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SOME FAST FINITE-DIFFERENCE SOLVERS FOR DIRICHLET PROBLEMS ON GENERAL DOMAINS

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Our aim is to prove the existence of asymptotic error expansion to some simple finite-difference schemes for Dirichlet problems on general domains which, by Richardson extrapolation, lead to fast finite-difference solvers for the problems mentioned.

1. THE DIFFERENTIAL PROBLEM

In order to simplify the notation we shall consider only the two-dimensional geometry. The result can be generalized to the n-dimensional case. Let D be a bounded domain in the (x, y)-plane with a boundary G. Let us consider the boundary value problem

$$Lu = f(x, y)\,, \quad (x, y) \in D\,,$$

$$u = g(x, y)\,, \quad (x, y) \in G\,,$$

where

$$Lu = \frac{\partial}{\partial u}\left(p(x, y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x, y)\frac{\partial u}{\partial y}\right) - c(x, y)\,u\,,$$

$$p \geqq p_0 > 0\,, \quad q \geqq q_0 > 0\,, \quad c \geqq 0\,,$$

p, q, c, f, g being given smooth enough functions, p_0, q_0 given positive numbers. Assume that this problem has a unique smooth enough solution $u(x, y)$.

2. THE GRID

Let $\{h\}$ and $\{k\}$ be two sequences of positive numbers tending simultaneously to zero and

$$0 < \text{const} < h/k < \text{const}.$$

For some x_0, y_0 the points

$$(x_i, y_j), \quad x_i = x_0 + ih, \quad y_j = y_0 + jk, \quad i, j = 0, \pm 1, \pm 2, \ldots,$$

form a grid over the (x, y)-plane. Now we describe the grid over D. The points (x_i, y_j) which belong to the interior of D are called interior grid points and denoted by D_h. The intersections of the boundary G with each grid line $x = x_i$ or $y = y_j$ are called boundary grid points and denoted by G_h. Each interior grid point $P(x_P, y_P)$ has four neighbour grid points, which are the closest to it on the grid lines $x = x_P$ and $y = y_P$. They are $(x_P + h_P^+, y_P)$, $(x_P - h_P^-, y_P)$, $(x_P, y_P + k_P^+)$, $(x_P, y_P - k_P^-)$. So we always have $h_P^+ \leqq h$, $h_P^- \leqq h$, $k_P^+ \leqq k$, $k_P^- \leqq k$. An interior grid point P is called strictly interior if $h_P^+ = h_P^- = h$ and $k_P^+ = k_P^- = k$. It is called a near-boundary one if at least one of the four following inequalities $h_P^+ < h$, $h_P^- < h$, $k_P^+ < k$, $k_P^- < k$ holds. We denote the set of strictly interior grid points by D_h^0 and the set of near-boundary ones by D_h^*. Then $D_h^0 \cup D_h^* = D_h$. We shall call the set $D_h \cup G_h$ a grid with grid spacings h and k over D. This grid is in general not uniform near the boundary.

3. THE DISCRETE PROBLEM

We consider the following discrete problem with respect to the unknown $v(x_P, y_P)$ defined on $D_h \cup G_h$:

$$\begin{aligned}
L_h v = {} & [2/(h_P^+ + h_P^-)]\,[p(x_P + h_P^+/2, y_P)\,(v(x_P + h_P^+, y_P) - v(x_P, y_P))/h_P^+ - \\
& - p(x_P - h_P^-/2, y_P)\,(v(x_P, y_P) - v(x_P - h_P^-, y_P))/h_P^-] + \\
& + [2/(k_P^+ + k_P^-)]\,[q(x_P, y_P + k_P^+/2)\,(v(x_P, y_P + k_P^+) - v(x_P, y_P))/k_P^+ - \\
& - q(x_P, y_P - k_P^-/2)\,(v(x_P, y_P) - v(x_P, y_P - k_P^-))/k_P^-] - \\
& - c(x_P, y_P)\,v(x_P, y_P) = f(x_P, y_P), \quad (x_P, y_P) \in D_h, \\
& v(x_P, y_P) = g(x_P, y_P), \quad (x_P, y_P) \in G_h.
\end{aligned}$$

It is clear that the operator L_h satisfies the maximum principle.

4. THE MAIN RESULT

Theorem 1. *Assume that $u(x, y) \in C^5(\bar{D})$, $p(x, y)$, $q(x, y) \in C^4(\bar{D})$ and that the problem*

$$Lw = F(x, y) \in C^m(\bar{D}), \quad (x, y) \in D,$$

$$w = 0 \qquad (x, y) \in G,$$

has a unique solution $w \in C^{m+2}(\bar{D})$. *Then for h and k small enough there exist two functions* $w_1(x, y)$ *and* $w_2(x, y)$ *independent of h and k such that*

$$(1) \qquad v(x_P, y_P) - u(x_P, y_P) = h^2\, w_1(x_P, y_P) + k^2\, w_2(x_P, y_P) + O(h^3 + k^3)\,.$$

