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SIMILARITY MOTIONS IN E_3 WITH PLANE TRAJECTORIES

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In this paper we shall classify all similarity motions in E_3 which have all trajectories plane curves. We shall also give explicit expressions for each equivalence class of such motions.

Let E_3, $\bar{E}_3$ be two Euclidean spaces of dimension 3. By a frame in E_3 or $\bar{E}_3$ we mean any sequence $\mathscr{R} = \{A, f_1, f_2, f_3\}$ or $\bar{\mathscr{R}} = \{\bar{A}, \bar{f}_1 \bar{f}_2 \bar{f}_3\}$, where $A \in E_3$ or $\bar{A} \in \bar{E}_3$ is a point and f_i or $\bar{f}_i$ $i = 1, 2, 3$, are pairwise orthogonal vectors in E_3 or $\bar{E}_3$, respectively, of the same length. Let us further fix a frame $\mathscr{R}_0 = \{A_0, (f_i)_0\}$ or $\bar{\mathscr{R}}_0 = \{\bar{A}_0, (\bar{f}_i)_0\}$, $i = 1, 2, 3$, in E_3 or E_3, respectively. The Lie group S_3 of all similarity transformations of E_3 or $\bar{E}_3$ can be regarded as the group of all 4×4 matrices $g = \begin{pmatrix} 1, & 0 \\ t, & \gamma \end{pmatrix}$, where t is a column with 3 entries, γ and E are 3×3 matrices, $\gamma\gamma^{\mathsf{T}} = \lambda E$, $\lambda \in \mathscr{R}$ and E denotes the identity matrix.

S_3 acts also naturally as the group of all similarity transformations from $\bar{E}_3$ into E_3 by the rule $g(\bar{\mathscr{R}}_0) = \mathscr{R}_0 \,.\, g$ for $g \in S_3$. A curve $g(t)$ on S_3 regarded as a one-parametric system of similarity transformations from $\bar{E}_3$ into E_3 is called a similarity motion in E_3 (and we shall always suppose a sufficient degree of differentiability of all functions, whether given or constructed). By a lift of a motion $g(t)$ we mean a set of pairs $[\mathscr{R}(t), \bar{\mathscr{R}}(t)]$ of frames such that $g(t)\,(\bar{\mathscr{R}}(t)) = \mathscr{R}(t)$, where $\bar{\mathscr{R}}(t)$ is a frame in $\bar{E}_3$ and $\mathscr{R}(t)$ is a frame in E_3.

Let further $[\mathscr{R}(t), \bar{\mathscr{R}}(t)]$ be any lift of a given motion $g(t)$. Denote $\mathscr{R}' = \mathscr{R}\varphi$, $\bar{\mathscr{R}}' = \bar{\mathscr{R}}\psi$, $2\omega = \varphi - \psi$, $2\eta = \varphi + \psi$. If $\mathfrak{S}_3$ is the Lie algebra of S_3, then $\omega, \eta \in \mathfrak{S}_3$.

Let $\bar{A}$ be a fixed point in $\bar{E}_3$. Then $\bar{A} = \bar{\mathscr{R}}X$ and $g(\bar{A}) = \mathscr{R}X$ is the trajectory of $\bar{A}$ in E_3 during the motion $g(t)$. X is the column of coordinates of $\bar{A}$ in $\bar{\mathscr{R}}$ and also the column of coordinates of a point of the trajectory of $\bar{A}$ expressed in the frame $\mathscr{R}$. Denote by Ω_k the operator of the k-th derivative of the trajectory of the point $\bar{A}$, which is defined by the formula $[g(\bar{A})]^{(k)} = 2\mathscr{R}\Omega_k X$.

