-1} = z_{-1} \tag{7.3}$$

where R_j expresses the error of the solution of each equation of the system (1.4). Then the following theorem is an immediate consequence of Theorem 3.1:

Theorem 7.3. *Let z_j be solution of* (1.4), (1.5) *and let $\bar{z}_j$ be solutions of* (7.2), (7.3). *Then*

$$\alpha\|z_j - \bar{z}_j\|^2 + |Z_j - \bar{Z}_j|^2 \leqq \frac{T}{1-h}\max|R_j|^2\, \mathrm{e}^{T\max(1,K_1)/(1-h)}$$

holds for $j = 1, \ldots, n$ ($\bar{Z}_j$ defined analogously as Z_j).

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Souhrn

ŘEŠENÍ ABSTRAKTNÍCH HYPERBOLICKÝCH ROVNIC ROTHEHO METODOU

MILAN PULTAR

V práci jsou řešeny abstraktní hyperbolické rovnice, ve kterých vystupuje eliptický operátor závislý na čase, pomocí tzv. Rotheho metody, tj. metody diskretizace dané úlohy v čase. Je zde dokázána existence a jednoznačnost řešení a některé jeho vlastnosti v závislosti na různých předpokladech, kterým vyhovují dané rovnice. Dále jsou uvedeny některé numerické aspekty této metody.

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