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THE FINITE ELEMENT SOLUTION OF SECOND ORDER ELLIPTIC PROBLEMS WITH THE NEWTON BOUNDARY CONDITION

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The majority of elliptic model problems for which the convergence of the finite element method has been analysed is restricted to homogeneous Dirichlet problems (see e.g. [1], [2], [6], [10], [11]). There are only a few exceptions when other boundary conditions have been treated (see e.g. [8], [9], [12]). Ženíšek [12] studied the 2-nd order 2-dimensional elliptic problem with nonhomogeneous Dirichlet, Neumann as well as Newton boundary conditions and analysed the convergence in the H^1 norm.

In this paper the convergence in both the H^1 and L_2 norms for the 2-nd order elliptic problem in the n-dimensional Euclidean space ($n \geqq 2$) with the Newton boundary condition is analysed. The discretisation is carried out by means of k-regular simplicial isoparametric finite elements (see [1], [2]). In Section 1 the k-regular triangulation is introduced and some properties of the finite element space are established. In Section 2 the problem and its approximate solution are defined and in Section 3 the convergence results are obtained.

The technique of proofs used in this paper is similar to that of Ciarlet and Raviart [2] and Nedoma [5], [6].

1. CONSTRUCTION OF THE FINITE ELEMENT SPACE. NOTATION

We consider the k-regular family $\{K\}_h$ of simplicial isoparametric finite elements K introduced by Ciarlet and Raviart [2]. First of all, we are given

(a) A set $\hat{\Sigma}_K = \bigcup_{i=1}^{\hat{N}_K} \{\hat{a}_{i,K}\}$ of $\hat{N}_K$ distinct points from R^n such that its convex hull $\hat{K}$ is a unit n-simplex.

(b) A finite dimensional space $\hat{P}_K$ of functions defined on $\hat{K}$ with $\dim \hat{P}_K = \hat{N}_K$ such that $\hat{\Sigma}_K$ is $\hat{P}_K$-unisolvent. We suppose $\hat{P}_K \subset C^{k+1}(\hat{K})$, $\hat{P}_K \supset \hat{P}_n(1)$. Here for any integers $r \geqq 0$, $s \geqq 1$, $\hat{P}_s(r)$ is the space of all polynomials of degree $\leqq r$ in s variables $\hat{x}_1, \ldots, \hat{x}_s$.

(c) A set $\Sigma_K = \bigcup_{i=1}^{\hat{N}_K} \{a_{i,K}\}$ of $\hat{N}_K$ distinct points from R^n.

Then the simplicial finite element $K \in \{K\}_h$ is the image of the set $\hat{K}$ through the unique mapping $F_K : \hat{K} \to R^n$ which satisfies

$$\hat{F}_K \in (\hat{P}_K)^n\,, \quad \hat{F}_K(\hat{a}_{i,K}) = a_{i,K} \quad \forall \hat{a}_{i,K} \subset \hat{\Sigma}_K\,.$$

We suppose

(d) For all h, the mapping F_K is a C^{k+1} – diffeomorphism and there exist constants c_i, $i = 0, \ldots, k + 1$, independent of h, such that for all h:

$$\sup_{\hat{x} \in \hat{K}} \max_{|\alpha| = i} |D^\alpha F_K(\hat{x})| \leqq c_i h^i\,, \quad 1 \leqq i \leqq k + 1\,, \tag{1.1}$$

$$0 < c_0 h^n \leqq |J_K(\hat{x})|\,, \tag{1.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $J_K(\hat{x})$ is the Jacobian of the mapping F_K at the point $\hat{x} \in \hat{K}$.

From (1.1) we immediately obtain

$$|J_K(\hat{x})| \leqq c h^n\,, \tag{1.3}$$

where c is a constant independent of h. Every element K is associated with the finite dimensional space $P_K(\dim P_K = \hat{N}_K)$ of functions

$$P_K = \{p_K \mid K \to R,\ p_K = \hat{p}_K(F_K^{-1}),\ \forall \hat{p}_K \in \hat{P}_K\}\,. \tag{1.4}$$

The K-interpolate $\pi_K u$ of a given function $u : K \to R$ is the unique function which satisfies

$$\pi_K u \in P_K\,, \quad \pi_K u(a_{i,K}) = u(a_{i,K}) \quad \forall a_{i,K} \in \Sigma_K\,. \tag{1.5}$$

For a k-regular family $\{K\}_h$ of finite elements the following interpolation theorem is true (see Ciarlet and Raviart [2], Theorem 2, p. 429).

Lemma 1.1. (*the interpolation theorem*). *Let a k-regular family $\{K\}_h$ of simplicial elements such that $\hat{P}_n(k) \subset \hat{P}_K$ be given. Then there exists a constant c independent of h such that for any integers i, s with $0 \leqq i \leqq s \leqq k + 1$, for any $K \in \{K\}_h$ and for any function $u \in W^{s,p}(K)$ with $p \geqq 1$, $ps > n$, we have*

$$|u - \pi_K u|_{i,p,K} \leqq c h^{s-i} \|u\|_{s,p,K}\,. \tag{1.6}$$ □

We are using the usual notation:

$W^{s,p}(A) = \{u \mid D^\alpha u \in L_p(A),\ \forall |\alpha| \leqq s\}$ is the Sobolev space with the norm defined for $1 \leqq p < \infty$ by

$$\|v\|_{s,p,A} = \left(\sum_{i=0}^{s} |v|^p_{i,p,A}\right)^{1/p} \quad \text{where} \quad |v|_{i,p,A} = \left(\sum_{|\alpha| = i} \int_A |v|^p \, dx\right)^{1/p},$$

and for $p = \infty$ by

$$\|v\|_{s,\infty,A} = \max_{0 \leq i \leq s} |v|_{i,\infty,A} \quad \text{where} \quad |v|_{i,\infty,A} = \max_{|\alpha| = i} \operatorname{ess\,sup}_{x \in A} |D^\alpha v(x)| .$$

Evidently $W^{0,p}(A) = L_p(A)$.

As usual we denote $H^s(A) = W^{s,2}(A)$, $\|\cdot\|_{s,A} = \|\cdot\|_{s,2,A}$, $|\cdot|_{s,A} = |\cdot|_{s,2,A}$. The scalar product in the space $H^s(A)$ is denoted by $(\cdot, \cdot)_{s,A}$.

Now we define the k-regular family $\{S\}_h$ of surface simplicial isoparametric finite elements S induced by the family $\{K\}_h$. We introduce the notation $\hat{x} = (\hat{x}_1, \dots, \hat{x}_{n-1}, \hat{x}_n) = (\hat{x}', \hat{x}_n)$. Let $\hat{S}$ be one of the $n + 1$ surface $(n - 1)$-simplexes of the unit simplex $\hat{K}$. Particularly, we will consider the simplex $\hat{S} = \hat{K} \cap \{\hat{x}_n = 0\}$. We can suppose that $\hat{a}_{i,K} \in \hat{S}$ for $i = 1, \dots, \hat{N}_S$. If we denote $\hat{a}'_{i,S} = \hat{a}'_{i,K}$, $i = 1, \dots, \hat{N}_S$, we define $\hat{\Sigma}_S = \bigcup_{i=1}^{\hat{N}_S} \{\hat{a}'_{i,S}\}$. Let us denote by $\hat{P}_S$ the restriction of $\hat{P}_K$ to $\hat{S}$. Evidently $\hat{P}_S \subset C^{k+1}(\hat{S})$, $\hat{P}_S \supset \hat{P}_{n-1}(1)$. Further we denote by $\hat{F}_S$ the restriction of $\hat{F}_K$ to $\hat{S}$, so that $F_S(\hat{x}') = F_K(\hat{x}', 0)$ for $\hat{x}' \in \hat{S}$. Let us suppose that the set $\hat{\Sigma}_S$ is $\hat{P}_S$-unisolvent. Then we define the surface simplicial finite element S as the image of the set $\hat{S}$ through the mapping F_S. We define $a_{i,S} = F_S(\hat{a}'_{i,S})$, $\Sigma_S = \bigcup_{i=1}^{\hat{N}_S} \{a_{i,S}\}$. From (1.1) it follows that for all h

$$\sup_{\hat{x}' \in \hat{S}} \max_{|\alpha'| = i} |D^{\alpha'} F_S(\hat{x}')| \leq c_i h^i, \quad i = 1, \dots, k + 1 , \tag{1.7}$$

where $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$, $|\alpha'| = \alpha_1 + \dots + \alpha_{n-1}$.

For $\hat{x} \in \hat{S}$ we define the function

$$J_S(\hat{x}) = \frac{\mathrm{d}S(\hat{x})}{\mathrm{d}\hat{S}(\hat{x})} , \tag{1.8}$$

where $\mathrm{d}S(\hat{x})$ and $\mathrm{d}\hat{S}(\hat{x})$ are elements of the surfaces S and $\hat{S}$, respectively. Evidently $\mathrm{d}\hat{S}(\hat{x}) = \mathrm{d}\hat{x}'$. In the sequel we will denote $J_S(\hat{x}', 0)$ by $J_S(\hat{x}')$. Since by the definition of $\mathrm{d}S(\hat{x})$,

$$\mathrm{d}S(\hat{x}) = (\sum_{i=1}^{n} |J_K^{(i,n)}(\hat{x}', 0)|^2)^{1/2} \, \mathrm{d}\hat{x}' ,$$

we obtain

$$J_S(\hat{x}') = (\sum_{i=1}^{n} |J_K^{(i,n)}(\hat{x}', 0)|^2)^{1/2} , \tag{1.9}$$

where $J_K^{(i,n)}$ are the cofactors of J_K. From (1.9) and (1.1) we get

$$|J_S(\hat{x}')| \leq c h^{n-1} \tag{1.10}$$

for a constant c independent of h. Moreover, there exists a constant c independent of h such that

$$c h^{n-1} \leq |J_S(\hat{x}')| . \tag{1.11}$$

Let us prove (1.11). Suppose the contrary. Then for every $\varepsilon_m > 0$ there exist $\hat{x}'_m \in \hat{S}$ and $h_m > 0$ such that $|J_S(\hat{x}'_m)| < \varepsilon_m h_m^{n-1}$. As

$$J_K(\hat{x}', 0) = \sum_{i=1}^{n} J_K^{(i,n)}(\hat{x}', 0) \frac{\partial F_{Ki}}{\partial \hat{x}_n}(\hat{x}', 0) \quad \text{and} \quad \left|\frac{\partial F_{Ki}}{\partial \hat{x}_n}(\hat{x}', 0)\right| \leqq c_1 h_m$$

by (1.1), we have $|J_K(\hat{x}', 0)| < nc_1 \varepsilon_m h_m^n$, which contradicts (1.2).

Every element S is associated with the finite dimensional space P_S (dim $P_S = \hat{N}_S$) of functions

$$P_S = \{p_S \mid p_S = \hat{p}_S(F_S^{-1}),\ \forall \hat{p}_S \in \hat{P}_S\}\,. \tag{1.12}$$

The only assumption we need in deriving the surface element S from the element K is the assumption that the set $\hat{\Sigma}_S$ is $\hat{P}_S$-unisolvent or, which is the same, that the geometrical shape of the element S is completely determined by the set Σ_S.

