Oldřich Kropáč

Some properties and applications of probability distributions based on MacDonald function

Aplikace matematiky, Vol. 27 (1982), No. 4, 285–302

Persistent URL: `http://dml.cz/dmlcz/103973`

SVAZEK 27 (1982) APLIKACE MATEMATIKY ČÍSLO 4

SOME PROPERTIES AND APPLICATIONS OF PROBABILITY DISTRIBUTIONS BASED ON MACDONALD FUNCTION

OLDŘICH KROPÁČ

(Received April 3, 1981)

1. INTRODUCTION

The introduction of more and more complicated probability models into different application fields gives rise to increasing demands on necessary mathematical tools. One feature of this general tendency consists in the need for some higher transcendental functions to be included in to the currently used analytical apparatus. In the probability theory the error integral is of basic importance and also the incomplete gamma function is often used. Considerable application possibilities can be found for the modified Bessel function of the second kind (or the modified Hankel function), which is also called the MacDonald function. The last denotation will be used in this paper particularly for the sake of brevity of expression. It is the aim of this paper to recall some basic properties of this function and of some probability distribution functions based on it. Some interesting applications in probability theory will be also discussed.

2. DEFINITION AND BASIC ANALYTICAL PROPERTIES OF THE MACDONALD FUNCTION

It is well known that the generating differential equation of Bessel functions has the form

$$u'' + (1/z)\,u' + (1 - n^2/z^2)\,u = 0\,, \tag{1}$$

the particular solution of which is the Bessel function of the first kind of order n

$$J_n(z) = \sum_{m=0}^{\infty} (-1)^m\,(z/2)^{2m+n}\,[\Gamma(m + 1)\,\Gamma(m + n + 1)]^{-1}\,. \tag{2}$$

Substituting in (1) iz for z we obtain the modified Bessel equation

(3) $$u'' + (1/z)\, u' - (1 + n^2/z^2)\, u = 0\,,$$

the general solution of which may be written in the form

(4) $$u = C_1\, \mathrm{I}_n(z) + C_2\, \mathrm{K}_n(z)\,,$$

where

(5) $$\mathrm{I}_n(z) = \exp(-n\pi i/2)\, \mathrm{J}_n[z \exp(\pi i/2)]$$

is the modified Bessel function of the first kind and

(6) $$\mathrm{K}_n(z) = \pi(2 \sin n\pi)^{-1}\, [\mathrm{I}_{-n}(z) - \mathrm{I}_n(z)]$$

is the modified Bessel function of the second kind or the MacDonald function.

The theory and analytical properties of Bessel functions are described in detail in Watson [17] and in a concise form in [1]. Here we shall recall some of the most important properties of the MacDonald function, which are interesting from the point of view of applications in the probability theory.

It follows from the definition that for n real and z positive, $\mathrm{K}_n(z)$ is real. Further, the following relations hold:

(7) $$\mathrm{K}_{-n}(z) = \mathrm{K}_n(z)\,,$$

(8) $$\mathrm{K}_{n-1}(z) - \mathrm{K}_{n+1}(z) = -(2n/z)\, \mathrm{K}_n(z)\,,$$

(9) $$\mathrm{K}'_n(z) = -(1/2)\, [\mathrm{K}_{n+1}(z) + \mathrm{K}_{n-1}(z)] = -(n/z)\, \mathrm{K}_n(z) - \mathrm{K}_{n-1}(z)\,,$$

(10) $$\int z^n\, \mathrm{K}_{n-1}(z)\, \mathrm{d}z = -z^n\, \mathrm{K}_n(z)\,.$$

One subclass of MacDonald functions which is important in applications is that containing functions with integer indices n. These functions and their derivatives may be expressed recurrently by means of $\mathrm{K}_0(z)$ and $\mathrm{K}_1(z)$, the values of which may be found in mathematical tables (see e.g. [6], [17]).

Another interesting subclass in formed by functions with the indices equal to halves of odd numbers. These functions may be expressed by means of elementary functions following the relation

(11) $$\mathrm{K}_{n+1/2}(z) = (\pi/(2z))^{1/2} \exp(-z) \sum_{m=0}^{n} (n + m)!\, [m!\, (n - m)!]^{-1}\, (2z)^{-m}$$

or the recurrent formula

(12) $$\mathrm{K}_{n+1/2}(z) = (-1)^n\, [\pi/(2z)]^{1/2}\, z^{n+1} [\mathrm{d}/(z\, \mathrm{d}z)]^n\, (\mathrm{e}^{-n}/z)\,.$$

Remember also that the function $\mathrm{K}_n(z)$ is continuous in n, thus

(13) $$\lim_{n \to p} \mathrm{K}_n(z) = \mathrm{K}_p(z)\,.$$

Concluding this review chapter we mention some important integral representations of the MacDonald function [2]:

$$\int_0^\infty x^{n-1} \exp(-a/x - bx)\,dx = 2(a/b)^{n/2} K_n(2\sqrt{(ab)}), \quad a > 0, \quad b > 0, \tag{14}$$

$$\int_0^\infty x^{n-1} \exp(-a/x^k - bx^k)\,dx = (2/k)(a/b)^{n/(2k)} K_{n/k}(2\sqrt{(ab)}), \quad a > 0, \quad b > 0, \tag{15}$$

(16)

$$\int_0^u x^{-2n}(u^2 - x^2)^{n-1} \exp(-a/x)\,dx = \pi^{-1/2}(2/a)^{n-1/2} u^{n-3/2} \Gamma(n) K_{n-1/2}(a/u), \quad a > 0, \quad u > 0,$$

$$\int_u^\infty x^{p-1}(x - u)^{p-1} \exp(-bx)\,dx = \tag{17}$$

$$= \pi^{-1/2}(u/b)^{p-1/2} \Gamma(p) \exp(-bu/2) K_{p-1/2}(bu/2), \quad u > 0, \quad p > 0, \quad b > 0,$$

$$\int_u^\infty (x^2 - u^2)^{n-1} \exp(-bx)\,dx = \pi^{-1/2}(2u/b)^{n-1/2} \Gamma(n) K_{n-1/2}(ub), \quad u > 0, \quad n > 0, \quad b > 0. \tag{18}$$

For applications in the probability theory the following definite integral of the MacDonald function will be useful:

$$\int_0^\infty x^p K_n(x/b)\,dx = 2^{p-1} b^{p+1} \Gamma[(1 + p + n)/2] \Gamma[(1 + p - n)/2], \quad b > 0, \quad (p + 1 \pm n) > 0. \tag{19}$$

3. PROBABILITY DISTRIBUTION FUNCTIONS OF THE TYPE $x^{n+p} K_n(x)$

In probability problems, the distribution functions based on the MacDonald function most often have the form $x^n K_n(x)$ or $x^{n+1} K_n(x)$. Using equation (19) we can easily calcuate the normalizing constants.