Direct computations give

$$(1) \qquad \Omega_1 = \omega\,, \quad \Omega_{k+1} = (\omega + \eta)\,\Omega_k + \Omega_k(\omega - \eta) + \Omega_k'\,.$$

To find all motions with all trajectories planar means to find all solutions $\omega(t)$ and $\eta(t)$ of the equation

$$(2) \qquad \det |\Omega_1 X, \Omega_2 X, \Omega_3 X| = 0 \quad \text{for all} \quad X.$$

Denote further

$$\omega = \begin{pmatrix} 0, & 0 \\ \omega_0, & \omega_1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0, & 0 \\ \eta_0, & \eta_1 \end{pmatrix}, \quad \Omega_k = \begin{pmatrix} 0, & 0 \\ \vartheta_k, & \Theta_k \end{pmatrix}.$$

Then

$$\begin{aligned} (3) \qquad & \vartheta_1 = \omega_0, \quad \vartheta_{k+1} = (\omega_1 + \eta_1)\vartheta_k + \Theta_k(\omega_0 - \eta_0) + \vartheta'_k, \\ & \Theta_1 = \omega_1, \quad \Theta_{k+1} = (\omega_1 + \eta_1)\Theta_k + \Theta_k(\omega_1 - \eta_1) + \Theta'_k, \end{aligned}$$

where

$$\omega_0 = \begin{pmatrix} \omega_{01} \\ \omega_{02} \\ \omega_{03} \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} \eta_{01} \\ \eta_{02} \\ \eta_{03} \end{pmatrix}, \quad \omega_1 = \begin{pmatrix} \omega_{11}, & \omega_{12}, & \omega_{13} \\ -\omega_{12}, & \omega_{11}, & \omega_{23} \\ -\omega_{13}, & -\omega_{23}, & \omega_{11} \end{pmatrix},$$

$$\eta_1 = \begin{pmatrix} \eta_{11}, & \eta_{12}, & \eta_{13} \\ -\eta_{12}, & \eta_{11}, & \eta_{23} \\ -\eta_{13}, & -\eta_{23}, & \eta_{11} \end{pmatrix}. \quad \text{Write also} \quad X = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

First, let $\omega_{11} = 0$. Then we can change the lift of $g(t)$ in such a way that also $\eta_{11} = 0$. This means that in this case we get all Euclidean motions in E_3 with plane trajectories (taken with respect to the equivalence in S_3). They are known, see for example [1]. All of them are cylindrical motions (the spherical image is only a rotation around a fixed axis) and there are only three cases possible: a) planar motion in parallel planes, b) composition of a rotation around a fixed axis with some translation along this axis, which gives elliptical trajectories in parallel planes, c) composition of the elliptical motion in a plane with some translation in the direction perpendicular to this plane — a motion originally described by Darboux. Its trajectories are again ellipses.

Consequently, from now on we can suppose $\omega_{11} \neq 0$. Then ω_1 is a regular matrix and so we can change the lift of $g(t)$ and the parameter t in such a way that

$$(4) \qquad \omega = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 1, & -\mu, & 0 \\ 0, & \mu, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix}, \quad \text{where} \quad \mu \geqq 0.$$

The subgroup of S_3 which preserves such lifts for $\mu \neq 0$ consists of all elements $g \in S_3$ of the form

$$(5) \qquad g = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & \alpha, & -\beta, & 0 \\ 0, & \beta, & \alpha, & 0 \\ 0, & 0, & 0, & \sqrt{(\alpha^2 + \beta^2)} \end{pmatrix}, \quad \alpha, \beta \in \mathrm{R}.$$

Let us suppose for a moment that $\eta_{13}^2 + \eta_{23}^2 \neq 0$. Let first $\mu \neq 0$. Then we can change the lift of $g(t)$ and get $\eta_{23} = 0$. Direct and uninteresting computation of (2) shows that then also $\eta_{13} = 0$, which is not possible. We leave the details out. If

$\mu = 0$, we change the lift of $g(t)$ to get $\eta_{13} = \eta_{23} = 0$ and the remaining group is again given by (5). So we can suppose $\eta_{13} = \eta_{23} = 0$.