The S-interpolate $\pi_S u$ of a given function $u : S \to R$ is the unique function which satisfies

$$\pi_S u \in P_S\,, \quad \pi_S\, u(a_{i,S}) = u(a_{i,S}) \quad \forall a_{i,S} \in \Sigma_S\,. \tag{1.13}$$

From (1.9), (1.11) it follows that we can and will suppose

$$ch^{n-1} \leqq |J_K^{(n,n)}(\hat{x}', 0)|\,. \tag{1.14}$$

We denote $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$, $F_S = (F_{S1}, \ldots, F_{Sn-1}, F_{Sn}) = (F'_S, F_{Sn})$. Then $J_K^{(n,n)}$ is the Jacobian of the mapping F'_S. From (1.7) we get

$$\sup_{\hat{x}' \in \hat{S}} \max_{|\alpha'| = i} |D^{\alpha'} F'_S(\hat{x}')| \leqq c_i h^i\,, \quad i = 1, \ldots, k + 1\,. \tag{1.15}$$

We define $S' = F'_S(\hat{S})$, $J'_S(\hat{x}') = J_K^{(n,n)}(\hat{x}', 0)$. S' is obviously the projection of S into the hyperplane $x_n = 0$. From (1.1) and (1.14) we obtain that there exists a constant c independent of h such that

$$c^{-1} h^{n-1} \leqq |J'_S(\hat{x}')| \leqq ch^{n-1}\,, \quad \hat{x}' \in \hat{S}\,. \tag{1.16}$$

We denote $a'_{i,S} = F'_S(\hat{a}'_{i,S})$, $\Sigma'_S = \bigcup_{i=1}^{\hat{N}_S} \{a'_{i,S}\}$. We associate the element S' with the finite dimensional space P'_S(dim $P'_S = \hat{N}_S$) of functions

$$P'_S = \{p'_S \mid S' \to R,\ p'_S = \hat{p}_S(F'^{-1}_S),\ \forall \hat{p}_S \in \hat{P}_S\}\,. \tag{1.17}$$

From the definition of S', P'_S, Σ'_S we deduce that S' is the k-regular simplicial isoparametric finite element (in $n - 1$ dimensions). The S'-interpolate $\pi'_S u$ of a given function $u : S' \to R$ is the unique function which satisfies

$$\pi'_S u \in P'_S\,, \quad \pi'_S u(a'_{i,S}) = u(a'_{i,S}) \quad \forall a'_{i,S} \in \Sigma'_S\,. \tag{1.18}$$

From Lemma 1.1 we immediately obtain

Lemma 1.2 (*the interpolation theorem*). *Let a k-regular family* $\{S\}_h$ *of surface simplicial elements such that* $\hat{P}_{n-1}(k) \subset \hat{P}_S$ *be given. Then there exists a constant c independent of h such that for any integers i, s with* $0 \leqq i \leqq s \leqq k+1$, *for any* $S \in \{S\}_h$ *and for any function* $u \in W^{s,p}(S')$ *with* $p \geqq 1$, $ps > n-1$, *we have*

$$|u - \pi'_S u|_{i,p,S'} \leqq ch^{s-i} \|u\|_{s,p,S'} . \tag{1.19}$$ □

We introduce the function

$$\psi_S(x') = F_{Sn}(F_S'^{-1}(x')) , \quad x' \in S' . \tag{1.20}$$

Differentiating (1.20) we get

$$\frac{\partial \psi_S(x')}{\partial x_i} = \sum_{j=1}^{n-1} \left(J_S'^{(i,j)}(\hat{x}') \frac{\partial F_{Sn}}{\partial \hat{x}_j}(\hat{x}') \right) \Big/ J'_S(\hat{x}') , \quad i = 1, \ldots, n-1 ,$$

where $J_S'^{(i,j)}$ is the cofactor of J'_S (for $n = 2$ we take $J_S'^{(1,1)} = 1$). Repeating the differentiation and using (1.15), (1.16) we get

$$\sup_{x' \in S'} \max_{|\alpha'| = i} |D^{\alpha'} \psi_S(x')| \leqq c , \quad i = 1, \ldots, k+1 , \tag{1.21}$$

where the constant c does not depend on h. From the definition of the function ψ_S it follows that $S = \psi_S(S')$.

For any function u defined on S we denote

$$_{\psi_S}u(x') = u(x', \psi_S(x')) , \quad x' \in S' . \tag{1.22}$$

From the definitions (1.13) and (1.18) of interpolants $\pi_S u$ and $\pi'_S u$ and the definition (1.20) of the function ψ_S we easily obtain for any function u defined on S

$$_{\psi_S}(\pi_S u) = \pi'_S(_{\psi_S}u) . \tag{1.23}$$

In addition, for $n \geqq 3$ we introduce the k-regular family $\{H\}_h$ of simplicial isoparametric edges H induced by the family $\{S\}_h$. We denote by $\hat{H}$ one of the n surface $(n-2)$-simplexes of the simplex $\hat{S}$ and by $\hat{P}_H$, $\hat{\Sigma}_H$, F_H the restrictions of $\hat{P}_S$, $\hat{\Sigma}_S$, F_S to $\hat{H}$. We suppose that the set $\hat{\Sigma}_H$ is $\hat{P}_H$-unisolvent. Then the simplicial edge H is the image of the set $\hat{H}$ through the mapping F_H. The projection H' of the edge H into the hyperplane $x_n = 0$ is obviously the k-regular surface simplicial element (in $n-1$ dimensions). If we define for $\hat{x}' \in \hat{H}$ the function

$$J'_H(\hat{x}') = \frac{\mathrm{d}H'(\hat{x}')}{\mathrm{d}\hat{H}(\hat{x}')} , \tag{1.24}$$

where $\mathrm{d}H'(\hat{x}')$ and $\mathrm{d}\hat{H}(\hat{x}')$ are elements of the edge H' and $\hat{H}$, respectively, then

$$c^{-1}h^{n-2} \leqq |J'_H(\hat{x}')| \leqq ch^{n-2} , \quad \hat{x}' \in H' , \tag{1.25}$$

with a constant c independent of h (compare with (1.10), (1.11)).

In the sequel, we mean by Ω a bounded domain in R^n with a sufficiently smooth boundary Γ. Following the usual definition of a smooth boundary, see e.g. [3], pp. 269–270, we can suppose that there exist R coordinate systems $\{x^r\} = \{(x_1^r, \ldots \ldots, x_n^r)\}$ such that every point of the boundary Γ can be described at least in one of this coordinate systems by an equation

$$x_n^r = \varphi^r(x'^r), \quad x'^r \in \Delta^r. \tag{1.26a}$$

Here $x'^r = (x_1^r, \ldots, x_{n-1}^r)$, Δ^r is an $(n-1)$-dimensional closed cube and φ^r is a smooth function on Δ^r. Following the way similar to that of Ciarlet and Raviart [2] we define a k-regular triangulation τ_h of Ω. Let Ω_h be the union of a finite number of simplicial elements $K \in \{K\}_h$. We denote by Γ_h the boundary of Ω_h. We say that a triangulation τ_h of Ω is k-regular if:

(a) The points $a_{i,K}$ of all elements $K \in \Omega_h$ belong to $\bar{\Omega}$, i.e. $\Sigma_K \in \bar{\Omega}$ $\forall K \in \Omega_h$.

(b) The geometric shape of any surface element S of any element $K \in \Omega_h$ is completely determined by those points $a_{i,K}$ which belong to S; this means that the surface elements S of all elements $K \in \Omega_h$ belong to a k-regular family $\{S\}_h$ of surface simplicial isoparametric finite elements.

(c) The points $a_{i,S}$ of all elements $S \in \Gamma_h$ belong to Γ, i.e. $\Sigma_S \in \Gamma$ $\forall S \in \Gamma_h$.

(d) For $n \geqq 3$ the geometric shape of any edge H of any surface element $S \in \Gamma_h$ is completely determined by those points $a_{i,S}$ which belong to H; this means that the edges H of all surface elements $S \in \Gamma_h$ belong to a k-regular family $\{H\}_h$ of simplicial isoparametric edges.

Let us denote $\Gamma^r = \{x \mid x = (x'^r, \varphi^r(x'^r)), x'^r \in \Delta^r\}$, $\Gamma_h^r = \{x \mid x = (x'^r, x_n^r), x'^r \in \Delta^r, x \in S \subset \Gamma_h$ such that $\Sigma_S \cap \Gamma^r \neq \{\emptyset\}\}$, see Fig. 1.1.

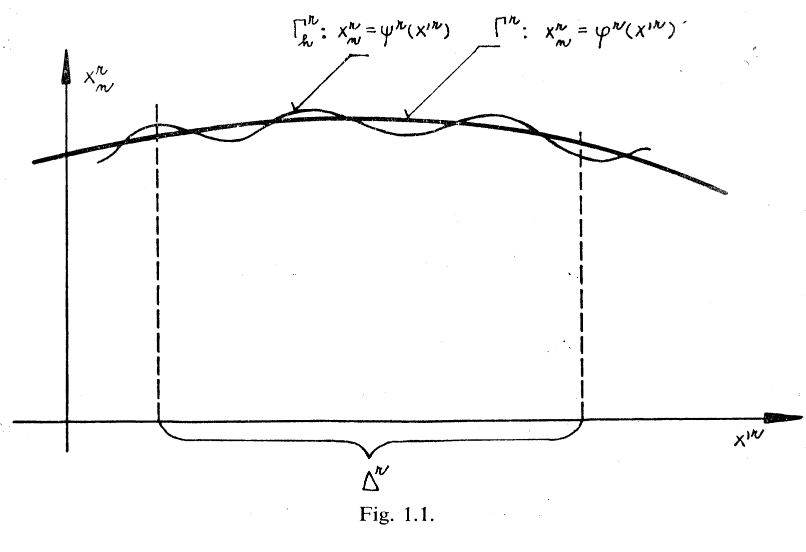

Fig. 1.1.

We can suppose that every element $S \in \Gamma_h$ belongs to any set Γ_h^r. If $S \cap \Gamma_h^r \neq \{\emptyset\}$ we denote by S'^r the projection of the element S into the hyperplane $x_n^r = 0$. Further we denote $\Gamma_S^r = \{x \mid x \in \Gamma^r,\ x'^r \in S'^r\}$, see Fig. 1.2. Due to the smoothness of the

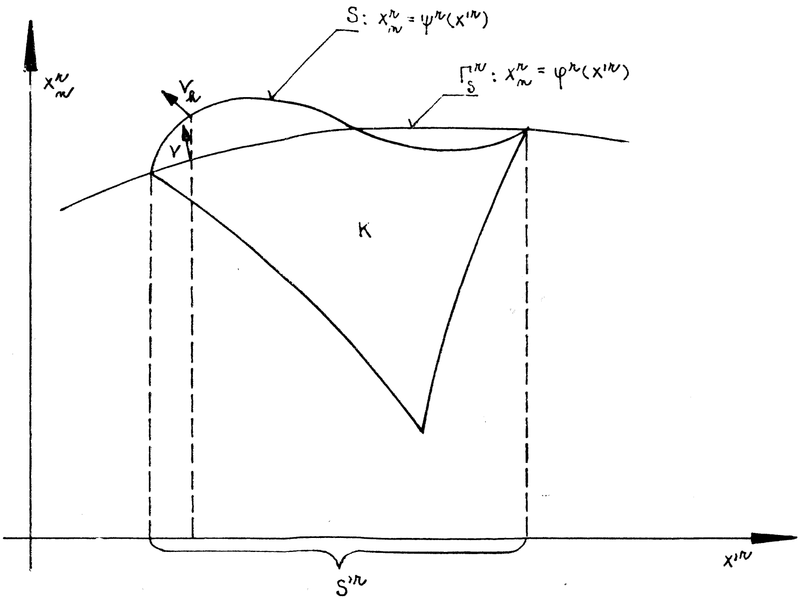

Fig. 1.2.

boundary Γ^r and to the assumption (c) in the definition of a k-regular triangulation there exists a function ψ^r defined on Δ^r such that Γ_h^r can be described for all h sufficiently small by the equation

(1.26b) $$x_n^r = \psi^r(x'^r), \quad x'^r \in \Delta^r,$$

see Fig. 1.1. Moreover, $\psi^r(x'^r) = \psi_S(x'^r)$ for $x'^r \in \Delta^r$, where ψ_S was defined by (1.20). For a function u defined on Γ^r and Γ_h^r we denote

$$_{\varphi^r}u(x'^r) = u(x'^r, \varphi^r(x'^r)), \quad x'^r \in \Delta^r$$

and

$$_{\psi^r}u(x'^r) = u(x'^r, \psi^r(x'^r)), \quad x'^r \in \Delta^r,$$

respectively. If it does not lead to an ambiguity we will drop the index r.

