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ON NUMERICAL INTEGRATION OF IMPLICIT ORDINARY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Consider the initial-value problem for the implicit ordinary differential equation (IODE)

(1) $$y'(x) = f(x, y(x), y'(x)), \quad x \in I := [x_0, x_0 + a],$$
$$y(x_0) = y_0 .$$

The initial-value problem of the form

$$F(x, y(x), y'(x)) = 0, \quad x \in I,$$
$$y(x_0) = y_0$$

under a suitable assumption on F can be reduced to the problem (1).

Assume that B is a Banach space with a norm $\|\cdot\|$ and

(H_1) the function $f : I \times B \times B \to B$ is continuous and satisfies the Lipschitz condition

$$\|f(x, y, z) - f(x, \bar{y}, \bar{z})\| \leqq L_1\|y - \bar{y}\| + L_2\|z - \bar{z}\|$$

with some $L_1 \geqq 0$, $0 \leqq L_2 < 1$, for $x \in I$, $y, \bar{y}, z, \bar{z} \in B$. The problem (1) has a unique solution $Y \in C^1(I, B)$ ($C^1(I, B)$ denotes the space of all functions from I into B with a continuous first derivative). Indeed, in view of (H_1) and the Banach contraction principle there exists a function $g : I \times B \to B$ such that (1) is equivalent to the problem

(2) $$y'(x) = g(x, y(x)), \quad x \in I,$$
$$y(x_0) = y_0 .$$

It is easy to check that g is continuous and satisfies the Lipschitz condition with the constant $L_1/(1 - L_2)$. From this existence and uniqueness follow by the well known Picard theorem (see for example [2]). A straightforward proof of this fact is also quite easy. Put $y'(x) = z(x)$. Then (1) is equivalent to the equation

$$z(x) = f(x, y_0 + \int_{x_0}^{x} z(s)\, ds, z(x)), \quad x \in I,$$

which can be considered in the Banach space $C(I, B)$ with the norm

$$\|z\|_C := \{\exp(-\lambda(x - x_0)) \|z(x)\| : x \in I\}, \quad \lambda > 0.$$

By (H_1), the operator defined by the right hand side of the above equation is a contraction for sufficiently large λ. Now the Banach contraction principle proves our assertion.

Existence and uniqueness of the solution of (1) is implied also by many more general results published before (see for example [5]).

Note that although the problem (1) can be reduced to (2), the function g appearing in (2) is not known explicitly. Therefore (1) cannot be solved numerically by solving (2).

For computing a numerical approximation to the solution of (1) a uniform step size h is used. Put $x_i = x_0 + ih$, $i = 0, 1. \ldots, N$, $Nh = a$, $I_{h_0} := [0, h_0]$, $h_0 > 0$. Suppose we are given functions $\alpha_s : \mathcal{N} \to R$, $s = 0, 1, \ldots, k$, and $\phi_i : I^{k+1} \times B^{k+1} \times \times B^{k+1} \times I_{h_0} \to B$, $i \in \mathcal{N} = \{0, 1, \ldots\}$. Assume that

(H_2) the family $\{\phi_i : i \in \mathcal{N}\}$ is equicontinuous;

(H_3) the family $\{\phi_i : i \in \mathcal{N}\}$ satisfies the Lipschitz condition (uniformly in i)

$$\begin{aligned} &\|\phi_i(u_0, \ldots, u_k, v_0, \ldots, v_k, w_0, \ldots, w_k, h) - \\ &- \phi_i(u_0, \ldots, u_k, \bar{v}_0, \ldots, \bar{v}_k, \bar{w}_0, \ldots, \bar{w}_k, h)\| \leqq \\ &\leqq L\left(\sum_{j=0}^{k} \|v_j - \bar{v}_j\| + \sum_{j=0}^{k} \|w_j - \bar{w}_j\|\right), \end{aligned}$$

for $u_j \in I$, $v_j, \bar{v}_j, w_j, \bar{w}_j \in B$, $j = 0, 1, \ldots, k$, $h \in I_{h_0}$, $i \in \mathcal{N}$;

(H_4) the functions α_j, $j = 0, 1, \ldots, k$, are bounded and $\alpha_k \equiv 1$.

For the numerical integration of the problem (1) consider a general nonstationary quasilinear multistep method of the form

$$(3) \qquad \sum_{s=0}^{k} \alpha_s(i)\, y_{i+s} = h\phi_i(x_{i+k}, \ldots, x_i, y_{i+k}, \ldots, y_i, z_{i+k}, \ldots, z_i, h),$$
$$z_{i+k} = f(x_{i+k}, y_{i+k}, z_{i+k}),$$

$i = 0, 1, \ldots, N - k$, where y_s, z_s for $s = 0, 1, \ldots, k - 1$ are given. Here y_i and z_i are numerical approximations to $Y(x_i)$ and $Y'(x_i)$, respectively (Y is the solution of (1)).

A special case of the method (3), namely the Euler method, was considered by Mamiedow [8].

Note that if the function f appearing in (1) does not depend on y' then (3) reduces to the method considered in [4].