Case 1. Let $\eta_{01}^2 + \eta_{02}^2 \neq 0$. Then we can change the lift of $g(t)$ to get η in the form

$$(6) \qquad \eta = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 1, & a, & -b, & 0 \\ 0, & b, & a, & 0 \\ \beta, & 0, & 0, & a \end{pmatrix}$$

and for $\mu \neq 0$ this lift is unique. Direct computation now gives

$$(7) \qquad \Omega_1 = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 1, & -\mu, & 0 \\ 0, & \mu, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0, & 0, & 0, & 0 \\ -1, & 2 - 2\mu^2, & -4\mu - \mu', & 0 \\ -\mu, & 4\mu + \mu', & 2 - 2\mu^2, & 0 \\ -\beta, & 0, & 0, & 2 \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 0 & 0, & 0, & 0 \\ 3\mu^2 - 3 + \mu b - a, & 4 - 12\mu^2 - 6\mu\mu', & 4\mu^3 - 12\mu - 6\mu' - \mu'', & 0 \\ -6\mu - b - a\mu - 2\mu', & -4\mu^3 + 12\mu + 6\mu' + \mu'', & 4 - 12\mu^2 - 6\mu\mu', & 0 \\ -3\beta - a\beta - \beta', & 0, & 0, & 4 \end{pmatrix}.$$

After a simplification, (2) takes the form

$$\begin{vmatrix} x_1 - \mu x_2, & -2\mu^2 x_1 - (2\mu + \mu')x_2 - 1, & -6\mu\mu' x_1 + (4\mu^3 + 4\mu - \mu'')x_2 + 3 + 3\mu^2 + \mu b - a \\ \mu x_1 + x_2, & (2\mu + \mu')x_1 - 2\mu^2 x_2 - \mu, & -(4\mu^3 + 4\mu - \mu'')x_1 - 6\mu\mu' x_2 - b - a\mu - 2\mu' \\ x_3, & -\mu & 3\beta - a\beta - \beta' \end{vmatrix} = 0 .$$

This leads to the following equations:

(8) 1. $2\mu[4\mu^2(1 + \mu^2) + 6\mu\mu' + 3(\mu')^2 - \mu\mu''] = 0$,
2. $2\mu' + (3\mu + b)(1 + \mu^2) = 0$,
3. $-5\mu^2\mu' + 2\mu(a - 1)(1 + \mu^2) - \mu'(3 + \mu b - a) - \mu'' = 0$,
4. $2\mu(b + 3\mu)(1 + \mu^2) + \mu'(4\mu + b + a\mu - \mu') = 0$,
5. $2\beta\mu\mu' - \beta'(1 + \mu^2) = 0$,
6. $\beta[(\mu^2 + 1)(2\mu + 3\mu') + 3\mu^2\mu' + \mu''] - (a\beta + \beta')[2\mu(1 + \mu^2) + \mu'] = 0$.

Substituting from 2 into 4 we obtain

4'. $\mu'(b + a\mu - \mu') = 0$.

a) Let $\mu \neq 0$. Suppose $\mu' = 0$. Then from 1 we get $\mu = 0$, which is a contradiction. So $\mu' \neq 0$. From 4' we get $b + a\mu - \mu' = 0$. Denote $\mu = m^{-1/2}$. Then

$$(9) \qquad a = 3\frac{m'}{2m}\left(\frac{3m + 1}{m + 1}\right), \quad b = m^{-1/2}\left(\frac{m'}{m + 1} - 3\right), \quad \beta = C\frac{m + 1}{m},$$

C is a constant.

From 1 we get $m'' - 6m' + 8m + 8 = 0$, so $m = C_1 e^{4t} + C_2 e^{2t} - 1$, where C_1 and C_2 are constants of integration.

b) Let $\mu = 0$. Then 2 of (8) implies $b = 0$. For the sake of convenience we change the lift of $g(t)$ to have

$$\mu = a = b = \eta_{01} = \eta_{02} = 0\,, \quad \eta_{03} \text{ is an arbitrary function}\,. \tag{10}$$

Case 2. $\eta_{01} = \eta_{02} = 0$, $\mu \neq 0$.