A given k-regular triangulation τ_h is associated with the finite dimensional space V_h of functions defined by

(1.27) $$V_h = \{v \mid v \in H^1(\Omega_h),\ v_K \in P_K,\ \forall K \in \Omega_h\},$$

where v_K is the restriction of the function v to the set K. From the definition of the k-regular triangulation it follows that the functions from the space V_h are Lipschitz continuous in $\bar{\Omega}_h$, i.e. $v \in V_h \Rightarrow v \in C^{0,1}(\bar{\Omega}_h)$.

Next, with any function v defined on $\bar{\Omega}$ we may associate its unique interpolate $\pi_\Omega v$ which satisfies

$$\pi_\Omega v = \pi_K v \quad \forall K \in \Omega_h .$$

Similarly, with any function v defined on Γ we may associate its unique interpolate $\pi_\Gamma v$ which satisfies

$$\pi_\Gamma v = \pi_S v \quad \forall S \in \Gamma_h .$$

Let $W^{s,p}(\Gamma)$ denote the Sobolev space of functions defined on the boundary Γ with the norm

$$\|v\|_{s,p,\Gamma} = \left(\sum_{r=1}^{R} \|_{\varphi^r} v\|_{s,p,\Delta^r}^p\right)^{1/p} \quad \text{for} \quad p < \infty ,$$

$$\|v\|_{s,\infty,\Gamma} = \max_{r=1,\ldots,R} \|_{\varphi^r} v\|_{s,\infty,\Delta^r} ,$$

see Kufner [3], p. 327. As usual we denote $H^s(\Gamma) = W^{s,2}(\Gamma)$, $\|\cdot\|_{s,\Gamma} = \|\cdot\|_{s,2,\Gamma}$. As $\Omega_h \in \mathscr{C}^{0,1}$ (for the definition of domains of this type see e.g. Kufner [3], pp. 269–270), we can define spaces $H^i(\Gamma_h)$, $i = 0, 1$.

For functions $v \in H^i(S)$ and $w \in H^i(\Gamma_h)$, $i = 0, 1$, we introduce the norms $\|v\|_{i,S} = \|_\psi v\|_{i,S'}$ and $\|w\|_{i,\Gamma_h} = \left(\sum_{S \in \Gamma_h} \|_\psi w\|_{i,S'}^2\right)^{1/2}$, respectively. We denote

$$(v, w)_{0,S} = \int_S vw \, \mathrm{d}S , \quad (v, w)_{0,\Gamma_h} = \int_{\Gamma_h} vw \, \mathrm{d}\Gamma_h .$$

Let $\tilde{\Omega}$ be a sufficiently smooth bounded domain containing Ω and Ω_h for all sufficiently small h.

In our paper we will suppose that $\hat{P}_K = \hat{P}_n(k)$ so that $\hat{P}_S = P_{n-1}(k)$ and $\hat{P}_H = \hat{P}_{n-2}(k)$. This restriction is not essential. It enables us to give simpler proofs.

Let $v(x)$ be any function defined on the element K. Then the function $v(F_K(\hat{x}))$ is defined on $\hat{K}$. We will denote it $\hat{v}(\hat{x})$. In an analogous way we denote $\hat{v}(\hat{x}') = v(F_S(\hat{x}'))$ for a function v defined on S and $\hat{v}(\hat{x}') = v(F'_S(\hat{x}'))$ for a function v defined on S'.

In the sequel the constants independent of h will be denoted by c. The notation is generic, i.e. c will not denote the same constant at any two places.

Now we introduce some lemmas.

Lemma 1.3. *Let a k-regular triangulation τ_h of the set Ω be given. Let S be any surface element belonging to Γ_h. Then for any integers i, s with $0 \leqq i \leqq s \leqq k + 1$ and for any real $p \geqq 1$ such that $ps > n - 1$ we have*

$$(1.28) \qquad |\varphi - \psi|_{i,p,S'} \leqq ch^{s-i} \|\varphi\|_{s,p,S'} .$$

The proof follows from Lemma 1.2 as $\psi = \pi'_S \varphi$. □

We denote by $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{v}_h = (v_{h1}, \ldots, v_{hn})$ the unit vectors of the outward normals to the boundary Γ and Γ_h, respectively. Then (see Fig. 1.2)

$$(1.29)\qquad \begin{aligned} {}_\varphi v_j &= -\frac{\partial\varphi}{\partial x_j}\left(1 + |\mathbf{grad}\,\varphi|^2\right)^{-1/2}, \quad {}_\psi v_{hj} = -\frac{\partial\psi}{\partial x_j}\left(1 + |\mathbf{grad}\,\psi|^2\right)^{-1/2}, \\ &\qquad\qquad j = 1, \ldots, n-1, \\ {}_\varphi v_n &= \left(1 + |\mathbf{grad}\,\varphi|^2\right)^{-1/2}, \qquad {}_\psi v_{hn} = \left(1 + |\mathbf{grad}\,\psi|^2\right)^{-1/2}. \end{aligned}$$

From this definition and Lemma 1.3 we easily obtain

Lemma 1.4. *Let a k-regular triangulation τ_h of the domain Ω be given. Let S be any surface element belonging to Γ_h. Then for any integers i, s with $0 \leqq i \leqq s \leqq k + 1$ and for any real $p \geqq 1$ such that $ps > n - 1$ we have*

$$(1.30)\qquad \|{}_\varphi v_j - {}_\psi v_{hj}\|_{0,p,S'} \leqq ch^k \|\varphi\|_{k+1,p,S'}, \quad j = 1, \ldots, n. \qquad \square$$

Lemma 1.5. (*the trace theorem*). *Let $\Omega \in \mathscr{C}^{1,1}$. Then for any function $v \in H^1(\Omega_h)$ and for all h sufficiently small we have*

$$(1.31)\qquad \|v\|_{0,\Gamma_h} \leqq c\|v\|_{1,\Omega_h}.$$

The proof follows from the proof of Theorem 1.2, p. 15 in [4]. □

Lemma 1.6. *Let a k-regular triangulation τ_h of the domain Ω be given. Then for any function $v \in H^i(K)$ and $w \in H^i(S')$ and for any integer $i = 0, \ldots, k + 1$ the following estimates are true:*

$$(1.32)\qquad |\hat{v}|_{i,\hat{K}} \leqq ch^{-\frac{1}{2}n+i}\|v\|_{i,K},$$

$$(1.33)\qquad |v|_{i,K} \leqq ch^{\frac{1}{2}n-i}\|\hat{v}\|_{i,\hat{K}},$$

$$(1.34)\qquad |\hat{w}|_{i,\hat{S}} \leqq ch^{-\frac{1}{2}(n-1)+i}\|w\|_{i,S'},$$

$$(1.35)\qquad |w|_{i,S'} \leqq ch^{\frac{1}{2}(n-1)-i}\|\hat{w}\|_{i,\hat{S}}.$$

Moreover, for $i = 1$ we can use semi-norms on the right hand sides of these inequalities.

Proof. Inequalities (1.32) and (1.34) follow from Lemma 1 in [2], p. 427. Inequalities (1.33) and (1.35) can be proved using the method of Ciarlet, see Theorems 4.3.2 and 4.3.3 in [1], pp. 232–241. □

Lemma 1.7. (*Friedrichs' inequality*). *For any function $v \in H^1(\Omega_h)$ there exists a constant c (independent of h, v) such that*

$$(1.36)\qquad \|v\|_{0,\Omega_h} \leqq c\left(|v|_{1,\Omega_h} + \|v\|_{0,\Gamma_h}\right).$$

The proof can be carried out similarly as in [7], pp. 201–204. □

2. APPROXIMATE SOLUTION OF THE ELLIPTIC PROBLEM

Let Ω be a bounded domain in R^n with a sufficiently smooth boundary Γ. We study the elliptic problem

$$-lu = f(x)\,, \quad x \in \Omega\,, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} + a(x)\, u = q(x)\,, \quad x \in \Gamma\,,$$

where $f(x)$, $a(x)$, $q(x)$ are sufficiently smooth functions and

$$l = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial}{\partial x_i} \right), \tag{2.2}$$

$$\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \nu_j\,. \tag{2.3}$$

We suppose that the functions $a_{ij}(x)$ are sufficiently smooth and

$$a_{ij}(x) = a_{ji}(x)\,. \tag{2.4}$$

Concerning the differential operator l we suppose that it is strongly elliptic, i.e. there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geqq c \sum_{i=1}^{n} \xi_i^2 \quad \forall x \in \bar{\Omega}\,, \quad (\xi_1, \ldots, \xi_n) \in R^n\,. \tag{2.5}$$

Concerning the function $a(x)$ we assume that there exists a constant $c > 0$ such that

$$a(x) \geqq c > 0 \quad \forall x \in \Gamma\,. \tag{2.6}$$

The variational formulation of the elliptic problem is:

Find a function $u \in H^1(\Omega)$ such that

$$b(u, v) = d(v) \quad \forall v \in H^1(\Omega)\,, \tag{2.7}$$

where
$$b(u, v) = a(u, v) + (au, v)_{0,\Gamma}\,,$$

$$a(u, v) = \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j} \right)_{0,\Omega}, \tag{2.8}$$

$$d(v) = (f, v)_{0,\Omega} + (q, v)_{0,\Gamma}\,.$$

It is well known that the problem (2.7) has a unique solution, which is sufficiently smooth if all the data of the problem are sufficiently smooth.