3.1. Type $x^n K_n(x)$, $x \in \langle 0, \infty)$

The probability density function of this distribution with the scale parameter b included has the form

$$f(x) = \pi^{-1/2} 2^{-n+1} [\Gamma(n + 1/2) b]^{-1} (x/b)^n K_n(x/b), \tag{20}$$

where the index n has the meaning of the shape parameter. The probability densities $f(x)$ for $n = 0(1/2)\,5/2$ and $b = 1$ are shown in Fig. 1.

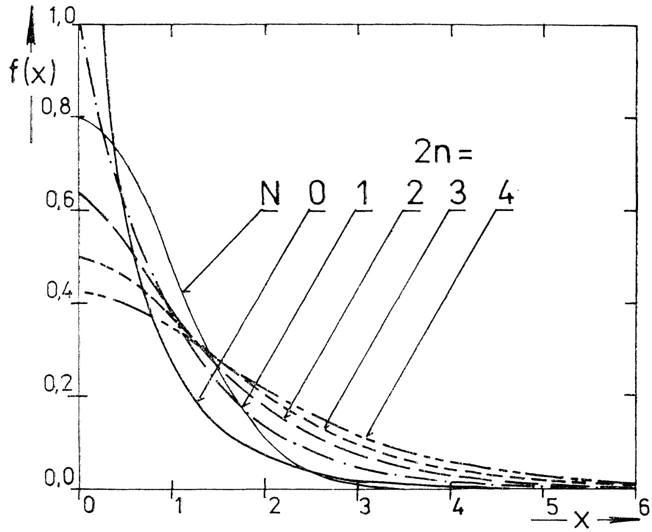

Fig. 1. Probability densities of distributions of the type $x^n\,\mathrm{K}_n(x)$, $x \geqq 0$, for $n = 0\,(\frac{1}{2})\,\frac{5}{2}$, $b = 1$. N — one-sided Gaussian distribution.

The distribution function $F(x)$ is easily expressible only for $n = k + 1/2$, k integer. Otherwise, $F(x)$ may be expressed only through $\mathrm{K}_n(x)$, $\mathrm{K}_{n-1}(x)$ and $\mathrm{L}_n(x)$, $\mathrm{L}_{n-1}(x)$, the latter being the modified Struve functions [17].

The k-th moment of (20) may be expressed via equation (19) in the form

$$m_k(n) = 2^k b^k\,\Gamma[(k+1)/2 + n]\,\Gamma[(k+1)/2]\,[\sqrt{\pi}\,\Gamma(n + 1/2)]^{-1} \tag{21}$$

and detailed specification for the expected value E, dispersion D, third and fourth central moments, μ_3 and μ_4, respectively, yields

$$E(n) = 2b\pi^{-1/2}\,\Gamma(n+1)\,[\Gamma(n+1/2)]^{-1}\,, \tag{21a}$$

$$D(n) = b^2\{(2n+1) - (4/\pi)\,[\Gamma(n+1)]^2\,[\Gamma(n+1/2)]^{-2}\}\,, \tag{21b}$$

$$\mu_3(n) = 2b^3\pi^{-1/2}\{-(2n-1)\,\Gamma(n+1)\,[\Gamma(n+1/2)]^{-1} + \\ + (8/\pi)\,[\Gamma(n+1)]^3\,[\Gamma(n+1/2)]^{-3}\}\,, \tag{21c}$$

$$\mu_4(n) = (b^4/\pi)\,\{3(2n+1)(2n+3) - 8(2n+5)\,[\Gamma(n+1)]^2\,. \\ .\,[\Gamma(n+1/2)]^{-2} - 48[\Gamma(n+1)]^4\,[\Gamma(n+1/2)]^{-4}\,. \tag{21d}$$

A graphical representation of dependencies of E, D (for $b = 1$), skew $A = \mu_3 D^{-3/2}$ and curtosis $B = \mu_4 D^{-2} - 3$ on n is given in Fig. 2.

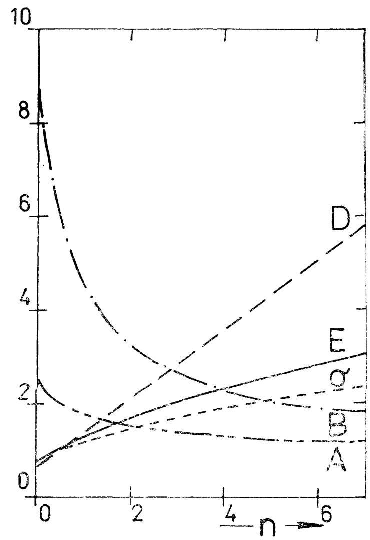

Fig. 2. Moments of distributions of the type $x^n \mathrm{K}_n(x)$, $x \geqq 0$, $b = 1$: dependence on n.

3.2. Type $|x|^n \mathrm{K}_n(|x|)$, $x \in (-\infty, \infty)$

In practical applications the random variables defined on $x \in (-\infty, \infty)$ often occur with a symmetric probability density

$$f(x) = \pi^{-1/2} 2^{-n} [\Gamma(n + 1/2)\, b]^{-1} (|x|/b)^n \mathrm{K}_n(|x|/b) \tag{22}$$

and with moments

$$m_{2k+1}(n) = 0\,, \quad k = 0, 1, 2, \ldots, \tag{23}$$

$$m_{2k}(n) = \pi^{-1/2} (2b)^{2k}\, \Gamma(k + n + 1/2)\, \Gamma(k + 1/2)\, [\Gamma(n + 1/2)]^{-1}\,,$$

particularly

$$D(n) = b^2(2n + 1)\,, \tag{23a}$$

$$\mu_4(n) = 3b^4(2n + 1)(2n + 3)\,, \tag{23b}$$

where of

$$B(n) = 6(2n + 1)^{-1}\,. \tag{23c}$$

The dependencies of D/b^2 and B on n are shown in Fig. 3.