a) $\eta_{03} \neq 0$. Then we can change the lift of $g(t)$ to get $\eta_{03} = 1$, $b = 0$. Equation (2) takes the form

$$\begin{vmatrix} x_1 - \mu x_2, & -2\mu^2 x_1 - (2\mu + \mu')\,x_2, & -6\mu\mu' x_1 + [4\mu(1 + \mu^2) - \mu'']\,x_2 \\ \mu x_1 + x_2, & (2\mu + \mu')\,x_1 - 2\mu^2 x_2, & -[4\mu(1 + \mu^2) - \mu'']\,x_1 - 6\mu\mu' x_2 \\ x_3, & -1, & 3 - a \end{vmatrix} = 0\,.$$

As a result we get equations 1 and 6 from (8) with $\beta = 1$; point 6 in (8) changes to

6'. $(2\mu + \mu' + 2\mu^3)(3\mu' + 3\mu - a\mu) = 0$.

So we get two solutions:

$$\mu = m^{-1/2}, \quad a = 3 - \frac{3m'}{2m}, \quad \beta = 1, \quad b = 0, \quad \eta_{01} = \eta_{02} = 0\,, \tag{11}$$

$$\mu = m^{-1/2} \text{ with } C_2 = 0\,, \quad \beta = 1\,, \quad b = 0\,, \quad \eta_{01} = \eta_{02} = 0\,, \quad a \text{ is arbitrary.} \tag{12}$$

b) $\eta_0 = 0$. We change the lift of $g(t)$ to $a = b = 0$, $\mu = m^{-1/2}$. So we obtain

$$\eta = 0\,, \quad \mu = m^{-1/2}\,. \tag{13}$$

Case 3. Here $\eta_{01} = \eta_{02} = \mu = 0$. This is b) of Case 1.

Now we must solve the differential equations

$$\mathscr{R}' = \mathscr{R}\varphi\,, \quad \bar{\mathscr{R}}' = \bar{\mathscr{R}}\psi\,, \quad \text{where} \quad \varphi = \eta + \omega\,, \quad \psi = \eta - \omega\,. \tag{14}$$

The vector part of (14) can be formally written in the form $\mathscr{T}' = \mathscr{T} . M$, where

$$M = \begin{pmatrix} A, & -B, & 0 \\ B, & A, & 0 \\ 0, & 0, & A \end{pmatrix}, \tag{15}$$

A, B are some functions, $\mathscr{T} = \{e_1, e_2, e_3\}$ is a base. We expect the solution in the form $\mathscr{T} = \mathscr{T}_0 . h$, where

$$h = \begin{pmatrix} u \cos v, & -u \sin v, & 0 \\ u \sin v, & u \cos v, & 0 \\ 0 & 0, & u \end{pmatrix}, \tag{16}$$

where $\mathscr{T}_0 = \{f_1, f_2, f_3\}$ is a fixed base.

Then $h^{-1}h' = M$ and so $u'u^{-1} = A$, $v' = B$. The solutions are the following ($\varepsilon = 1$ for the fixed frame, $\varepsilon = -1$ for the moving frame):

$(9): A = a + \varepsilon\,, \quad B = b + \varepsilon\mu\,, \quad u = m^{-1/2}(m+1)^{-1}\exp\left[(3+\varepsilon)\,t\right],$

$\quad v = (\varepsilon - 3)\,H(t) + 2\arctan\sqrt{m},\ H = \int_{t_0}^{t}\mu(\bar{t})\,\mathrm{d}\bar{t}$ for a suitable t_0.

$(10): A = \varepsilon\,, \quad B = 0\,, \quad u = e^{\varepsilon t}\,, \quad v = 0\,.$

$(11): A = 3 + \varepsilon - \dfrac{3m'}{2m}\,, \quad B = \varepsilon\mu\,, \quad u = m^{-3/2}\exp\left[(3+\varepsilon)\,t\right], \quad v = \varepsilon H\,.$

$(12): A = a + \varepsilon\,, \quad B = \varepsilon\mu\,, \quad u = \exp\left[F(t) + \varepsilon t\right], \quad v = \varepsilon H \quad \text{with} \quad C_2 = 0\,,$
$\quad F(t) = \int_{t_0}^{t} a\,\mathrm{d}\bar{t}.$