We extend the functions $a_{ij}(x)$, $f(x)$ to the larger domain $\tilde{\Omega}$ so that the conditions (2.4) and (2.5) are again satisfied. In this way we obtain functions $A_{ij}(x)$, $F(x)$. We denote

(2.9) $$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A_{ij}(x)\frac{\partial}{\partial x_i}\right),$$

(2.10) $$\frac{\partial}{\partial v_h} = \sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial}{\partial x_i} v_{hj}\,.$$

Now we formulate the following discrete problem:

Find a function $u_h(x) \in V_h$ such that

(2.11) $$b_h(u_h, v) = d_h(v) \quad \forall v \in V_h\,,$$

where $$b_h(u_h, v) = a_h(u_h, v) + (\pi_\Gamma a u_h, v)_{0,\Gamma_h}\,,$$

(2.12) $$a_h(u_h, v) = \sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u_h}{\partial x_i}, \frac{\partial v}{\partial x_j}\right)_{0,\Omega_h},$$

$$d_h(v) = (f, v)_{0,\Omega_h} + (\pi_\Gamma q, v)_{0,\Gamma_h}\,.$$

From the following lemma we deduce that there exists a unique solution of the problem (2.11):

Lemma 2.1. *The bilinear form $b_h(v, w)$ is uniformly V_h-elliptic, i.e. there exists a constant c ($c > 0$ and independent of h) such that*

(2.13) $$b_h(v, v) \geqq c\|v\|^2_{1,\Omega_h} \quad \forall v \in V_h\,.$$

Proof. If we prove that there exists a constant $c > 0$, independent of h and such that

(2.14) $$\pi_\Gamma\, a(x) \geqq c > 0 \quad \forall x \in \Gamma_h\,,$$

(2.13) follows immediately from (2.12), (2.5), (2.14) and (1.36). For an element $S \in \Gamma_h$ we get from (1.23) and (1.19)

$$\begin{aligned} {}_\varphi a|_{S'} &= {}_\psi\pi_S a + {}_\varphi a|_{S'} - {}_\psi\pi_S a = {}_\psi\pi_S a + {}_\varphi a|_{S'} - \pi'_S({}_\psi a) = \\ &= {}_\psi\pi_S a + {}_\varphi a|_{S'} - \pi'_S({}_\varphi a) \leqq {}_\psi\pi_S a + |{}_\varphi a - \pi'_S({}_\varphi a)|_{0,\infty,S'} \leqq \\ &\leqq {}_\psi\pi_S a + ch\|{}_\varphi a\|_{1,\infty,S'}\,, \end{aligned}$$

so that

$${}_\psi\pi_S a \geqq {}_\varphi a|_{S'} - ch\|a\|_{1,\infty,\Gamma}\,.$$

Using (2.6) we obtain (2.14) for all h sufficiently small. □

Remark 2.1. The condition (2.6) for a function $a(x)$ can be weakened as follows:

(2.6a) $$\begin{aligned} a(x) &\geqq 0 && \forall x \in \Gamma\,, \\ a(x) &\geqq c > 0 && \forall x \in \Gamma^* \subset \Gamma\,, \quad \operatorname{meas} \Gamma^* \neq 0\,. \end{aligned}$$

To prove the uniform V_h-ellipticity of the bilinear form b_h we need the following discrete form of Friedrichs' inequality:

$$\|v\|_{1,\Omega_h} \leqq c(|v|_{1,\Omega_h} + \|v\|_{0,\Gamma_h^*}), \tag{2.15}$$

where $\Gamma_h^* = \{x \mid x \in S \text{ where } \Sigma_S \subset \Gamma^*\}$ and c does not depend on h and v.

The proof of the inequality (2.15) for $n = 2$ follows from Ženíšek's paper [13], see Theorem 1 and remarks to it; for $n \geqq 3$ it will appear elsewhere. □

Since it is practically impossible to evaluate exactly integrals $(\cdot, \cdot)_{0,\Omega_h}$ and $(\cdot, \cdot)_{0,\Gamma_h}$, it is necessary to take into account the approximate integration for their computation. Following Ciarlet and Raviart [2], we could introduce the numerical isoparametric integration on both "volume" and surface elements K and S and analyse the obtained fully discrete problem similarly as Nedoma [6]. As it would be rather a technical matter we omit the analysis of the numerical integration in this paper and in Remark 3.2 we introduce only the final results.

3. ERROR ESTIMATES

Let us suppose that the solution $u(x)$ of the problem (2.1) belongs to $H^s(\Omega)$ for an integer $s \geqq 2$. By the Calderon theorem there exists an extension U of the function u onto $\tilde{\Omega}$ such that

$$\|U\|_{s,\tilde{\Omega}} \leqq c\|u\|_{s,\Omega}. \tag{3.1}$$

It is quite natural to take

$$F = -LU. \tag{3.2}$$

Evidently F is an extension of the function f. Substituting (3.2) into (2.11) we get

$$b_h(u_h, v) = -(LU, v)_{0,\Omega_h} + (\pi_\Gamma q, v)_{0,\Gamma_h} \quad \forall v \in V_h \tag{3.3}$$

and from the Green theorem we obtain

$$b_h(U - u_h, v) = \left(\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q, v\right)_{0,\Gamma_h} \quad \forall v \in V_h. \tag{3.4}$$

The equation (3.4) is the starting point for the estimate of the discretisation error $u - u_h$. Before coming to this estimate we give some lemmas.

Lemma 3.1. Let $U \in W^{1,\infty}(\tilde{\Omega})$. Then

$$\|{}_\varphi U - {}_\psi U\|_{0,p,S'} \leqq ch^{k+1}\|U\|_{1,\infty,\tilde{\Omega}} (\text{meas } S')^{1/p}. \tag{3.5}$$

Proof. Using the Cauchy inequality we obtain for $p < \infty$

$$\|{}_\varphi U - {}_\psi U\|^p_{0,p,S'} = \int_{S'} [U(x', \varphi(x')) - U(x', \psi(x'))]^p \,\mathrm{d}x' =$$

$$= \int_{S'} \left[\frac{\partial}{\partial x_n} U(x', \xi(x'))\right]^p [\varphi(x') - \psi(x')]^p \,\mathrm{d}x' \leqq |\varphi - \psi|^p_{0,\infty,S'} \|U\|^p_{1,\infty,\tilde{\Omega}} \operatorname{meas} S' .$$

Hence and from (1.28) we get (3.5) for $p < \infty$. (3.5) for $p = \infty$ is obtained similarly. □

Lemma 3.2. *Let τ_h be a k-regular triangulation of the domain Ω with $2(k + 1) > n$. Let $U \in H^{k+3}(\tilde{\Omega})$ and $\partial U/\partial \nu + aU = q$ on Γ. Then there exists a constant c (independent of h and U) such that*

$$\left\|\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q\right\|_{0,\Gamma_h} \leqq ch^k \|U\|_{k+3,\tilde{\Omega}} . \tag{3.6}$$

Proof. For any element $S \in \Gamma_h$ we have

$$\left\|\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q\right\|_{0,S} = \left\|{}_\psi\left[\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q\right]\right\|_{0,S'} = \tag{3.7}$$

$$= \left\|{}_\psi\left[\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q\right] - {}_\varphi\left[\frac{\partial U}{\partial \nu} + aU - q\right]\right\|_{0,S'} \leqq$$

$$\leqq c\left(\left\|{}_\psi\left[\frac{\partial U}{\partial \nu_h}\right] - {}_\varphi\left[\frac{\partial U}{\partial \nu}\right]\right\|_{0,S'} + \|{}_\psi[\pi_\Gamma a U] - {}_\varphi[aU]\|_{0,S'} + \|{}_\psi \pi_\Gamma q - {}_\varphi q\|_{0,S'}\right).$$

From (2.3) and (2.10) we infer

$$\left\|{}_\psi\left[\frac{\partial U}{\partial \nu_h}\right] - {}_\varphi\left[\frac{\partial U}{\partial \nu}\right]\right\|_{0,S'} = \left\|{}_\psi\left[\sum_{i,j=1}^{n} A_{ij}\nu_{hj}\frac{\partial U}{\partial x_i}\right] - {}_\varphi\left[\sum_{i,j=1}^{\prime} a_{ij}\nu_j \frac{\partial U}{\partial x_i}\right]\right\|_{0,S'} \leqq$$

$$\leqq \sum_{i,j=1}^{n}\left(\left\|{}_\psi\left[A_{ij}\frac{\partial U}{\partial x_i}\right] - {}_\varphi\left[A_{ij}\frac{\partial U}{\partial x_i}\right]\right\|_{0,\infty,S'} \|{}_\psi \nu_{hj}\|_{0,S'} + \right.$$

$$\left. + \left\|{}_\varphi\left[A_{ij}\frac{\partial U}{\partial x_i}\right]\right\|_{0,\infty,S'} \|{}_\psi \nu_{hj} - {}_\varphi \nu_j\|_{0,S'}\right).$$

Hence, from (3.5), the Sobolev lemma and (1.30) we get

$$\left\|{}_\psi\left[\frac{\partial U}{\partial \nu_h}\right] - {}_\varphi\left[\frac{\partial U}{\partial \nu}\right]\right\|_{0,S'} \leqq ch^k (\operatorname{meas} S')^{1/2} \|U\|_{k+3,\tilde{\Omega}} . \tag{3.8}$$

From the Sobolev lemma, (1.23), (1.19) and (3.5) we see that

$$\|{}_\psi[\pi_\Gamma a U] - {}_\varphi[aU]\|_{0,S'} \leqq \|{}_\psi \pi_S A_\psi U - {}_\varphi A_\psi U\|_{0,S'} + \tag{3.9}$$

$$+ \|{}_\varphi A_\psi U - {}_\varphi A_\varphi U\|_{0,S'} \leqq \|U\|_{0,\infty,\tilde{\Omega}} \|\pi'_S({}_\varphi A) - {}_\varphi A\|_{0,S'} +$$

$$+ \|{}_\varphi A\|_{0,S'} \|{}_\varphi U - {}_\psi U\|_{0,\infty,S'} \leqq ch^{k+1} \|{}_\varphi a\|_{k+1,S'} \|U\|_{k+2,\tilde{\Omega}} .$$

From (1.23) and (1.19) we conclude

$$\|{}_{\psi}\pi_{\Gamma}q - {}_{\varphi}q\|_{0,S'} = \|\pi'_S({}_{\varphi}q) - {}_{\varphi}q\|_{0,S'} \leqq ch^{k+1}\|{}_{\varphi}q\|_{k+1,S'} .$$

Substituting from (3.8), (3.9) and from the last inequality into (3.7), summing over all elements $S \in \Gamma_h$ and using the trace theorem we obtain

$$\left\|\frac{\partial U}{\partial \nu_h} + \pi_\Gamma aU - \pi_\Gamma q\right\|_{0,\Gamma_h} \leqq ch^k(\|U\|_{k+3,\tilde{\Omega}} + \|q\|_{k+1,\Gamma}) \leqq ch^k\|U\|_{k+3,\tilde{\Omega}} . \qquad \square$$

Let us denote in the usual way

$$(3.10) \qquad \|w\|_{-1,\Gamma_h} = \sup_{v \in H^1(\Gamma_h)} \frac{|(w, v)_{0,\Gamma_h}|}{\|v\|_{1,\Gamma_h}} .$$

Lemma 3.3. *Let τ_h be a k-regular triangulation of the domain Ω with $2(k + 1) > n$. Let $U \in H^{k+3}(\tilde{\Omega})$ and $\partial U/\partial \nu + aU = q$ on Γ. Then there exists a constant c (independent of h and U) such that*

$$(3.11) \qquad \left\|\frac{\partial U}{\partial \nu_h} + \pi_\Gamma aU - \pi_\Gamma q\right\|_{-1,\Gamma_h} \leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}} .$$

Proof. We cover the boundary Γ by the set $\{\gamma^r\}_{r=1}^R$ of mutually disjoint pieces $\gamma^r \subset \Gamma^r$ with sufficiently smooth boundaries $\partial\gamma^r$. We denote by δ^r the projection of γ^r into the hyperplane $x_n^r = 0$, i.e. $\delta^r = \{x'^r | (x'^r, \varphi^r(x'^r)) \in \gamma^r\}$. Further, we denote

$$\delta_h^r = \{x'^r | \; x'^r \in S'^r \text{ where } S \in \Gamma_h^r \text{ and } S'^r \cap \delta^r \neq \{\emptyset\}\} , \; \gamma_h^r = \{x^r | \; x'^r \in \delta_h^r\} .$$