3.3. Type $x^{n+1} \mathrm{K}_n(x)$, $x \in \langle 0, \infty)$

The probability density of this distribution is

$$f(x) = 2^{-n}[\Gamma(n + 1)\, b]^{-1}\, (x/b)^{n+1}\, \mathrm{K}_n(x/b) \tag{24}$$

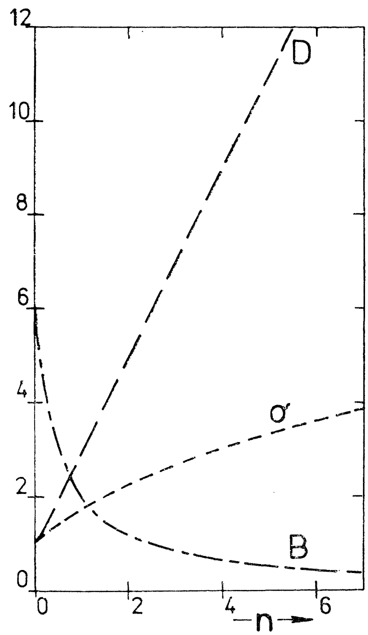

Fig. 3. Moments of distributions of the type $|x|^n\, \mathrm{K}_n(|x|)$, $x \in \mathrm{R}$, $b = 1$: dependence on n.

and the distribution function may be expressed by using (10) in the form

$$F(x) = 1 - 2^{-n}[\Gamma(n + 1)]^{-1}\, (x/b)^{n+1}\, \mathrm{K}_{n+1}(x/b)\,. \tag{25}$$

The probability densities $f(x)$ for $n = -1/2(1/2)\, 2$ are shown in Fig. 4.

For general moments the relation

$$m_k(n) = 2^k b^k\, \Gamma(n + 1 + k/2)\, \Gamma(k/2 + 1)\, [\Gamma(n + 1)]^{-1} \tag{26}$$

may be easily derived with the specific values

$$E(n) = \pi^{1/2} b\, \Gamma(n + 3/2)\, [\Gamma(n + 1)]^{-1}\,, \tag{26a}$$

$$D(n) = b^2\{4(n + 1) - \pi[\Gamma(n + 3/2)]^2\, [\Gamma(n + 1)]^{-2}\}\,, \tag{26b}$$

$$\mu_3(n) = 2\pi^{1/2} b^3\{-3(n + 1/2)\, \Gamma(n + 3/2)\, [\Gamma(n + 1)]^{-1} + \pi[\Gamma(n + 3/2)]^3\, [\Gamma(n + 1)]^{-3}\}\,, \tag{26c}$$

(26d) $$\mu_4(n) = b^4\{32(n+1)(n+2) - 12\pi[\Gamma(n+3/2)]^2[\Gamma(n+1)]^{-2} - 3\pi^2[\Gamma(n+3/2)]^4[\Gamma(n+1)]^{-4}\}.$$

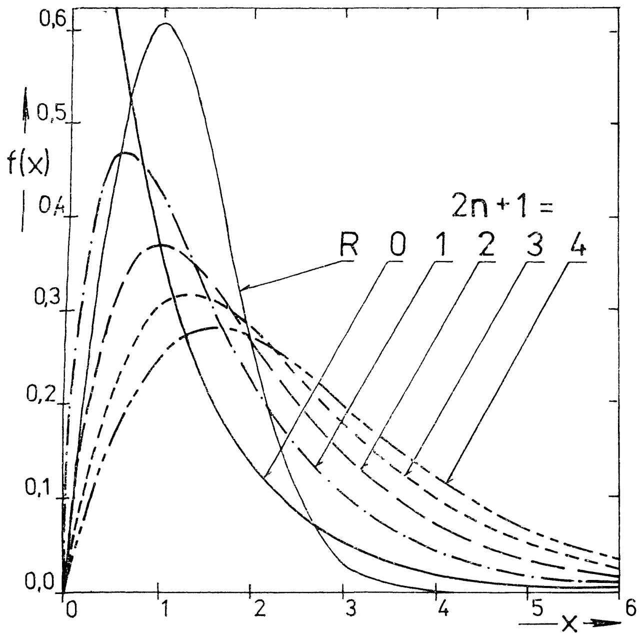

Fig. 4. Probability densities of distributions of the type x^{n+1} $K_n(x)$, $x \geqq 0$, for $n = -\frac{1}{2}(\frac{1}{2})\ 2$, $b = 1$.
R — Rayleigh distribution.

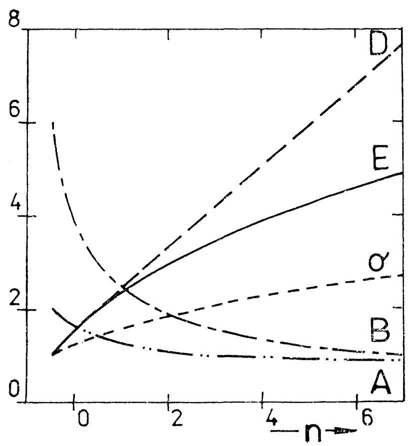

Fig. 5. Moments of distributions of the type x^{n+1} $K_n(x)$, $x \geqq 0$, $b = 1$; dependence on n.

The dependencies of E, D (for $b = 1$) and of A, B on n are shown in Fig. 5.

3.4. Type $x^{n+p}\,\mathrm{K}_n(x)$, $x \in \langle 0, \infty)$

This is a generalization of the preceding types. The probability density of this distribution is

$$(27)\qquad f(x) = 2^{-n-p+1}\{b\,\Gamma[n + (p+1)/2]\,\Gamma[(p+1)/2]\}^{-1}\,(x/b)^{n+p}\,\mathrm{K}_n(x/b)\,.$$

Its distribution function $F(x)$ cannot be expressed generally in a simple form. For general moments the following relation holds:

$$(28)\qquad m_k(n, p) = 2^k b^k\,\Gamma[n + (k+p+1)/2]\,\Gamma[(k+p+1)/2]\,.$$
$$.\,\{\Gamma[n + (p+1)/2]\,\Gamma[(p+1)/2]\}^{-1}\,,$$

which may be easily specified for the values of k desired.