$(13): A = \varepsilon\,, \quad B = \varepsilon\mu\,, \quad u = e^{\varepsilon t}\,, \quad v = \varepsilon H\,.$

In all cases we can write the solution of (14) in the form

$$g = \begin{pmatrix} 1, & 0 \\ T, & \gamma \end{pmatrix}, \quad \text{where} \quad g = g_1 g_2^{-1}\,, \quad \mathscr{R} = \mathscr{R}_0 g_1\,, \quad \bar{\mathscr{R}} = \bar{\mathscr{R}}_0 g_2\,,$$

$$g_1 = \begin{pmatrix} 1, & 0 \\ T_1, & \gamma_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1, & 0 \\ T_2, & \gamma_2 \end{pmatrix}, \quad \bar{\mathscr{R}}_0, \mathscr{R}_0 \text{ are fixed frames.}$$

Then $\gamma = \gamma_1\gamma_2^{-1}$, $T = T_1 - \gamma T_2$.

Denote by u_1, v_1 the expressions for u and v in the case $\varepsilon = 1$, by u_2, v_2 in the case $\varepsilon = -1$. Then

$$\gamma_i = \begin{pmatrix} u_i\cos v_i, & -u_i\sin v_i, & 0 \\ u_i\sin v_i, & u_i\cos v_i, & 0 \\ 0, & 0, & u_i \end{pmatrix}, \quad i = 1, 2,$$

$$\gamma = \begin{pmatrix} u_1u_2^{-1}\cos(v_1 - v_2), & -u_1u_2^{-1}\sin(v_1 - v_2), & 0 \\ u_1u_2^{-1}\sin(v_1 - v_2), & u_1u_2^{-1}\cos(v_1 - v_2), & 0 \\ 0, & 0, & u_1u_2^{-1} \end{pmatrix}.$$

Now we return to the cases (9)–(13) again:

$$(9): \qquad \gamma = (2\delta)^{-1}\begin{pmatrix} 2 - C_2e^{2t}, & -2\sqrt{m}, & 0 \\ 2\sqrt{m}, & 2 - C_2e^{2t}, & 0 \\ 0, & 0, & 2\delta e^{2t} \end{pmatrix},$$

where $\delta = (C_1 + \frac{1}{4}C_2^2)^{1/2}$. Further, $A' = \sum_{i=1}^{3}\eta_{0i}e_i$, $\bar{A}' = \sum_{i=1}^{3}\eta_{0i}\bar{e}_i$, where $\mathscr{R} = \{A, e_i\}$, $\bar{\mathscr{R}} = \{\bar{A}, \bar{e}_i\}$. Substitution from γ_1 and γ_2 gives

$$A' = e^{2t}\delta^{-1}(m+1)^{-2}\left\{m^{-1/2}\left[m + 1 + \tfrac{1}{2}C_2e^{2t}(m-1)\right]f_1 + C_1e^{4t}f_2\right\} + $$
$$+ Cm^{-3/2}e^{4t}f_3\,,$$

$$\bar{A}' = e^{2t}m^{-1/2}(m+1)^{-2}\left[1 + m\left(1 - \tfrac{1}{2}C_2^2\delta^{-2}\right)\right]\bar{f}_1 - \tfrac{1}{4}\delta^{-2}(m+1)^{-2}\, C_2 m'\bar{f}_2 + \\ + Cm^{-3/2}e^{2t}\bar{f}_3\,,$$

where $\mathscr{R}_0 = \{A_0, f_i\}$, $\bar{\mathscr{R}}_0 = \{\bar{A}_0, \bar{f}_i\}$, $i = 1, 2, 3$. The solution is

$$A = \tfrac{1}{2}\delta^{-1}(m+1)^{-1}\, m^{1/2}e^{2t}f_1 - \tfrac{1}{2}\delta^{-1}(m+1)^{-1}\, e^{2t}f_2 + \\ + \tfrac{1}{4}C\delta^{-2}m^{-1/2}(C_2e^{2t} - 2)\, f_3 + A_0\,,$$

$$\bar{A} = \tfrac{1}{8}\delta^{-2}(m+1)^{-1}\, m^{1/2}m'e^{-2t}\bar{f}_1 + \tfrac{1}{4}\delta^{-2}(m+1)^{-1}\, C_2\bar{f}_2 - \\ - \tfrac{1}{8}C\delta^{-2}m^{-1/2}m'e^{-2t}\bar{f}_3 + \bar{A}_0\,.$$