We see that $\Gamma_h \subset \bigcup_{r=1}^R \gamma_h^r$. Hence and from (1.21) we get for any function $v \in H^1(\Gamma_h)$

$$(3.12) \quad \left|\left(\frac{\partial U}{\partial \nu_h} + \pi_\Gamma aU - \pi_\Gamma q, v\right)_{0,\Gamma_h}\right| \leqq \sum_{r=1}^R \left|\int_{\gamma_h^r} \left(\frac{\partial U}{\partial \nu_h} + \pi_\Gamma aU - \pi_\Gamma q\right) v \, d\gamma_h^r\right| =$$

$$= \sum_{r=1}^R \left|\int_{\delta_h^r} \left\{{}_{\psi^r}\!\left[\frac{\partial U}{\partial \nu_h} + \pi_\Gamma aU - \pi_\Gamma q\right] - {}_{\varphi^r}\!\left[\frac{\partial U}{\partial \nu} + aU - q\right]{}_{\psi^r}v \sqrt{(1 + |\mathbf{grad}\, \psi^r|^2)}\, dx'^r\right.\right| \leqq$$

$$\leqq \sum_{r=1}^R \left|\int_{\delta_h^r} \left\{{}_{\psi^r}\!\left[\frac{\partial U}{\partial \nu_h}\right] - {}_{\varphi^r}\!\left[\frac{\partial U}{\partial \nu}\right]\right\} {}_{\psi^r}v \sqrt{(1 + |\mathbf{grad}\, \psi^r|^2)}\, dx'^r\right. +$$

$$+ c \sum_{r=1}^R \{\|{}_{\psi^r}[\pi_\Gamma aU] - {}_{\varphi^r}[aU]\|_{0,\delta_h^r} + \|{}_{\psi^r}\pi_\Gamma q - {}_{\varphi^r}q\|_{0,\delta_h^r}\} \, \|{}_{\psi^r}v\|_{0,\delta_h^r} .$$

Similarly as in Lemma 3.2 we obtain the estimates

$$(3.13) \qquad \|{}_{\psi^r}[\pi_\Gamma aU] - {}_{\varphi^r}[aU]\|_{0,\delta_h^r} \leqq ch^{k+1}\|{}_{\varphi^r}a\|_{k+1,\delta_h^r}\|U\|_{k+3,\tilde{\Omega}} ,$$

$$\|{}_{\psi^r}\pi_\Gamma q - {}_{\varphi^r}q\|_{0,\delta_h^r} \leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}} .$$

If we prove the inequality

$$(3.14) \qquad \left| \int_{\delta_h r} \left\{ {}_{\psi^r}\left[\frac{\partial U}{\partial v_h}\right] - {}_{\varphi^r}\left[\frac{\partial U}{\partial v}\right] \right\} {}_{\psi^r}v \sqrt{(1 + |\mathbf{grad}\, \psi^r|^2} \,\mathrm{d}x'^r \right| \leqq$$

$$\leqq c h^{k+1} \|U\|_{k+3,\tilde{\Omega}} \|{}_{\psi^r}v\|_{1,\delta_h r},$$

then (3.11) will follow from (3.12), (3.13), (3.14) and (3.10).

In the proof of the inequality (3.14) we drop the index r. Then using (2.3) and (2.10) we get for $S' \in \delta_h$

$$(3.15) \qquad \int_{S'} \left\{ {}_{\psi}\left[\frac{\partial U}{\partial v_h}\right] - {}_{\varphi}\left[\frac{\partial U}{\partial v}\right] \right\} {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)} \,\mathrm{d}x' =$$

$$= \int_{S'} \sum_{i,j=1}^{n} \left\{ {}_{\psi}\left[A_{ij} \frac{\partial U}{\partial x_i}\right] - {}_{\varphi}\left[A_{ij} \frac{\partial U}{\partial x_i}\right] \right\} {}_{\psi}v_{hj}\, {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)} \,\mathrm{d}x' +$$

$$+ \int_{S'} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} {}_{\varphi}\left[A_{ij} \frac{\partial U}{\partial x_i}\right] ({}_{\psi}v_{hj} - {}_{\varphi}v_j) \right\} {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)} \,\mathrm{d}x'.$$

From (3.5) and (1.21) we obtain

$$(3.16) \quad \left| \int_{S'} \sum_{i,j=1}^{n} \left\{ {}_{\psi}\left[A_{ij} \frac{\partial U}{\partial x_i}\right] - {}_{\varphi}\left[A_{ij} \frac{\partial U}{\partial x_i}\right] \right\} {}_{\psi}v_{hj}\, {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)} \,\mathrm{d}x' \right| \leqq$$

$$\leqq c h^{k+1} \|U\|_{k+3,\tilde{\Omega}} (\mathrm{meas}\, S')^{1/2} \|{}_{\psi}v\|_{0,S'}.$$

Let us denote

$$(3.17) \qquad z_j(x') = \sum_{i=1}^{n} {}_{\varphi}\left[A_{ij} \frac{\partial U}{\partial x_i}\right](x'), \quad x' \in \delta_h, \quad j = 1, \ldots, n.$$

Then (3.15), (3.16) and (3.17) imply

$$(3.18) \qquad \left| \int_{\delta_h} \left\{ {}_{\psi}\left[\frac{\partial U}{\partial v_h}\right] - {}_{\varphi}\left[\frac{\partial U}{\partial v}\right] \right\} {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)} \,\mathrm{d}x' \right| \leqq$$

$$\leqq c h^{k+1} \|U\|_{k+3,\tilde{\Omega}} \sum_{S' \in \delta_h} (\mathrm{meas}\, S')^{1/2} \|{}_{\psi}v\|_{0,S'} +$$

$$+ \sum_{j=1}^{n} \left| \int_{\delta_h} ({}_{\psi}v_{hj} - {}_{\varphi}v_j)\, z_j\, {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)} \,\mathrm{d}x' \right|.$$

Since due to (1.21) and (1.31) (we choose $v = 1$) we have

$$(3.19) \qquad \mathrm{meas}\, \delta_h \leqq c\, \mathrm{meas}\, \gamma_h \leqq c\, \mathrm{meas}\, \Gamma_h \leqq c\, \mathrm{meas}\, \Omega_h \leqq c\, \mathrm{meas}\, \tilde{\Omega} \leqq c,$$

we obtain using the Cauchy inequality

$$(3.20) \qquad \sum_{S' \in \delta_h} (\mathrm{meas}\, S')^{1/2} \|{}_{\psi}v\|_{0,S'} \leqq c \|{}_{\psi}v\|_{0,\delta_h}.$$

Then (3.18) and (3.20) implies that to prove (3.14) it suffices to prove the inequality

$$(3.21)\quad \left|\int_{\delta_h} ({}_{\psi}v_{hj} - {}_{\varphi}v_j)\, z_j\, {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)}\, dx'\right| \leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}}\, \|{}_{\psi}v\|_{1,\delta_h}\,,$$

$$j = 1, \ldots, n\,.$$

In proving (3.21) we restrict overselves to the case $j < n$ (for $j = n$, (3.21) can be proved similarly).

Let us denote

$$(3.22)\qquad \Phi(x') = (1 + |\mathbf{grad}\, \varphi(x')|^2)^{1/2}\,, \quad \Psi(x') = (1 + |\mathbf{grad}\, \psi(x')^2)^{1/2}\,.$$

Then from (1.29) we obtain after simple calculations

$$(3.23)\qquad {}_{\psi}v_{hj} - {}_{\varphi}v_j = -\frac{\partial \psi}{\partial x_j}\Psi^{-1} - \left(-\frac{\partial \varphi}{\partial x_j}\Phi^{-1}\right) =$$

$$-\frac{\partial}{\partial x_j}(\psi - \varphi)\Psi^{-1} + \frac{\partial \varphi}{\partial x_j}\Phi^{-1}\Psi^{-1}(\Phi + \Psi)^{-1}\sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}(\psi - \varphi)\frac{\partial}{\partial x_i}(\psi + \varphi)\,.$$

Hence

$$(3.24)\qquad \int_{S'} ({}_{\psi}v_{hj} - {}_{\varphi}v_j)\, z_j\, {}_{\psi}v \sqrt{(1 + |\mathbf{grad}\, \psi|^2)}\, dx' =$$

$$= \int_{S'}\left\{-\frac{\partial}{\partial x_j}(\psi - \varphi) + \frac{\partial \varphi}{\partial x_j}\Phi^{-1}(\Phi + \Psi)^{-1}\sum_{i=1}^{n-1}\frac{\partial}{\partial x_i}(\psi - \varphi)\frac{\partial}{\partial x_i}(\psi + \varphi)\right\} z_j\, {}_{\psi}v\, dx'\,.$$

Let us first consider the case $n = 2$. For the end points $a'_S < b'_S$ of the interval S' we have $\varphi(a'_S) = \psi(a'_S)$, $\varphi(b'_S) = \psi(b'_S)$. Applying the integration by parts to the right hand side of the equation (3.24) we obtain (we drop the index j and write $'$ instead of d/dx and x instead of x')

$$\int_{S'} ({}_{\psi}v_h - {}_{\varphi}v)\, z\, {}_{\psi}v \sqrt{(1 + \psi'^2)}\, dx =$$

$$= \left[-(\psi - \varphi) + \varphi'(\psi - \varphi)(\psi + \varphi)'\,\Phi^{-1}(\Phi + \Psi)^{-1}\right] z\, {}_{\psi}v\,\Big|_{a_S'}^{b_S'} +$$

$$+ \int_{S'} (\psi - \varphi)(z\, {}_{\psi}v)'\, dx - \int_{S'} (\psi - \varphi)\left[\varphi'(\varphi + \psi)'\,\Phi^{-1}(\Phi + \Psi)^{-1}\, z\, {}_{\psi}v\right]'\, dx =$$

$$= \int_{S'} (\psi - \varphi)(z\, {}_{\psi}v)'\, dx - \int_{S'} (\psi - \varphi)\left[\varphi'(\varphi + \psi)'\,\Phi^{-1}(\Phi + \Psi)^{-1}\, z\, {}_{\psi}v\right]'\, dx\,.$$

Since by (1.28) and (3.22)

$$\|\varphi'(\varphi + \psi)'\,\Phi^{-1}(\Phi + \Psi)^{-1}\|_{1,\infty,S'} \leqq c\|\varphi\|_{2,\infty,S'} \leqq c\,,$$

we get using the Cauchy inequality, (1.28) and (3.17)

$$\left|\int_{S'} ({}_\psi v_h - {}_\varphi v)\, z\, {}_\psi v \sqrt{(1 + \psi'^2)}\, \mathrm{d}x\right| \leqq c(\operatorname{meas} S')^{1/2} \|\varphi - \psi\|_{0,\infty,S'} \|z\|_{1,\infty,S'} \|{}_\psi v\|_{1,S'} \leqq$$

$$\leqq ch^{k+1}\|U\|_{k+3,\bar\Omega}(\operatorname{meas} S')^{1/2} \|{}_\psi v\|_{1,S'}\,.$$

Summing over all elements $S \in \delta_h$ and applying the Cauchy inequality and (3.19) we see that we have proved (3.21) for $n = 2$.