Proof. First, Taylor's formula yields

$$L_h u(x_P, y_P) = Lu(x_P, y_P) + h^2\, a(x_P, y_P) + k^2\, b(x_P, y_P) + O(h^3 + k^3)\,, \quad P \in D_h^0\,,$$

$$L_h u(x_P, y_P) = Lu(x_P, y_P) + O(h + k)\,, \quad P \in D_h^*\,,$$

where

$$a(x, y) = (1/24)\frac{\partial^3}{\partial x^3}\left(p(x, y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(p(x, y)\frac{\partial^3 u}{\partial x^3}\right),$$

$$b(x, y) = (1/24)\frac{\partial^3}{\partial y^3}\left(q(x, y)\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(q(x, y)\frac{\partial^3 u}{\partial y^3}\right).$$

Then for any $w_1(x, y)$ and $w_2(x, y) \in C^3(\bar{D})$ we put

$$z = v - u - h^2 w_1 - k^2 w_2\,,$$

and we have

$$L_h z = h^2[-Lw_1 - a(x_P, y_P)] + k^2[-Lw_2 - b(x_P, y_P)] + O(h^3 + k^3)\,, \quad P \in D_h^0\,,$$

$$L_h z = O(h + k)\,, \quad P \in D_h^*\,.$$

We choose w_1 and w_2 so that

$$Lw_1 = -a(x, y)\,, \quad (x, y) \in D\,; \quad w_1 = 0\,, \quad (x, y) \in G\,,$$

$$Lw_2 = -b(x, y)\,, \quad (x, y) \in D\,; \quad w_2 = 0\,, \quad (x, y) \in G\,,$$

which exist by assumption. Thus we have

$$L_h z = O(h^3 + k^3)\,, \quad P \in D_h^0\,,$$

$$L_h z = O(h + k)\,, \quad P \in D_h^*\,.$$

$$z = 0\,, \quad P \in G_h\,.$$

Hence our theorem immediately follows from the following lemma.

Lemma. *If z satisfies*

$$L_h z = \varphi\,, \quad P \in D_h\,; \quad z = 0\,, \quad P \in G_h\,,$$

then for h and k small enough we have

$$\max_{D_h} |z| \leqq M\{\max_{D_h^0} |\varphi| + \max_{D_h^*} |\varphi| \,.\, (h^2 + k^2)\}\,,$$

where M denotes a constant independent of h and k.

Proof of the lemma. We set $z = z_1 + z_2$, where

$$L_h z_1 = \varphi, \quad P \in D_h^0,$$
$$L_h z_1 = 0, \quad P \in D_h^*,$$
$$z_1 = 0, \quad P \in G_h,$$
$$L_h z_2 = 0, \quad P \in D_h^0,$$
$$L_h z_2 = \varphi, \quad P \in D_h^*,$$
$$z_2 = 0, \quad P \in G_h.$$

To evaluate z_1 let $B(x, y)$ be the unique smooth enough solution of the differential problem

$$LB = -2, \quad (x, y) \in D, \quad B = 0, \quad (x, y) \in G,$$

which exists by assumption. We have

$$0 \leqq B(x, y) \leqq M_1,$$

where M_1 denotes a constant. At the same time

$$L_h B = LB + O(h^2 + k^2), \quad P \in D_h^0,$$
$$L_h B = LB + O(h + k), \quad P \in D_h^*.$$

Thus for h and k small enough we have

$$L_h B \leqq -1.$$

Now let us consider the problem

$$LA(x, y) = -2K, \quad (x, y) \in D, \quad A(x, y) = 0, \quad (x, y) \in G,$$

where

$$K = \max_{D_h^0} |\varphi|.$$

Thus we have on the one hand

$$A = KB, \quad 0 \leqq \max_D A = K \max_D B \leqq M_1 \max_{D_h^0} |\varphi|$$

and on the other hand, for h and k small enough,

$$L_h A = K\, L_h B \leqq -K.$$

Then

$$L_h(A \pm z_1) \leqq 0, \quad P \in D_h,$$
$$A \pm z_1 = 0, \quad P \in G_h.$$

We deduce $A \pm z_1 \geqq 0$, that is $|z_1| \leqq A$. Hence

$$\max_{D_h} |z_1| \leqq M_1 \max_{D_h^0} |\varphi| . \tag{2}$$

The evaluate z_2 we first consider the problem

$$L_h Z = 0 , \qquad P \in D_h^0 ,$$
$$L_h Z = -|\varphi| , \quad P \in D_h ,$$
$$Z = 0 , \qquad P \in G_h .$$