Further computation gives

$$T = \begin{pmatrix} \tfrac{1}{4}C_2\delta^{-3}m^{1/2} \\ -\tfrac{1}{8}\delta^{-3}e^{-2t}m' \\ \tfrac{1}{2}\delta^{-2}Cm^{1/2} \end{pmatrix}.$$

In the end we substitute $e^{2t} = s$ and change the constants of integration. As a result we can write $g(s)$ in the final form

$$g = \delta^{-1}\begin{pmatrix} 1, & 0, & 0, & 0 \\ C_2\sqrt{m}, & 1 - \tfrac{1}{2}C_2 s, & -\sqrt{m}, & 0 \\ -2C_1 s - C_2, & \sqrt{m}, & 1 - \tfrac{1}{2}C_2 s, & 0 \\ C\sqrt{m}, & 0, & 0, & \delta s \end{pmatrix}, \tag{17}$$

where $m = C_1s^2 + C_2s - 1$ is defined for s such that $m > 0$.

Remark. The differential equation for the trajectory $X(t)$ of a point is in the case (9) : $X''' - (6 - 3m'/m)\,X'' + (8 - 3m'/m)\,X' = 0$. The general solution of this equation can be written as $X = A_0 + f_1e^{2t} + f_2\sqrt{m}$, where A_0 is any point and f_1, f_2 are any two independent vectors. Hence the trajectories are conic sections. They are ellipses for $C_1 < 0$, hyperbolas for $C_1 > 0$ and parabolas for $C_1 = 0$.

The solution in the case (10) is

$$g = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & s, & 0, & 0 \\ 0, & 0, & s, & 0 \\ G(s), & 0, & 0, & s \end{pmatrix}, \tag{18}$$

where $G(s)$ is an arbitrary function. Any plane curve can be a trajectory of this motion for suitable $G(s)$.

(11) is treated in a similar way as (9). As a result we get

$$g = \delta^{-1}\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1 - \tfrac{1}{2}C_2 s, & \sqrt{m}, & 0 \\ 0, & \sqrt{m}, & 1 - \tfrac{1}{2}C_2 s, & 0 \\ \sqrt{m}, & 0, & 0, & s \end{pmatrix}. \tag{19}$$

The differential equation for the trajectory is the same as in the case (9).

Now the case (12): We can change the parameter t to get $C_1 = 1$, because $C_2 = 0$ and so $C_1 > 0$. Then $\delta = 1$ and we can write γ_1 and γ_2. Further, $A' = e^t F(t) f_3$, $\bar{A}' = e^{-t} F(t) \bar{f}_3$. Integration gives $A = A_0 + F_1(t) f_3$, $\bar{A} = \bar{A}_0 + F_2(t) \bar{f}_3$, where $F_1' = e^t F$, $F_2' = e^{-t} F$. Finally, we get for the components of $T: T_1 = T_2 = 0$ and $T_3 = F_1 - e^{2t} F_2 = \int e^t F(t)\,\mathrm{d}t - e^{2t} \int e^{-t} F(t)\,\mathrm{d}t = \frac{1}{2} \int s^{-1/2} F(s)\,\mathrm{d}s - s/2 \int s^{-3/2}$. $. F(s)\,\mathrm{d}s = -\left[\int(\int s^{-3/2} F(s)\,\mathrm{d}s)\,\mathrm{d}s\right]_{s=e^{2t}}$, where the substitution $s = e^{2t}$ and integration by parts was used. This shows that T_3 can be an arbitrary function; let us denote it by $G(s)$. The final form of $g(s)$ is then

$$(20) \qquad g(s) = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & -\sqrt{(s^2 - 1)}, & 0 \\ 0, & \sqrt{(s^2 - 1)}, & 1, & 0 \\ G(s), & 0, & 0, & s \end{pmatrix}.$$

Any plane curve can be locally generated by a suitable motion of this type as its trajectory. To see it we compute for instance the trajectory of the point $x_0 = z_0 = 0$, $y_0 = -1$. We get $x = \sqrt{(s^2 - 1)}$, $y = -1$, $z = G(s)$.