To prove the inequality (3.21) for $n > 2$ we apply the Green theorem to the right hand side of the inequality (3.24). Then we get

$$\int_{S'} ({}_\psi v_{hj} - {}_\varphi v_j)\, z_j\, {}_\psi v \sqrt{(1 + |\boldsymbol{grad}\, \psi|^2)}\, \mathrm{d}x' =$$

$$= \int_{\partial S'} \left[-(\psi - \varphi)\, v'_{Sj} + \frac{\partial \varphi}{\partial x_j} \Phi^{-1}(\Phi + \Psi)^{-1} \sum_{i=1}^{n-1} v'_{Si}(\psi - \varphi) \frac{\partial}{\partial x_i}(\psi + \varphi) \right] z_j\, {}_\psi v\, \mathrm{d}(\partial S') +$$

$$+ \int_{S'} (\psi - \varphi) \frac{\partial}{\partial x_j}(z_j\, {}_\psi v)\, \mathrm{d}x' -$$

$$- \int_{S'} (\psi - \varphi) \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \left[\frac{\partial \varphi}{\partial x_j} \Phi^{-1}(\Phi + \Psi)^{-1} \frac{\partial}{\partial x_i}(\varphi + \psi)\, z_j\, {}_\psi v \right] \mathrm{d}x'\,,$$

where $v'_S = (v'_{S1}, \ldots, v'_{Sn-1})$ is the unit vector of the outward normal to the boundary $\partial S'$. Similarly as in the proof of the inequality (3.21) for $n = 2$ we obtain

$$\left| \sum_{S' \in \delta_h} \int_{S'} (\psi - \varphi) \frac{\partial}{\partial x_j}(z_j\, {}_\psi v)\, \mathrm{d}x' - \right.$$

$$\left. - \int_{S'} (\psi - \varphi) \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \left[\frac{\partial \varphi}{\partial x_j} \Phi^{-1}(\Phi + \Psi)^{-1} \frac{\partial}{\partial x_i}(\psi + \varphi)\, z_j\, {}_\psi v \right] \mathrm{d}x' \right| \leqq$$

$$\leqq ch^{k+1}\|U\|_{k+3,\bar\Omega} \|{}_\psi v\|_{1,\delta_h}\,.$$

Therefore, to get (3.21) for $n > 2$ it suffices to prove the inequality

$$\left| \sum_{S' \in \delta_h} \int_{\partial S'} y_S\, \mathrm{d}(\partial S') \right| \leqq ch^{k+1}\|U\|_{k+3,\bar\Omega} \|{}_\psi v\|_{1,\delta_h}\,, \tag{3.25}$$

where y_S is the function defined on $\partial S'$ by the equation

(3.26)

$$y_S = \left[-(\psi - \varphi)\, v'_{Sj} + \frac{\partial \varphi}{\partial x_j} \Phi^{-1}(\Phi + \Psi_S)^{-1} \sum_{i=1}^{n-1} v'_{Si}(\psi - \varphi) \left(\frac{\partial \psi_S}{\partial x_i} + \frac{\partial \varphi}{\partial x_i} \right) \right] z_j\, {}_\psi v$$

and where $\partial\psi_S/\partial x_i$ denotes the trace of the function $\partial\psi/\partial x_i$ on $\partial S'$ and $\Psi_S = (1 + \sum_{j=1}^{n-1} (\partial\psi_S/\partial x_j)^2)^{1/2}$. Obviously

$$\sum_{S' \in \delta_h} \int_{\partial S'} y_S \, d(\partial S') = \sum_{S' \in \delta_h} \sum_{H' \in \partial S'} \int_{H'} y_S \, dH' . \tag{3.27}$$

We divide the sum $\sum_{S' \in \delta_h} \sum_{H' \in \partial S'}$ into the sum $\sum_{H'}^B = \sum_{H' \in \partial\delta_h}$ over the boundary edges and the sum $\sum_{H'}^I = \sum_{S' \in \delta_h} \sum_{H' \in \partial S'} - \sum_{H'}^B$ over the remaining inside edges. Then we have

$$\left| \sum_{S' \in \delta_h} \sum_{H' \in \partial S'} \int_{H'} y_S \, dH' \right| \leqq \left| \sum_{H'}^I \int_{H'} y_S \, dH' \right| + \left| \sum_{H'}^B \int_{H'} y_S \, dH' \right| . \tag{3.28}$$

Let us denote $\|v\|_{0,H'} = (\int_{H'} v^2 \, dH')^{1/2}$, $\|v\|_{0,\partial\delta_h} = (\int_{\partial\delta_h} v^2 \, d(\partial\delta_h))^{1/2}$, $\|v\|_{0,\partial\delta} = (\int_{\partial\delta} v^2 \, d(\partial\delta))^{1/2}$ and estimate the terms on the right hand side of the inequality (3.28).

1. The error estimate of the term $|\sum_{H'}^I \int_{H'} y_S \, dH'|$. The integral $\int_{H'} y_S \, dH'$ over the inside edge H' appears in the sum $\sum_{H'}^I$ once as a contribution from the element S'_+ and for the second time as a contribution from the element S'_- where $H' = S'_+ \cap S'_-$. Therefore

$$\sum_{H'}^I \int_{H'} y_S \, dH' = \sum_{H'}^I{}_* \int_{H'} (y_{S_+} + y_{S_-}) \, dH' , \tag{3.29}$$

where $\sum_{H'}^I{}_*$ denotes the sum over all inside edges $H' \in \{ \bigcup_{S' \in \delta_h} \partial S' - \partial\delta_h \}$. Since $v'_{S_+ i} = -v'_{S_- i}$ and since the functions φ, ψ, z_j and ${}_\psi v$ are continuous on δ_h, it follows from (3.26) that

$$\sum_{H'}^I{}_* \int_{H'} (y_{S_+} + y_{S_-}) \, dH' = \sum_{H'}^I{}_* \int_{H'} (\psi - \varphi) \frac{\partial\varphi}{\partial x_j} z_j \, {}_\psi v \Phi^{-1} \sum_{i=1}^{n-1} v'_{S_+ i} \omega_i \, dH' , \tag{3.30}$$

where

$$\omega_i = \left(\frac{\partial\psi_{S_+}}{\partial x_i} + \frac{\partial\varphi}{\partial x_i} \right) (\Phi + \Psi_{S_+})^{-1} - \left(\frac{\partial\psi_{S_-}}{\partial x_i} + \frac{\partial\varphi}{\partial x_i} \right) (\Phi + \Psi_{S_-})^{-1} . \tag{3.31}$$

The term ω_i can be rewritten in the form

$$\omega_i = \left(\frac{\partial\psi_{S_+}}{\partial x_i} - \frac{\partial\psi_{S_-}}{\partial x_i} \right) (\Phi + \Psi_{S_-})^{-1} + \left(\frac{\partial\psi_{S_+}}{\partial x_i} + \frac{\partial\varphi}{\partial x_i} \right) \times$$

$$\times \sum_{j=1}^{n-1} \left(\frac{\partial\psi_{S_-}}{\partial x_j} - \frac{\partial\psi_{S_+}}{\partial x_j} \right) \left(\frac{\partial\psi_{S_-}}{\partial x_j} + \frac{\partial\psi_{S_+}}{\partial x_j} \right) (\Phi + \Psi_{S_+})^{-1} (\Phi + \Psi_{S_-})^{-1} (\Psi_{S_+} + \Psi_{S_-})^{-1} .$$

Using (1.28) we obtain

$$\left|\frac{\partial\psi_{S_+}}{\partial x_j} - \frac{\partial\psi_{S_-}}{\partial x_j}\right| = \left|\left(\frac{\partial\psi_{S_+}}{\partial x_j} - \frac{\partial\varphi}{\partial x_j}\right) + \left(\frac{\partial\varphi}{\partial x_j} - \frac{\partial\psi_{S_-}}{\partial x_j}\right)\right| \leqq$$
$$\leqq |\psi - \varphi|_{1,\infty,S_+'} + |\psi - \varphi|_{1,\infty,S_-'} \leqq ch^k .$$

Hence and from (1.28) we get

$$\|\omega_i\|_{0,\infty,H'} \leqq ch^k , \quad i = 1, \ldots, n - 1 . \tag{3.32}$$

Then from (3.30), (1.28), (3.17), (3.32) and from the Cauchy inequality we have

$$\left|\sum_{H'}{}^I_* \int_{H'} (y_{S_+} + y_{S_-}) \,\mathrm{d}H'\right| \leqq ch^{2k+1}\|U\|_{k+2,\tilde{\Omega}} \sum_{H'}{}^I_* (\operatorname{meas} H')^{1/2} \|{}_\psi v\|_{0,H'} . \tag{3.33}$$

From (1.25), the trace theorem and (1.34) we conclude

$$\|{}_\psi v\|^2_{0,H'} \leqq ch^{n-2}\|\hat{v}\|^2_{0,\hat{H}} \leqq ch^{n-2}\|\hat{v}\|^2_{1,\hat{S}} \leqq ch^{-1}\|{}_\psi v\|^2_{1,S_+'} . \tag{3.34}$$

If we take ${}_\psi v = 1$ then obviously

$$\operatorname{meas} H' \leqq ch^{-1} \operatorname{meas} S'_+ . \tag{3.35}$$

Substituting from (3.34) and (3.35) into (3.33) and using the Cauchy inequality, (3.19) and (3.29) we get the needed estimate:

$$\left|\sum_{H'}{}^I \int_{H'} y_S \,\mathrm{d}H'\right| \leqq ch^{2k}\|U\|_{k+2,\tilde{\Omega}} \|{}_\psi v\|_{1,\delta_h} . \tag{3.36}$$

2. The error estimate of the term $|\sum_{H'}^B \int_{H'} y_S \,\mathrm{d}H'|$.

From (3.26), (1.28), (3.17) and the Cauchy inequality we easily obtain

$$\left|\int_{H'} y_S \,\mathrm{d}H'\right| \leqq ch^{k+1}\|U\|_{k+2,\tilde{\Omega}} (\operatorname{meas} H')^{1/2} \|{}_\psi v\|_{0,H'} ,$$

so that, using Cauchy's inequality, we have

$$\left|\sum_{H'}{}^B \int_{H'} y_S \,\mathrm{d}H'\right| \leqq ch^{k+1}\|U\|_{k+2,\tilde{\Omega}} (\operatorname{meas} \partial\delta_h)^{1/2} \|{}_\psi v\|_{0,\partial\delta_h} . \tag{3.37}$$

If we prove that there exists a constant c independent of h such that

$$\|w\|_{0,\partial\delta_h} \leqq c\|w\|_{1,\delta_h} \quad \forall w \in H^1(\delta_h) , \tag{3.38}$$

then from (3.37), (3.38) and (3.19) we have

$$\left|\sum_{H'}{}^B \int_{H'} y_S \,\mathrm{d}H'\right| \leqq ch^{k+1}\|U\|_{k+2,\tilde{\Omega}} \|{}_\psi v\|_{1,\delta_h}$$

and (3.25) follows from (3.28) and (3.36). So let us prove (3.38).

Let the edge $H' \subset \partial\delta_h$, see Fig. 3.1. Then there exists an element $S' \in \delta_h$ such that $H' \subset \partial S'$. Due to the smoothness of the boundary $\partial\delta$ we can choose the coordinate system $(x_1, \ldots, x_{n-2}, x_{n-1}) = (x'', x_{n-1})$ in such a way that the part $\partial\delta^*$ of the

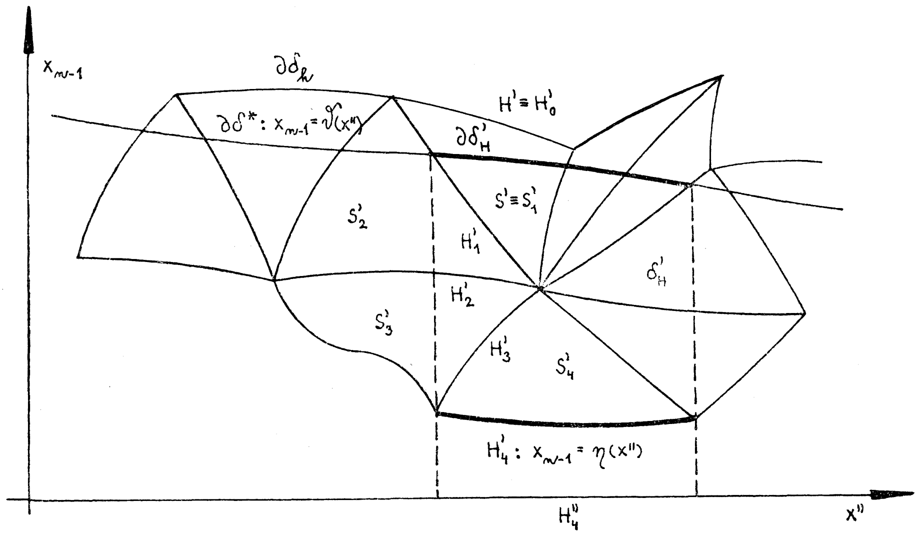

Fig. 3.1.

boundary $\partial\delta$ containing $\partial\delta \cap S'$ is described in this coordinate system by an equation $x_{n-1} = \vartheta(x'')$ for $x'' \in \Delta^*$. We can and will suppose that if $x' = (x'', x_{n-1}) \in \delta$ and $x'' \in \Delta^*$ then $x_{n-1} \leqq \vartheta(x'')$, see Fig. 3.1. We construct the set $\{S'_i\}_{i=1}^{I}$ of elements S'_i with the following properties:

1) $S'_1 = S'$;

2) $S'_i \in \delta_h$, $i = 1, \ldots, I$;

3) $S'_I \in \delta$;

4) S'_i and S'_{i+1} have a common "face" $H'_i = S'_i \cap S'_{i+1}$, $i = 1, \ldots, I - 1$;

5) $\bigcap_{i=1}^{I} S'_i \neq \{\emptyset\}$.