The probabilities for the case $n = 0$, $p = 0(1)\,4$ are shown in Fig. 6, and for the case $n = 1$, $p = 0(1)\,4$, in Fig. 7. Let us recall that the case $n = 1/2$ may be expressed by means of the gamma distribution.

3.5. Some convergence properties

Let us discuss the asymptotic behaviour of the above mentioned distributions for $n \to \infty$. This analysis is simplest for the symmetric type (22). It follows from

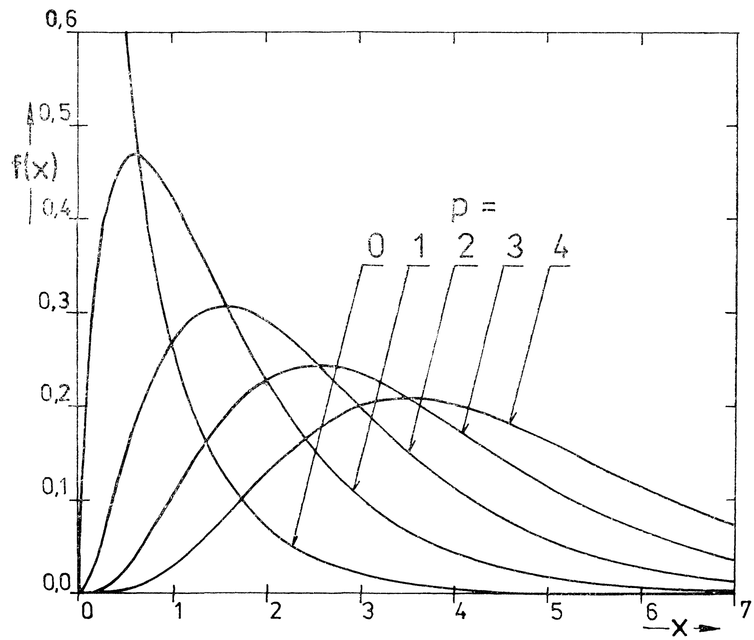

Fig. 6. Probability densities of distributions of the type $x^p\,\mathrm{K}_0(x)$, $x \geqq 0$, for $p = 0(1)\,4$, $b = 1$.

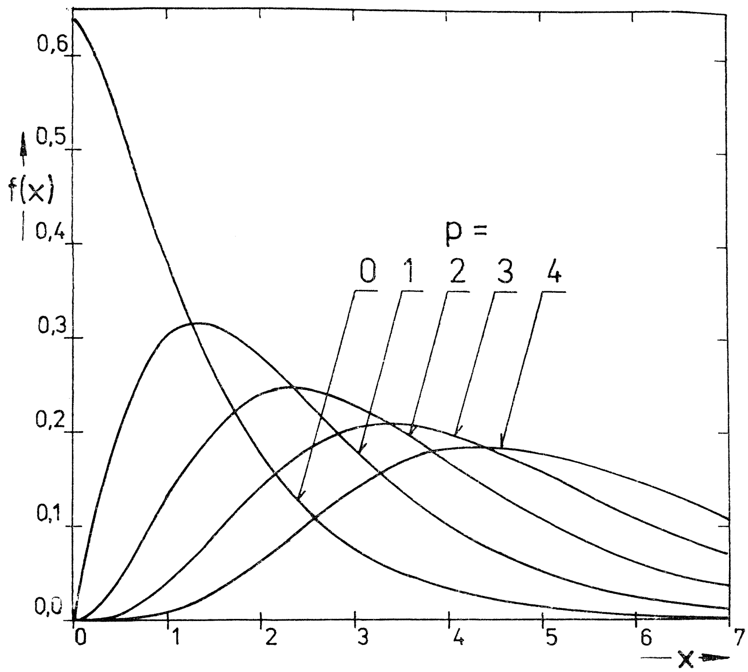

Fig. 7. Probability densities of distributions of the type $x^{p+1}\,\mathrm{K}_1(x)$, $x \geqq 0$, for $p = 0(1)\,4$, $b = 1$.

(23c) that $\lim_{n\to\infty} B(n) = 0$ and all higher even semiinvariants for n increasing also asymptotically approach zero. Thus the distribution (22) with increasing n approaches the normal distribution. Using the simple relation between (20) and (22) we may state that the distribution (20) for increasing n approaches the one-sided Gaussian distribution. Finally, it may be shown that between the random variable X described by means of $f_x(x)$ following (22) and the random variable Y described by means of $f_y(y)$ following (24) the relation

$$(29) \qquad f_x^{(n)}(x) = \pi^{-1} \int_x^{\infty} f_y^{(n-1/2)}(y)\,(y^2 - x^2)^{-1/2}\,\mathrm{d}y$$

holds (see e.g. [8], [10], [11]), where the probability density functions f_x, f_y have been assigned the corresponding values of indices. After passing to the limit $n \to \infty$ we obtain on the left-and right-hand sides of (29) the normal and the Rayleigh distribution, respectively. Thus the distribution (24) with increasing n approaches the Rayleigh distribution.

The classes of distribution functions considered and their convergence for increasing n may be vizualized in a graph with axes (A^2, B) introduced by Pearson in a slightly different arrangement of the axes (Fig. 8). In this graphical representation the rapid convergence of the type (22) towards the normal distribution is distinctly visible. The one-sided distribution (20) forms a continuous sequence of functions from the