Then by the maximum principle

$$Z \geqq 0 , \quad |z_2| \leqq Z .$$

Now we have to evaluate Z. It is clear that Z attains its maximum value on D_h, but cannot attain it on D_h^0 (because here the right hand member is zero). Let $Q \in D_h^*$ be the grid point at which Z attains its maximum value. Then the difference equation $L_h Z = -|\varphi|$ written at Q leads to an equality where the right hand member is $|\varphi(Q)| = |\varphi(x_Q, y_Q)|$ and the left hand member is the sum of four nonnegative differences between the value of Z at Q and the values of Z at the four neighbour grid points of Q, and one nonnegative term cu at Q. Therefore, at least one neighbour grid point of Q lies on G. Let S be this point. The value of Z at S must be zero. Then if S lies on the grid line $x = x_Q$ we have

$$[2/(h_Q^+ + h_Q^-)]\,[p(x_Q + \tfrac{1}{2}h_Q^+, y_Q)\,(Z(x_Q, y_Q) - 0)/h_Q^+] \leqq |\varphi(Q)|$$

or

$$[2/(h_Q^+ + h_Q^-)]\,[p(x_Q - \tfrac{1}{2}h_Q^-, y_Q)\,(Z(x_Q, y_Q) - 0)/h_Q^-] \leqq |\varphi(Q)| .$$

If S lies on the grid line $y = y_Q$ we have

$$[2/(k_Q^+ + k_Q^-)]\,[q(x_Q, y_Q + \tfrac{1}{2}k_Q^+)\,(Z(x_Q, y_Q) - 0)/k_Q^+] \leqq |\varphi(Q)|$$

or

$$[2/(k_Q^+ + k_Q^-)]\,[q(x_Q, y_Q - \tfrac{1}{2}k_Q^-)\,(Z(x_Q, y_Q) - 0)/k_Q^-] \leqq |\varphi(Q)| .$$

Hence we deduce

$$\min\{p_0, q_0\} \,.\, Z(x_Q, y_Q) \leqq |\varphi(Q)| \,.\, (h^2 + k^2) ,$$

that is, we have

$$0 \leqq Z(x_P, y_P) \leqq Z(x_Q, y_Q) \leqq M_2(h^2 + k^2) \max_{D_h^*} |\varphi| \tag{3}$$

for all $P \in D_h$, with $M_2 = 1/\min\{p_0, q_0\}$. Then the lemma follows from $|z| \leqq |z_1| + |z_2|$ and the inequalities (2), (3) with $M = \max\{M_1, M_2\}$.

Note 1. If $p = \text{const} > 0$, $q = \text{const} > 0$ the theorem holds without assuming that h and k are small enough because in the proof of the lemma we can take $A = K(R^2 - x^2 - y^2)$, where R denotes the radius of a circle having the centre at $0(0, 0)$ and containing D.

Note 2. The theorem is still available if the term cu in the differential equation is replaced by $c(x, y, u)$ with $(\partial c/\partial u) \geqq 0$.

5. CONSEQUENCE

Theorems 1 leads to a simple process for accelerating the convergence of the method by Richardson extrapolation. Assume that we want to calculate the approximate value of $u(x_P, y_P)$ at a grid point P which is common to three grids with grid spacings (h, k), $(h/2, k)$, $(h, k/2)$. We denote the value obtained on the grid with the grid spacing (h, k) by $v^{h,k}(x_P, y_P) = v^{h,k}$ and $u(x_P, y_P) = u$. Then by (1) we have

$$v^{h,k} \; - u = h^2\, w_1(x_P, y_P) + k^2\, w_2(x_P, y_P) + O(h^3 + k^3)\,,$$

$$v^{h/2,k} - u = (h/2)^2\, w_1(x_P, y_P) + k^2\, w_2(x_P, y_P) + O(h^3 + k^3)\,,$$

$$v^{h,k/2} - u = h^2\, w_1(x_P, y_P) + (k/2)^2\, w_2(x_P, y_P) + O(h^3 + k^3)\,.$$

By eliminating $w_1(x_P, y_P)$ and $w_2(x_P, y_P)$ from these relations we obtain

$$\tfrac{4}{3}(v^{h/2,k} + v^{h,k/2}) - \tfrac{5}{3}v^{h,k} = u + O(h^3 + k^3)\,,$$

which yields a more accurate approximate value of $u(x_P, y_P)$ than any of $v^{h,k}$, $v^{h/2,k}$, $v^{h,k/2}$. Our algorithm is much simpler than that of [1].

Reference

[1] *V. Pereyra, W. Proskurowski, O. Widlund:* High order fast Laplace solvers for Dirichlet problem on general domains. Math. Comp. 31, 137 (1977), 1—17.

Souhrn

RYCHLÉ ŘEŠENÍ DIRICHLETOVA PROBLÉMU NA OBECNÉ OBLASTI METODOU KONEČNÝCH DIFERENCÍ

Ta Van Dinh

Autor dokazuje existenci mnohoparametrického asymptotického rozvoje pro chybu obvyklého pětibodového diferenčního schématu pro Dirichletův problém pro lineární a semilineární eliptickou parciální diferenciální rovnici na obecných oblastech. Tento rozvoj dává s použitím Richardsonovy extrapolace jednoduchý způsob zrychlení konvergence dané metody. Postup je ilustrován na numerickém příkladě.

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