Let us denote by H'_I one of the "faces" of the element S'_I which can be described by the equation $x_{n-1} = \eta(x'')$ for $x'' \in H''_I$ such that

$$\sup_{x'' \in H_I''} \left|\frac{\partial}{\partial x_i}\, \eta(x'')\right| \leqq c\,, \quad i = 1, \ldots, n-2\,,$$

where H''_I is the projection of the element H'_I into the hyperplane $x_{n-1} = 0$. The existence of such a face H_I follows from the regularity assumptions (1.15), (1.16) of the element S_I.

Further, we denote

$$\delta'_H = \{(x'', x_{n-1}) \mid x'' \in H''_I,\ \eta(x'') \leqq x_{n-1} \leqq \vartheta(x'')\},$$
$$\partial\delta'_H = \{(x'', x_{n-1}) \mid x'' \in H''_I,\ x_{n-1} = \vartheta(x'')\},$$

see Fig. 3.1. Then it is possible to prove the inequalities

$$\|w\|_{0,H_{i-1}'} \leqq c(\|w\|_{0,H_i'} + |w|_{1,S_i'}), \quad i = 1, \ldots, I, \tag{3.39}$$

$$\|w\|_{0,H_I'} \leqq c(\|w\|_{0,\partial\delta_H'} + |w|_{1,\delta_H'}), \tag{3.40}$$

where $H'_0 \equiv H$. The proof will be given later.

If we denote by S'_H the set $\delta'_H \cup \bigcup_{i=1}^{I} S'_i$, then (3.39) and (3.40) yields

$$\|w\|_{0,H'} \leqq c(\|w\|_{0,\partial\delta_H'} + |w|_{1,S_H'}),$$

where we have used the fact that I does not depend on h (it follows from the regularity property (1.16) of the elements S'). Summing over all elements $H' \in \partial\delta_h$ we get

$$\|w\|_{0,\partial\delta_h} \leqq c(\|w\|_{0,\partial\delta} + |w|_{1,\delta_h}),$$

where we have used the regularity of the elements S' again. Hence and from the trace theorem the inequality (3.38) follows.

The proof of the inequality (3.39).

Let $\hat{H}_i$, $\hat{H}_{i-1}$ be the images of H'_i, H'_{i-1} in the mapping F'_S. It can be easily proved that

$$\|\hat{w}\|^2_{0,\hat{H}_{i-1}} \leqq c(\|\hat{w}\|^2_{0,\hat{H}_i} + |\hat{w}|^2_{1,\hat{S}}).$$

Hence, from (1.25) and (1.34) we get

$$\|w\|^2_{0,H_{i-1}'} \leqq ch^{n-2}\|\hat{w}\|^2_{0,\hat{H}_{i-1}} \leqq ch^{n-2}(\|\hat{w}\|^2_{0,\hat{H}_i} + |\hat{w}|^2_{1,\hat{S}}) \leqq$$
$$\leqq c(\|w\|^2_{0,H_i'} + h|w|^2_{1,S_i'}),$$

which proves (3.39).

The proof of the inequality (3.40).

For every point $x'' \in H''_I$ we have

$$w(x'', \eta(x'')) = w(x'', \vartheta(x'')) + \int_{\vartheta(x'')}^{\eta(x'')} \frac{\partial}{\partial x_{n-1}} w(x'', \tau)\, d\tau.$$

Squaring, using Cauchy's inequality and integrating over the set H''_I we obtain

$$\int_{H_I''} w^2(x'', \eta(x''))\, dx'' \leqq c\Bigg(\int_{H_I''} w^2(x'', \vartheta(x''))\, dx'' +$$
$$+ \int_{H_I''} \left|\int_{\vartheta(x'')}^{\eta(x'')} \left[\frac{\partial}{\partial x_{n-1}} w(x'', \tau)\right]^2 d\tau\right| dx''\Bigg) \leqq c(\|w\|^2_{0,\partial\delta_H'} + |w|^2_{1,\delta_H'}).$$

Since by our assumption $|\textbf{grad}\, \eta(x'')| \leqq c \ \forall x'' \in H_I''$, (3.40) follows from the last two inequalities.

Then (3.38) is true and the lemma is proved. □

Remark 3.1. Let us consider the general Newton type boundary condition

$$g(u(x)) = q(x)\,, \quad x \in \Gamma\,,$$

where

$$g(v) = \frac{\partial v}{\partial \nu} + \sum_{|\alpha| \leqq 1} a_\alpha(x)\, D^\alpha v$$

with functions a_α and q sufficiently smooth on Γ. Let us denote

$$g_h(v) = \frac{\partial v}{\partial \nu_h} + \sum_{|\alpha| \leqq 1} \pi_\Gamma a_\alpha D^\alpha v\,, \quad q_h(v) = \pi_\Gamma q\,.$$

Then arguing similarly as in Lemmas 3.2, 3.3 we can prove that there exists a constant c (independent of h and w) such that the inequality

$$\|g_h(w) - q_h\|_{-i,\Gamma_h} \leqq c h^{k+i} \|w\|_{k+3,\tilde{\Omega}}\,, \quad i = 0, 1$$

holds for any function $w \in H^{k+3}(\tilde{\Omega})$ satisfying the boundary condition $g(w) = q$ on Γ. □

Lemma 3.4. *Let τ_h be a k-regular triangulation of the domain Ω with $2(k + 1) > n$. Let $Y \in H^2(\tilde{\Omega})$ and $\partial Y/\partial \nu + aY = 0$ on Γ. Then there exists a constant c (independent of h and Y) such that*

$$\left\| \frac{\partial Y}{\partial \nu_h} + \pi_\Gamma a Y \right\|_{0,\Gamma_h} \leqq c h^{(k+1)/2} \|Y\|_{2,\tilde{\Omega}}\,. \tag{3.41}$$

The proof is similar to the proof of Lemma 3.2. Therefore we leave it to the reader. □

Lemma 3.5. *To every function $Y \in H^l(\tilde{\Omega})$* there exists a mollifier $Y^h \in H^i(\tilde{\Omega})$ *with $i \geqq l$ such that*

$$|Y - Y^h|_{s,\tilde{\Omega}} \leqq c h^{l-s} |Y|_{l,\tilde{\Omega}}\,, \quad 0 \leqq s \leqq l\,, \tag{3.42}$$

$$|Y^h|_{s,\tilde{\Omega}} \leqq c h^{l-s} |Y|_{l,\tilde{\Omega}}\,, \quad l \leqq s \leqq i\,.$$

The proof follows from Theorem 2 in [9], p. 93 and from the inequality (19) in [10], p. 237.

Now we are able to formulate and to prove the main result of this paper, namely the estimate of the discretization error $u - u_h$ in the H^1 and L_2 norms.

Theorem 3.1. *Let u be the solution of the elliptic problem (2.1) with sufficiently smooth functions f, a_{ij}, a, q satisfying the conditions (2.4), (2.5) and (2.6). Let τ_h be a k-regular $(2(k+1) > n)$ triangulation of the domain Ω with sufficiently smooth boundary Γ. Then the discrete problem (2.11) has a unique solution u_h and there exists a constant c (independent of h and U) such that*

$$\|u - u_h\|_{1,\Omega\cap\Omega_h} \leqq ch^k \|u\|_{k+3,\Omega}, \tag{3.43}$$

$$\|u - u_h\|_{0,\Omega\cap\Omega_h} \leqq ch^{k+1} \|u\|_{k+3,\Omega}. \tag{3.44}$$

Proof. The existence and uniqueness of the solution u_h follows from the fact that u_h is the solution of the linear system of equations with a positive definite matrix.

Let U be the extension of the function u introduced at the beginning of this section, see (3.1). Let $v \in V_h$. Then

$$\|U - u_h\|_{1,\Omega_h} \leqq \|U - v\|_{1,\Omega_h} + \|v - u_h\|_{1,\Omega_h}. \tag{3.45}$$

From (1.36) and (2.5), (2.6) we have

$$\|v - u_h\|^2_{1,\Omega_h} \leqq c(\|v - u_h\|^2_{1,\Omega_h} + \|v - u_h\|^2_{0,\Gamma_h}) \leqq c b_h(v - u_h, v - u_h). \tag{3.46}$$

Using (3.4), the continuity assumption, Cauchy's inequality and (1.31) we get

$$b_h(v - u_h, v - u_h) = b_h(U - u_h, v - u_h) + b_h(v - U, v - u_h) \leqq$$

$$\leqq c\left[\left|\left(\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q, v - u_h\right)_{0,\Gamma_h}\right| + |v - U|_{1,\Omega_h} |v - u_h|_{1,\Omega_h} + \right.$$

$$\left. + \|v - U\|_{0,\Gamma_h} \|v - u_h\|_{0,\Gamma_h}\right] \leqq$$

$$\leqq c\left(\left\|\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q\right\|_{0,\Gamma_h} + \|U - v\|_{1,\Omega h}\right) \|v - u_h\|_{1,\Omega_h}.$$

Hence and from (3.45), (3.46) we obtain the abstract error estimate

$$\|U - u_h\|_{1,\Omega_h} \leqq c\left(\left\|\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q\right\|_{0,\Gamma_h} + \inf_{v\in V_h} \|U - v\|_{1,\Omega_h}\right).$$

Choosing $v = \pi_\Omega U$ and using (1.6), (3.6) we immediately get

$$\|U - u_h\|_{1,\Omega_h} \leqq ch^k \|U\|_{k+3,\tilde{\Omega}}. \tag{3.47}$$

Hence and from (3.1) we get (3.43).