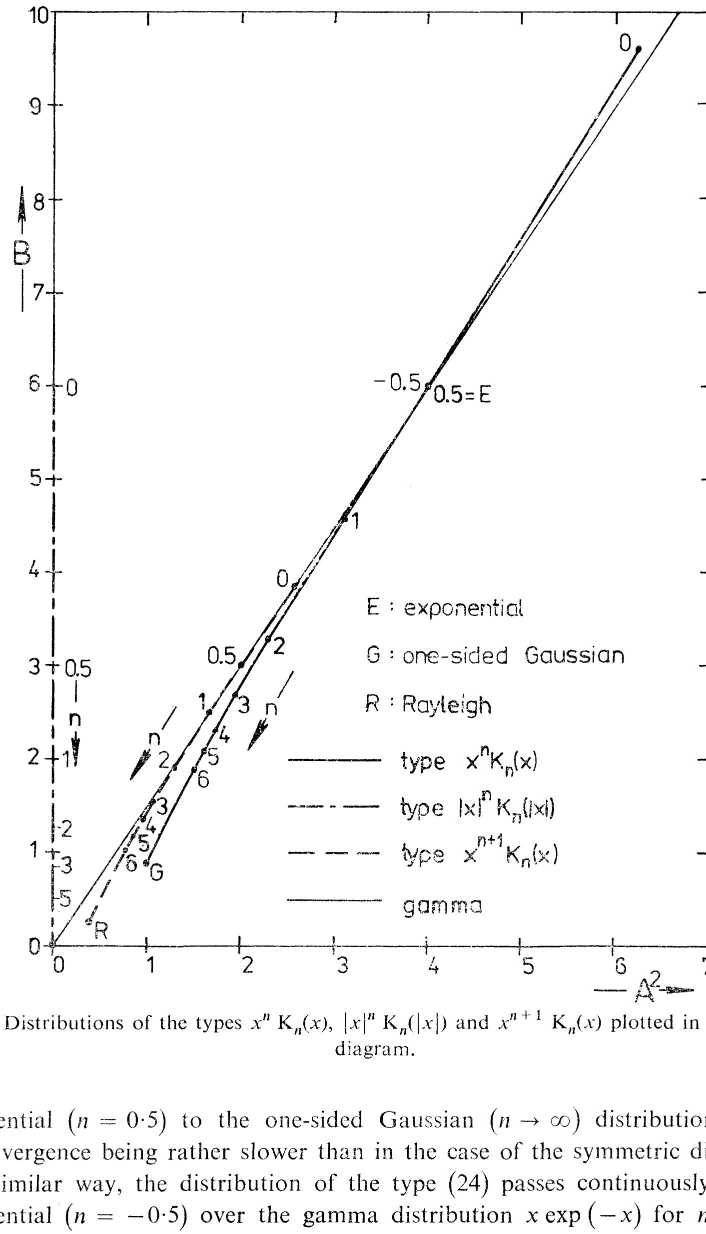

Fig. 8. Distributions of the types x^n $K_n(x)$, $|x|^n$ $K_n(|x|)$ and x^{n+1} $K_n(x)$ plotted in the $A^2 - B$ diagram.

exponential ($n = 0{\cdot}5$) to the one-sided Gaussian ($n \to \infty$) distribution, the rate of convergence being rather slower than in the case of the symmetric distribution. In a similar way, the distribution of the type (24) passes continuously from the exponential ($n = -0{\cdot}5$) over the gamma distribution $x \exp(-x)$ for $n = 0{\cdot}5$ up to the Rayleigh distribution ($n \to \infty$). The rate of convergence is approximately of the same order as for the one-sided distribution (20).

4. APPLICATIONS

The application of the MacDonald function in probability theory is connected with different types of composed distributions, the analytical expressions of which in form of certain integrals lead just to the MacDonald function. In this chapter we shall give some typical examples.

1. Let X, Y be independent normal random variables with zero expected values and standard deviations b_x, b_y, respectively. Then the random variable $Z = XY$ has the probability density

$$f(z) = (\pi b_x b_y)^{-1} \mathrm{K}_0[z/(b_x b_y)] .$$

When $E_x \neq 0$, $E_y \neq 0$ then the probability density $f(z)$ will be expressed by means of an infinite series, the terms of which contain products of the MacDonald function K_n and the modified Bessel function of the first kind I_{2n}. Even a more complicated relation will be obtained for X and Y correlated (for details see [4]). The MacDonald function appears also in distributions of a product $Z = XY$ when X, Y are independent gamma, Pearson χ or generalized gamma distributed. General expressions with some interesting particular cases are summarized in Table 1. All random variables X, Y and Z considered in this table are defined on $\langle 0, \infty)$.

2. Let X, Y be independent random variables with probability densities of the gamma (Pearson III) type, i.e.

$$f_x(x) = [\Gamma(p + 1)]^{-1} x^p \exp(-x) ,$$

$$f_y(y) = [\Gamma(q + 1)]^{-1} y^q \exp(-y) .$$

Then it may be shown (see e.g. [12]) that the random variable $U = X - Y$ has the probability density

$$f_u(u) = 1/2(u/2)^{(p+q)/2} [\Gamma(p + 1)]^{-1} \mathrm{W}_{(p-q)/2,(p+q+1)/2}(2u) ,$$

where $\mathrm{W}_{k,m}(u)$ is the confluent hypergeometric (Whittaker) function which for $p = q$ reduces to the MacDonald function, thus yielding

$$f_u(u) = \pi^{-1/2}[\Gamma(p + 1)]^{-1} (u/2)^{p+1/2} \mathrm{K}_{p+1/2}(u) .$$

3. To express the distributions of mean values of random samples from exponential and Laplace populations, the functions of the form $\exp(-b|x|) \cdot |x|^n \mathrm{K}_n(|x|)$ have been used in [14].

4. The MacDonald function also appears in composed distributions created from conditional distributions of the exponential type, the scale parameter of which is considered to be a random gamma distributed variable [16].

Table 1. Some distributions of the product $Z = XY$ (X, Y independent) containing the MacDonald function.