At last we get to the case (13): The motion can be written in the form

$$(21) \qquad g(s) = \delta^{-1} \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1 - \frac{1}{2}C_2 s, & -\sqrt{m}, & 0 \\ 0, & \sqrt{m}, & 1 - \frac{1}{2}C_2 s, & 0 \\ 0, & 0, & 0, & s \end{pmatrix}.$$

The differential equation for the trajectory is the same as in the case (9).

Theorem. *Non Euclidean similarity motions in E_3 with only plane trajectories are, up to the equivalence, exactly those given by* (17)–(21). *All of them are cylindrical (they preserve one direction) and the trajectories are affinely equivalent conic sections in the cases* (17), (19) and (21), *and arbitrary plane curves in the cases* (18) *and* (20).

Remark. Euclidean motions with only planar trajectories (apart from the case of the planar motion in parallel planes) have the following properties: They are cylindrical, trajectories are affinely equivalent and all of them are ellipses. The present paper shows that if we increase the group of transformations, such motions remain cylindrical, but the other properties are lost. It can be shown that in larger groups of transformations of the space even the last property will not be preserved.

Remark. Each cylindrical similarity motion can be written as a product of two motions, a plane similarity motion in a plane α and a similarity motion in a line perpendicular to the plane α. Let us study out motions (17)–(21) from this point of view.

In the cases (18) and (20) the motion is a product of a plane similarity motion with one fixed point, let us denote it by P, with an arbitrary similarity motion in a straight line. The planar motion in these two cases has all trajectories straight lines, in (18) all of them pass through the fixed point P, in (20) the trajectory of a point X is the straight line through P perpendicular to $\overline{PX}$.

In the cases (19) and (21) the planar motion is the same. It has again a fixed point P, all trajectories are similar conic sections with one axis passing through P.

Finally, we shall concentrate on the general case (17). The corresponding plane similarity motion has conic sections as poloids. The trajectory of any point $Q = \bar{A}_0 + x_0\bar{f}_1 + y_0\bar{f}_2$ is a conic section which degenerates in a straight line if

$$x_0^2 + (y_0 + 2\delta^2 C_2^{-1} - C_2)^2 = 4\delta^4 C_2^{-2} \text{ for } C_2 \neq 0 \text{ and } x_0 = \text{const. for } C_2 = 0\,.$$

All these straight line trajectories pass through a fixed point if $C_2 \neq 0$, they are parallel for $C_2 = 0$. This shows that the plane motion in the case (17) is a similarity analogue of the elliptical motion in the Euclidean plane which occurs in the Euclidean space motion discussed (the Darboux motion). We notice also that the motions (18)–(21) preserve a straight line which corresponds to the so called "special" Darboux motion in the Euclidean case, while (17) corresponds to the "general" one.

References

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Souhrn

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V článku jsou nalezeny a explicitně popsány všechny podobnostní (ekviformní) pohyby v prostoru, které mají pouze rovinné trajektorie. Nejdříve jsou odvozeny diferenciální rovnice pro ekviformní pohyby s pouze rovinnými trajektoriemi. Tyto rovnice jsou použitím metody specialisace repéru zjednodušeny tak, že je lze explicitně řešit. Integrováním Frenetových formulí pro uvažované pohyby je pak získáno jejich explicitní vyjádření. Všechny tyto pohyby jsou cylindrické (zachovávají alespoň jeden směr) a dělí se na 5 různých typů. U 3 typů jsou trajektoriemi kuželosečky u zbývajících dvou typů může libovolná rovinná křivka být trajektorií.

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