We prove now the inequality (3.44) by means of the technique similar to that used by Ciarlet and Raviart [2] and Nedoma [6]. Let us denote

$$z = \begin{cases} U - u_h & \text{for } x \in \bar{\Omega}_h, \\ 0 & \text{for } x \in \tilde{\Omega} - \bar{\Omega}_h. \end{cases}$$

Let y be a solution of the homogeneous Newton problem

$$(3.48) \qquad -ly = z \quad \text{in } \Omega\,,$$

$$\frac{\partial y}{\partial \nu} + ay = 0 \quad \text{on } \Gamma\,.$$

If Γ is smooth enough then $y \in H^2(\Omega)$ and

$$(3.49) \qquad \|y\|_{2,\Omega} \leqq c\|z\|_{0,\Omega} \leqq c\|z\|_{0,\tilde{\Omega}} = c\|z\|_{0,\Omega_h}\,.$$

Using the Calderon theorem we extend the function y from Ω onto $\tilde{\Omega}$. In this way we obtain a function $Y \in H^2(\tilde{\Omega})$ such that

$$\|Y\|_{2,\tilde{\Omega}} \leqq c\|y\|_{2,\Omega}\,.$$

Therefore, (3.49) implies

$$(3.50) \qquad \|Y\|_{2,\tilde{\Omega}} \leqq c\|z\|_{0,\Omega_h}\,.$$

By simple calculation we get

$$(3.51) \qquad \|z\|^2_{0,\Omega_h} = \int_{\Omega_h - \Omega} z(z + LY)\,\mathrm{d}x - \int_{\Omega_h} zLY\,\mathrm{d}x\,.$$

Our aim is to bound both terms on the right hand side of the inequality (3.51) by $ch^{k+1}\|U\|_{k+3,\tilde{\Omega}}\|z\|_{0,\Omega_h}$. The Cauchy inequality and (3.50) give

$$(3.52) \qquad \left|\int_{\Omega_h - \Omega} z(z + LY)\,\mathrm{d}x\right| \leqq \|z\|_{0,\Omega_h - \Omega}\left(\|z\|_{0,\Omega_h - \Omega} + \|LY\|_{0,\Omega_h - \Omega}\right) \leqq$$

$$\leqq c\|z\|_{0,\Omega_h - \Omega}\left(\|z\|_{0,\Omega_h} + \|Y\|_{2,\Omega_h}\right) \leqq c\|z\|_{0,\Omega - \Omega_h}\|z\|_{0,\Omega_h}\,.$$

Let the element $K \in \Omega_h$ have a non empty intersection $K^* = K \cap (\Omega_h - \Omega)$ with the set $\Omega_h - \Omega$. For a point $x = (x', x_n) \in K^*$ we have

$$z(x) = z(x', x_n) = z(x', \psi(x')) + \int_{\psi(x')}^{x_n} \frac{\partial}{\partial \tau} z(x', \tau)\,\mathrm{d}\tau\,.$$

Squaring and using Cauchy's inequality we obtain

$$z^2(x) \leqq c\left({}_\psi z^2(x) + |x_n - \psi(x')|\left|\int_{\psi(x')}^{x_n}\left[\frac{\partial}{\partial \tau} z(x', \tau)\right]^2 \mathrm{d}\tau\right)\right.\,.$$

Integrating over K^* and using (1.28) we get

$$\|z\|^2_{0,K^*} \leqq c\left(\int_{S'}\left|\int_{\varphi(x')}^{\psi(x')} {}_\psi z^2(x')\,\mathrm{d}x_n\right|\mathrm{d}x' + \right.$$

$$\left. + \int_{S'}\left|\int_{\varphi(x')}^{\psi(x')} |x_n - \psi(x')| \int_{\psi(x')}^{x_n}\left[\frac{\partial}{\partial \tau} z(x', \tau)\right]^2 \mathrm{d}\tau\,\mathrm{d}x_n\right|\mathrm{d}x'\right) \leqq$$

$$\leqq c\|\psi - \varphi\|_{0,\infty,S'}\left(\|{}_\psi z\|^2_{0,S'} + \|\psi - \varphi\|_{0,\infty,S'}\,\|z\|^2_{1,K}\right) \leqq$$

$$\leqq ch^{k+1}\left(\|z\|^2_{0,S} + \|z\|^2_{1,K}\right).$$

Summing over all elements K^* and making use of (1.31) and (3.47) we see that $\|z\|^2_{0,\Omega_h-\Omega} \leqq ch^{3k+1}\|U\|^2_{k+3,\tilde{\Omega}}$. Hence and from (3.52) we have

$$(3.53) \qquad \left|\int_{\Omega_h-\Omega} z(z + LY)\,dx\right| \leqq ch^{1/2(3k+1)}\|U\|_{k+3,\tilde{\Omega}}\|z\|_{0,\Omega_h}.$$

The Green theorem yields

$$(3.54) \quad -\int_{\Omega_h} zLY\,dx = a_h(z, Y) - \left(z, \frac{\partial Y}{\partial \nu_h}\right)_{0,\Gamma_h} = b_h(z, Y) - \left(\frac{\partial Y}{\partial \nu_h} + \pi_\Gamma a Y, z\right)_{0,\Gamma_h}.$$

Let Y^h be the mollifier satisfying (3.42) with some $i \geqq k + 1$. Then

$$(3.55) \qquad b_h(z, Y) = b_h(z, Y - Y^h) + b_h(z, Y - \pi_\Omega Y^h) + b_h(z, \pi_\Omega Y^h).$$

From (3.47), (3.42) and (3.50) we get

$$(3.56) \quad |b_h(z, Y - Y^h)| \leqq c\|z\|_{1,\Omega_h}\,\|Y - Y^h\|_{1,\Omega_h} \leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}}\,\|Y\|_{2,\tilde{\Omega}} \leqq$$
$$\leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}}\,\|z\|_{0,\Omega_h}.$$

Similarly (3.47), (1.6), (3.42) and (3.50) yield

$$(3.57) \qquad |b_h(z, Y^h - \pi_\Omega Y^h)| \leqq c\|z\|_{1,\Omega_h}\,\|Y^h - \pi_\Omega Y^h\|_{1,\Omega_h} \leqq$$
$$\leqq ch^{2k}\|U\|_{k+3,\tilde{\Omega}}\,\|Y^h\|_{k+1,\Omega_h} \leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}}\,\|Y\|_{2,\tilde{\Omega}} \leqq$$
$$\leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}}\,\|z\|_{0,\Omega_h}.$$

(3.4), (3.11) and (3.10) give

$$(3.58) \qquad |b_h(z, \pi_\Omega Y^h)| = \left|\left(\frac{\partial U}{\partial \nu_h} + \pi_\Gamma a U - \pi_\Gamma q, \pi_\Omega Y^h\right)_{0,\Gamma_h}\right| \leqq$$
$$\leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}}\,\|\pi_\Omega Y^h\|_{1,\Gamma_h}.$$

Using (1.23) we obtain for an element $S \in \Gamma_h$

$$(3.59) \qquad \|{}_\psi\pi_\Omega Y^h\|_{1,S'} = \|{}_\psi\pi_S Y^h\|_{1,S'} = \|\pi'_S({}_\psi Y^h)\|_{1,S'} = \|\pi'_S({}_\varphi Y^h)\|_{1,S'} \leqq$$
$$\leqq \|{}_\varphi Y^h\|_{1,S'} + \|{}_\varphi Y^h - \pi'_S({}_\varphi Y^h)\|_{1,S'}.$$

Let $\varkappa = [\frac{1}{2}(n + 3)]$. Then $2(\varkappa - 1) > n - 1$, $\varkappa - 1 \leqq k + 1$ and consequently from (1.19) we have

$$\|{}_\varphi Y^h - \pi'_S({}_\varphi Y^h)\|_{1,S'} \leqq ch^{\varkappa-2}\|{}_\varphi Y^h\|_{\varkappa-1,S'}.$$

Hence, from (3.59), the trace theorem, (3.42) and (3.50) we get

$$\|\pi_\Omega Y^h\|^2_{1,\Gamma_h} = \sum_{S\in\Gamma_h}\|{}_\psi\pi_\Omega Y^h\|^2_{1,S'} \leqq c\sum_{S\in\Gamma_h}\left(\|{}_\varphi Y^h\|^2_{1,S'} + h^{2(\varkappa-2)}\|{}_\varphi Y^h\|^2_{\varkappa-1,S'}\right) \leqq$$
$$\leqq c\left(\|Y^h\|^2_{1,\Gamma} + h^{2(\varkappa-2)}\|Y^h\|^2_{\varkappa-1,\Gamma}\right) \leqq c\left(\|Y^h\|^2_{2,\Omega} + h^{2(\varkappa-2)}\|Y^h\|^2_{\varkappa,\Omega}\right) \leqq$$
$$\leqq c\|Y\|^2_{2,\tilde{\Omega}} \leqq c\|z\|^2_{0,\Omega_h}.$$

This inequality together with (3.58) gives

$$|b_h(z, \pi_\Omega Y^h)| \leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}} \|z\|_{0,\Omega_h} . \tag{3.60}$$

Applying the Cauchy inequality, (1.31), (3.41), (3.47) and (3.50) we obtain

$$\left|\left(\frac{\partial Y}{\partial \nu_h} + \pi_\Gamma a Y, z\right)_{0,\Gamma_h}\right| \leqq \left\|\frac{\partial Y}{\partial \nu_h} + \pi_\Gamma a Y\right\|_{0,\Gamma_h} \|z\|_{0,\Gamma_h} \leqq \tag{3.61}$$

$$\leqq ch^{(k+1)/2}\|Y\|_{2,\tilde{\Omega}} \|z\|_{1,\Omega_h} \leqq ch^{(3k+1)/2}\|U\|_{k+3,\tilde{\Omega}} \|z\|_{0,\Omega_h} .$$

Then from (3.54), (3.55), (3.56), (3.57), (3.60) and (3.61) we get

$$\left|-\int_{\Omega_h} zLY \, \mathrm{d}x\right| \leqq ch^{k+1}\|U\|_{k+3,\tilde{\Omega}} \|z\|_{0,\Omega_h} \tag{3.62}$$

and (3.44) follows from (3.51), (3.53), (3.62) and (3.1). □

Remark 3.2. Let us use the isoparametric numerical integration, see [2], [6], for approximate computation of the integrals $(\cdot, \cdot)_{0,K}$ and $(\cdot, \cdot)_{0,S}$ appearing in the forms b_h, d_h, see (2.12). We obtain new forms B_h, D_h and solve the problem

$$B_h(U_h, v) = D_h(v) \quad \forall v \in V_h . \tag{3.63}$$

Let the quadrature formula on the reference set $\hat{K}$ be of degree $d_K \geqq \max(1, 2k - 2)$ and let the quadrature formula on the reference set $\hat{S}$ be of degree $d_S \geqq 2k - 1$ with positive weights and with the $\hat{P}_S$-unisolvent set of integration nodes. Then under the hypotheses of Theorem 3.1 we have

$$\|u - U_h\|_{1,\Omega\cap\Omega_h} \leqq ch^k\|u\|_{k+3,\Omega} , \tag{3.64}$$

$$\|u - U_h\|_{0,\Omega\cap\Omega_h} \leqq ch^{k+1}\|u\|_{k+3,\Omega} . \tag{3.65}$$

We leave the proof of this assertion to the reader. □

Remark 3.3. Starting from the results contained in Theorem 3.1 we can analyse the parabolic problem

$$p(x)\frac{\partial w}{\partial t} + lw = f(x, t) , \quad x \in \Omega , \quad t \in (0, T] , \tag{3.66}$$

$$\frac{\partial w}{\partial \nu} + a(x) w = q(x, t) , \quad x \in \Omega , \quad t \in (0, T] ,$$

$$w(x, 0) = w_0(x) , \quad x \in \Omega$$

and following Nedoma's paper [6] we can obtain the optimal estimate of the discretisation error in the L_2 norm. □

Remark 3.4. It is possible to discretize the problem (2.1) by means of k-regular quadrilateral isoparametric finite elements (see e.g. [1], [2]) and to prove results analogous to those given in Theorem 3.1. □

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Souhrn

ŘEŠENÍ ELIPTICKÝCH PROBLÉMŮ DRUHÉHO ŘÁDU S NEWTONOVOU OKRAJOVOU PODMÍNKOU METODOU KONEČNÝCH PRVKŮ

Libor Čermák

V práci se analyzuje konvergence přibližného řešení eliptického problému druhého řádu s Newtonovou okrajovou podmínkou v n-rozměrné ohraničné oblasti ($n \geqq 2$) získaného metodou konečných prvků. Používají se simpliciální izoparametrické elementy. Jsou dokázány odhady diskretizační chyby a to jak v H^1 tak i v L_2 normě.

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