No.	$f_x(x)$	$f_y(y)$	$f_z(z)$
1	$\frac{1}{b_x\,\Gamma(m)}\left(\frac{x}{b_x}\right)^{m-1}\exp\left(-\frac{x}{b_x}\right)$ gamma $x \geqq 0,\ b_x > 0,\ m > 0$	$\frac{1}{b_y\,\Gamma(n)}\left(\frac{y}{b_y}\right)^{n-1}\exp\left(-\frac{y}{b_y}\right)$ gamma $y \geqq 0,\ b_y > 0,\ n > 0$	$\frac{2}{b_x b_y\,\Gamma(m)\,\Gamma(n)}\left(\frac{z}{b_x b_y}\right)^{(m+n)/2-1} \mathrm{K}_{\lvert m-n\rvert}\left(2\sqrt{\frac{z}{b_x b_y}}\right)$
1a	$\frac{1}{b_x}\exp\left(-\frac{x}{b_x}\right)$ exponential $x \geqq 0,\ b_x > 0$	$\frac{1}{b_y}\exp\left(-\frac{y}{b_y}\right)$ exponential $y \geqq 0,\ b_y > 0$	$\frac{2}{b_x b_y}\mathrm{K}_0\left(2\sqrt{\frac{z}{b_x b_y}}\right)$
2	$\frac{2^{-m/2+1}}{b_x\,\Gamma(m/2)}\left(\frac{x}{b_x}\right)^{m-1}\exp\left(-\frac{x^2}{2b_x^2}\right)$ Pearson χ $x \geqq 0,\ b_x > 0,\ m > 0$	$\frac{2^{-n/2+1}}{b_y\,\Gamma(n/2)}\left(\frac{y}{b_y}\right)^{n-1}\exp\left(-\frac{y^2}{2b_y^2}\right)$ Pearson χ $y \geqq 0,\ b_y > 0,\ n > 0$	$\frac{2^{-m/2-n/2+2}}{b_x b_y\,\Gamma(m/2)\,\Gamma(n/2)}\left(\frac{z}{b_x b_y}\right)^{(m+n)/2-1} \mathrm{K}_{\lvert m-n\rvert/2}\left(\frac{z}{b_x b_y}\right)$
2a	$(2/\pi)^{1/2}\,b_x^{-1}\exp\left(-\frac{x^2}{2b_x^2}\right)$ one-sided Gaussian $x \geqq 0$, $b_x > 0$	$(2/\pi)^{1/2}\,b_y^{-1}\exp\left(-\frac{y^2}{2b_y^2}\right)$ one-sided Gaussian $y \geqq 0$, $b_y > 0$	$\frac{2}{\pi b_x b_y}\mathrm{K}_0\left(\frac{z}{b_x b_y}\right)$
2b	$\frac{x}{b_x^2}\exp\left(-\frac{x^2}{2b_x^2}\right)$ Rayleigh $x \geqq 0,\ b_x > 0$	$\frac{y}{b_y^2}\exp\left(-\frac{y^2}{2b_y^2}\right)$ Rayleigh $y \geqq 0,\ b_y > 0$	$\frac{z}{b_x^2 b_y^2}\mathrm{K}_0\left(\frac{z}{b_x b_y}\right)$

Table 1 (continued)

No.	$f_x(x)$	$f_y(y)$	$f_z(z)$
2c	$(2/\pi)^{1/2} b_x^{-1} \exp\left(-\frac{x^2}{2b_x^2}\right)$ one-sided Gaussian $x \geqq 0$, $b_x > 0$	$\frac{y}{b_y^2} \exp\left(-\frac{y^2}{2b_y^2}\right)$ Rayleigh $y \geqq 0$, $b_y > 0$	$\frac{1}{b_x b_y} \exp\left(-\frac{z}{b_x b_y}\right)$ exponential
3	$\frac{k}{b_x \Gamma(m/k)} \left(\frac{x}{b_x}\right)^{m-1} \exp\left[-\left(\frac{x}{b_x}\right)^k\right]$ generalized gamma $x \geqq 0$ $b_x > 0$, $m > 0$, $k > 0$	$\frac{k}{b_y \Gamma(n/k)} \left(\frac{y}{b_y}\right)^{n-1} \exp\left[-\left(\frac{y}{b_y}\right)^k\right]$ generalized gamma $y \geqq 0$, $b_y > 0$, $n > 0$, $k > 0$	$\frac{2k}{b_x b_y \Gamma(m/k) \Gamma(n/k)} \left(\frac{z}{b_x b_y}\right)^{(m+n)/2-1} \cdot$ $\cdot \mathrm{K}_{\lvert m-n \rvert/k} \left[2\left(\frac{z}{b_x b_y}\right)^{k/2}\right]$
3a	$\frac{k}{b_x} \left(\frac{x}{b_x}\right)^{k-1} \exp\left[-\left(\frac{x}{b_x}\right)^k\right]$ Weibull $x \geqq 0$, $b_x > 0$, $k > 0$	$\frac{k}{b_y} \left(\frac{y}{b_y}\right)^{k-1} \exp\left[-\left(\frac{y}{b_y}\right)^k\right]$ Weibull $y \geqq 0$, $b_y > 0$, $k > 0$	$\frac{2}{b_x b_y} \left(\frac{z}{b_x b_y}\right)^{k-1} \mathrm{K}_0 \left[2\left(\frac{z}{b_x b_y}\right)^{k/2}\right]$
4	$\frac{1}{b_x} \exp\left(-\frac{x}{b_x}\right)$ exponential $x \geqq 0$, $b_x > 0$	$\frac{2}{\pi} (b_y^2 - y^2)^{-1/2}$ arcsinus $y \in \langle -b_y, b_y \rangle$	$\frac{2}{\pi b_x b_y} \mathrm{K}_0 \left(\frac{z}{b_x b_y}\right)$

Denoting the probability density of a conditional variable X as $f(x/\lambda)$, the scale parameter Λ having the probability density $g(\lambda)$, we express the probability density of the unconditioned variable X^* as

$$f^*(x) = \int_0^\infty f(x/\lambda)\, g(\lambda)\, \mathrm{d}\lambda\,. \tag{30}$$

A detailed discussion of this topic with a number of corresponding functions $f(x/\lambda) - g(\lambda) - f^*(x)$ is given in [9]. Some interesting cases leading to distributions discussed in Chapter 3 are summarized in Table 2.

5. A very important class of distributions is obtained when we consider the conditional normal distribution with the random variance which is gamma distributed, or with the standard deviation which is Pearson χ distributed. The extraordinary significance of these distributions belonging to a class called elliptically symmetric distributions [3], [13] lies in the possibility of expressing their joint probability density in a very simple way [10], [11]. Following the definition of the total (unconditioned) probability for the joint probability density, we may derive that for the marginal probability density $f_1^*(x)$ given in Table 2, line 1, the correponding joint (two-component) probability density has the form $f_{12}^*(R, \varrho)$ given in Table 2, line 2, where

$$R = [(x_1^2 + x_2^2 - 2\varrho x_1 x_2)/(1 - \varrho^2)]^{1/2}$$

and ϱ is the correlation coefficients of the generating normal distribution. It follows from the relation between $f_1^*(x)$ and $f_{12}^*(x_1, x_2)$ that the analytical form of the Laplace distribution is to be understood as the distribution of the type $|x|^n\, \mathrm{K}_n(|x|)$ with $n = 1/2$. This also offers new views on the incorporation of Laplace (or exponential) distribution into the system of distributions (cf. also Fig. 8). Thus, exponential distribution may be considered to be a special case of the gamma or of the Weibull distribution and of two classes of distributions based on the MacDonald function, types $x^n\, \mathrm{K}_n(x)$ and $x^{n+1}\, \mathrm{K}_n(x)$.

The commonly known difficulties occurring when formulating analytical expressions for joint distributions with given marginal distributions are to be mentioned. So e.g. for the exponential (or Laplace) distribution just mentioned, several ways for expressing the joint distribution have been proposed (a review of them see e.g. [7], Chapter 41.3) which, of course, in many cases have some undesirable properties. On the other hand, the analytical expression based on the MacDonald function yields a unified expression for both the marginal and joint distributions of arbitrary orders. The parameter entering the joint probability density is equivalent with the correlation coefficient ϱ (the normalized centred moment $\mu^{(1,1)}/\mu^{(2)}$) derived from the joint normal probability density. In a similar way, the marginal and joint probability density functions may be expressed for other values of the parameter n. Thus we obtain a sufficiently rich subclass of distributions with elements continuously

Table 2. MacDonald function describing one subclass of elliptically symmetric random processes.

Line	$f_1(x \mid \lambda)$	$g(\lambda)$	$f_1^*(x) = \int_0^\infty f_1(x \mid \lambda)\, g(\lambda)\, \mathrm{d}\lambda$
1	$\frac{1}{\sqrt{(2\pi)}\,\lambda} \exp\left(-\frac{x^2}{2\lambda^2}\right)$ normal $x \in (-\infty, \infty)$, $\lambda > 0$	$\frac{2^{-n/2+1}}{\beta\,\Gamma(n/2)} \left(\frac{\lambda}{\beta}\right)^{n-1} \exp\left(-\frac{\lambda^2}{2\beta^2}\right)$ Pearson χ $\lambda \geqq 0$, $\beta > 0$, $n > 0$	$\frac{2^{-(n-1)/2}}{\sqrt{(\pi)}\,\beta\,\Gamma(n/2)} \left(\frac{\lvert x\rvert}{\beta}\right)^{(n-1)/2} \mathrm{K}_{(n-1)/2}\left(\frac{\lvert x\rvert}{\beta}\right)$ generalized Laplace
	$f_{12}(x_1, x_2, \varrho \mid \lambda) \equiv f_{12}(R, \varrho \mid \lambda)$ *)	$g(\lambda)$	$f_{12}^*(R, \varrho) = \int_0^\infty f_{12}(R, \varrho \mid \lambda)\, g(\lambda)\, \mathrm{d}\lambda$ *)
2	$\frac{1}{2\pi\lambda^2 \sqrt{(1-\varrho^2)}} \exp\left(-\frac{R^2}{2\lambda^2}\right)$ joint normal $R \geqq 0$, $\lambda > 0$	$\frac{2^{-n/2+1}}{\beta\,\Gamma(n/2)} \left(\frac{\lambda}{\beta}\right)^{n-1} \exp\left(-\frac{\lambda^2}{2\beta^2}\right)$	$\frac{2^{-n/2}}{\pi\,\beta^2\,\Gamma(n/2)\sqrt{(1-\varrho^2)}} \left(\frac{R}{\beta}\right)^{n/2-1} \mathrm{K}_{n/2-1}\left(\frac{R}{\beta}\right)$ joint generalized Laplace
	$f_y(y \mid \lambda)$	$g(\lambda)$	$f_y^*(y) = \int_0^\infty f_y(y \mid \lambda)\, g(\lambda)\, \mathrm{d}\lambda$
3	$\frac{y}{\lambda^2} \exp\left(-\frac{y^2}{2\lambda^2}\right)$ Rayleigh $y \geqq 0$, $\lambda > 0$	$\frac{2^{-n/2+1}}{\beta\,\Gamma(n/2)} \left(\frac{\lambda}{\beta}\right)^{n-1} \exp\left(-\frac{\lambda^2}{2\beta^2}\right)$	$\frac{2^{-n/2+1}}{\beta\,\Gamma(n/2)} \left(\frac{y}{\beta}\right)^{n/2} \mathrm{K}_{n/2-1}\left(\frac{y}{\beta}\right)$ distribution of associated envelope

*) $R^2 = (x_1^2 + x_2^2 - 2\varrho x_1 x_2)/(1 - \varrho^2)$

moving from $(\pi b)^{-1}\,\mathrm{K}_0(|x|/b)$ with $B = 6$ over the Laplace distribution $(B = 3)$ up to the normal distribution $(B = 0)$.

Considering the Rayleigh distribution for $f(x/\lambda)$ in (30) and assuming $g(\lambda)$ to be Pearson χ-distributed we arrive at distributions of the type $x^{n+1}\,\mathrm{K}_n(x)$, see Table 2, line 3. It has been shown in (29) that a connection exists with the type $|x|^n\,\mathrm{K}_n(|x|)$. From the point of view of applications in the random process theory, the distribution $x^{n+1}\,\mathrm{K}_n(x)$ describes the envelope of a narrow-band vibratory random stationary process the state variable of which has the distribution of the type $|x|^{n+1/2}\,\mathrm{K}_{n+1/2}(|x|)$ [11]. Thus the above discussed system of both interconnected subclasses offers a valuable mathematical tool for the description and analysis of random vibratory processes (see also [8]).

6. CONTRIBUTIONS TO MATHEMATICAL ANALYSIS

From the relation between the joint and marginal probability densities some interesting expressions for definite integrals containing the MacDonald function may be deduced, which have general application possibilities in mathematical analysis. Two of these expressions are given below:

$$(31)\qquad \int_u^\infty x^{n+1}(x^2 - u^2)^{-1/2}\,\mathrm{K}_n(x/b)\,\mathrm{d}x = (\pi b/2)^{1/2}\,|u|^{n+1/2}\,\mathrm{K}_{n+1/2}(|u|/b)\,,$$

$$u \in \mathrm{R}\,,\quad b > 0\,,\quad n \geqq 0\,,$$

$$(32)\quad \int_{-\infty}^\infty (x^2 + y^2 - 2rxy)^{n/2}\,\mathrm{K}_n[(x^2 + y^2 - 2rxy)^{1/2}\,b^{-1}(1 - r^2)^{-1/2}]\,\mathrm{d}x =$$

$$= (2\pi b)^{1/2}\,(1 - r^2)^{(n+1)/2}\,|y|^{n+1/2}\,\mathrm{K}_{n+1/2}(|y|/b)$$

$$x, y \in \mathrm{R}\,,\quad b > 0\,,\quad r \in \langle 0, 1\rangle\,,\quad n \geqq 0\,.$$

7. TABLES OF PROBABILITY DISTRIBUTIONS BASED ON MACDONALD FUNCTION

While the MacDonald function $\mathrm{K}_n(x)$ is tabulated in some comprehensive mathematical tables [6], [17], usually for n integer, the probability densities $f(x)$ of the type $x^n\,\mathrm{K}_n(x)$, $x \geqq 0$ are tabulated only in [5] for $n = 0(1/2)23/2$, and the corresponding distribution functions $F(x)$ in [15]. The probability densities and distribution functions for the type $x^{p+n}\,\mathrm{K}_n(x)$ are not available in current references but they may be easily adapted from [5].

8. CONCLUSIONS

In this paper the basic analytical properties of the MacDonald function (the modified Bessel function of the second kind) have been summarized and then properties of some subclasses of distribution functions based on the MacDonald function, especially of the types $x^n \mathrm{K}_n(x)$, $x \geqq 0$, $|x|^n \mathrm{K}_n(|x|)$, $x \in \mathrm{R}$ and $x^{n+1} \mathrm{K}_n(x)$, $x \geqq 0$ have been discussed. The distribution functions mentioned are useful for analytical modelling of composed (mixed) distributions, especially for products of random variables having distributions of the exponential type. Extensive and useful applications may be found in the field of non-Gaussian random processes, the marginal and joint probability densities of which and of their envelopes may be described by means of the types discussed.

References

[1] *H. Bateman, A. Erdélyi:* Higher transcendental functions. Vol. 2. New York, McGraw-Hill, 1953.

[2] *H. Bateman., A. Erdélyi:* Tables of integral transforms, Vol. I, II. New York, McGraw-Hill, 1954.

[3] *K. C. Chu:* Estimation and decision for linear systems with elliptical random processes. IEEE Trans. Autom. Control, AC-18, 1973, pp. 499—505.

[4] *C. C. Craig:* On the frequency function xy. Ann. Math. Statist., 7, 1936, pp. 1—15.

[5] *E. M. Elderton:* Table of the product moment T_m function. Biometrika, 21, 1929, pp. 194 to 201.

[6] Jahnke-Emde-Lösch: Tafeln höherer Funktionen. Stuttgart, Teubner, 1960.

[7] *N. L. Johnson, S. Kotz:* Continuous multivariate distributions. New York, J. Wiley, 1972.

[8] *O. Kropáč:* Relations between distributions of random vibratory processes and distributions of their envelopes. Aplik. Matem., 17, 1972, pp. 75—112.

[9] *O. Kropáč:* Rozdělení s náhodnými parametry a jejich inženýrské aplikace. Strojn. Čas. 31, 1980, pp. 597—622.

[10] *O. Kropáč:* A unified model for non-stationary and/or non-Gaussian random processes. J. Sound Vibr., 79, 1981, pp. 11—21.

[11] *O. Kropáč:* Some general properties of elliptically symmetric random processes. Kybernetika, 17, 1981, pp. 401—412.

[12] *S. Kullback:* The distribution laws of the difference and quotient of variables independently distributed in Pearson III laws. Ann. Math. Statist., 7, 1936, pp. 51—53.

[13] *D. K. McGraw, J. F. Wagner:* Elliptically symmetric distributions. IEEE Trans. Inform. Theory, IT-14, 1968, pp. 110—120.

[14] *K. Pearson, G. B. Jeffery, E. M. Elderton:* On the distribution of the first product moment-coefficient, in samples drawn from an indefinitely large normal population. Biometrika, 21, 1929, pp. 164—193.

[15] *K. Pearson, S. A. Stouffer, F. N. David:* Further applications in statistics of the $T_m(x)$ Bessel function. Biometrika, 24, 1932, pp. 293—350.

[16] *D. Teichrow:* The mixture of normal distributions with different variances. Ann. Math. Statist., 28, 1957, pp. 510—512.

[17] *G. N. Watson:* A treatise on the theory of Bessel functions. Cambridge, Univ. Press, 1944 (2nd edit.).

Souhrn

NĚKTERÉ VLASTNOSTI A POUŽITÍ ROZDĚLENÍ PRAVDĚPODOBNOSTI OBSAHUJÍCÍCH MACDONALDOVU FUNKCI

OLDŘICH KROPÁČ

V článku jsou nejprve stručně shrnuty základní analytické vlastnosti MacDonaldovy funkce (modifikované Besselovy funkce druhého druhu). Dále jsou diskutovány vlastnosti několika podtříd rozdělení pravděpodobnosti obsahujících MacDonaldovu funkci, zejména typy $x^n \mathrm{K}_n(x)$, $x \geqq 0$, $|x|^n \mathrm{K}_n(|x|)$, $x \in \mathrm{R}$ a $x^{n+1} \mathrm{K}_n(x)$, $x \geqq 0$. Uvedená rozdělení se uplatňují při analytickém popisu složených rozdělení, zejména součinu náhodných veličin s rozděleními exponenciálního typu. Zvlášť rozsáhlé a přínosné jsou aplikace pro popis a analýzu negaussovských marginálních a sdružených rozdělení náhodných procesů a jejich obálek.

Author's address: Ing. *Oldřich Kropáč,* CSc., Výzkumný a zkušební letecký ústav, 199 05 Praha 9 - Letňany.