In this paper a general convergence theorem is stated for the method (3). Next we quote special cases of (3), namely the linear multistep methods, Runge-Kutta methods, Rosenbrock methods and second derivative methods. The theory of these methods in the case of ordinary differential equations (ODEs) may be found in [3], [6], [7]; the second derivative methods are considered in [1]. In this paper it is shown that all these methods can be adapted in a simple way to the case of IODEs. Moreover, the resulting methods are of the same order as the corresponding methods for ODEs.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF (3)

Later on we shall need the following

Lemma 1. *Assume that* $G : B \times B \to B$ *satisfies the inequality*

$$\|G(x, y) - G(\bar{x}, \bar{y})\| \leqq q \,.\, \max \{\|x - \bar{x}\|, \ \|y - \bar{y}\|\} \tag{4}$$

with some $0 \leqq q < 1$. *Then the sequences*

$$x_{n+1} = G(x_n, x_n), \qquad x_0 \in B,$$

$$\tilde{x}_{n+1} = G(\tilde{x}_n, \tilde{x}_{n-1}), \quad \tilde{x}_0, \tilde{x}_1 \in B,$$

are convergent to the unique solution x^* *of the equation* $x = G(x, x)$. *Moreover,*

$$\|x_n - x^*\| \leqq \frac{q^n}{1 - q} \|x_1 - x_0\|, \quad n \geqq 0, \tag{5}$$

$$\|\tilde{x}_n - x^*\| \leqq \frac{(\sqrt{q})^{n-1}}{1 - \sqrt{q}} \max \{\|\tilde{x}_2 - \tilde{x}_1\|, \ \|\tilde{x}_1 - \tilde{x}_0\|\}, \quad n \geqq 1. \tag{6}$$

Proof. The existence of a unique solution x^* of the equation $x = G(x, x)$, the convergence of $\{x_n\}_{n=0}^{\infty}$ to x^* and the inequality (5) follow from the Banach contraction principle. To prove that the sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ is convergent note that

$$\|\tilde{x}_{n+1} - \tilde{x}_n\| \leqq q^{n/2} \quad \max \{\|\tilde{x}_2 - \tilde{x}_1\|, \|\tilde{x}_1 - \tilde{x}_0\|\} \quad \text{for } n \text{ even},$$

$$\|\tilde{x}_{n+1} - \tilde{x}_n\| \leqq q^{(n-1)/2} \max \{\|\tilde{x}_2 - \tilde{x}_1\|, \|\tilde{x}_1 - \tilde{x}_0\|\} \quad \text{for } n \text{ odd}.$$

These inequalities leads to

$$\|\tilde{x}_{n+m} - \tilde{x}_n\| \leqq q^{(n-1)/2} \frac{1 - q^{m/2}}{1 - q^{1/2}} \max \{\|\tilde{x}_2 - \tilde{x}_1, \|\tilde{x}_1 - \tilde{x}_0\|\}$$

for all $n \in \mathcal{N}$, $m \geqq n$. This proves the convergence of $\{\tilde{x}_n\}_{n=0}^{\infty}$. Now if m passes to the infinity we obtain (6). This completes the proof.

Rewrite the system (3) in the form

$$(7) \qquad \begin{aligned} y_{i+k} &= \psi(y_{i+k}, z_{i+k}), \\ z_{i+k} &= F(y_{i+k}, z_{i+k}), \end{aligned}$$

$i = 0, 1, \ldots, N - k$, where

$$\psi(y_{i+k}, z_{i+k}) := -\sum_{s=0}^{k-1} \alpha_s(i)\, y_{i+s} + h\phi_i(x_{i+k}, \ldots, x_i, y_{i+k}, \ldots, y_i, z_{i+k}, \ldots, z_i, h),$$

$$F(y_{i+k}, z_{i+k}) := f(x_{i+k}, y_{i+k}, z_{i+k}).$$

We have the following

Lemma 2. *Assume that* (H_1), (H_2) *and* (H_3) *hold. Then the system* (3) *has a unique solution* (y_{i+k}, z_{i+k}) *for sufficiently small* h. *This solution can be determined as the limit of the simple iterations*

$$(8) \qquad \begin{aligned} y_{i+k}^{[n+1]} &= \psi(y_{i+k}^{[n]}, z_{i+k}^{[n]}), \quad & y_{i+k}^{[0]} &= y_{i+k-1}, \\ z_{i+k}^{[n+1]} &= F(y_{i+k}^{[n]}, z_{i+k}^{[n]}), \quad & z_{i+k}^{[0]} &= z_{i+k-1}, \end{aligned}$$

$n = 0, 1, \ldots,$ *or as the limit of the modified iterations*

$$(9) \qquad \begin{aligned} \tilde{y}_{i+k}^{[n+1]} &= \psi(\tilde{y}_{i+k}^{[n]}, \tilde{z}_{i+k}^{[n]}), \quad & \tilde{y}_{i+k}^{[0]} &= y_{i+k-1}, \\ \tilde{z}_{i+k}^{[n+1]} &= F(\tilde{y}_{i+k}^{[n+1]}, \tilde{z}_{i+k}^{[n]}), \quad & \tilde{z}_{i+k}^{[0]} &= z_{i+k-1}, \end{aligned}$$

$n = 0, 1, \ldots$

Proof. Clearly (7) is equivalent to the system

$$(10) \qquad \begin{aligned} y_{i+k} &= \psi(y_{i+k}, z_{i+k}), \\ z_{i+k} &= F(\psi(y_{i+k}, z_{i+k}), z_{i+k}). \end{aligned}$$

Define a function $G : (B \times B) \times (B \times B) \to B \times B$ by

$$G(u, v) = (\psi(u), F(\psi(v), \lambda(u)))^{\mathsf{T}}$$

for $u, v \in B \times B$, where $\lambda : B \times B \to B$ is given by $\lambda(u) = u_2$, $u = (u_1, u_2)^{\mathsf{T}}$, $u_1, u_2 \in B$ and T, stands for transportion. Now the system (10) can be written in the form

$$(11) \qquad u_{i+k} = G(u_{i+k}, u_{i+k}),$$

where $u_{i+k} = (y_{i+k}, z_{i+k})^{\mathsf{T}}$. It follows from (H_1) and (H_3) that for sufficiently small h the function G satisfies the condition (4), so the equation (11) has a unique solution

(y_{i+k}, z_{i+k}). This solution can be determined by two-point iterations

$$u_{i+k}^{[n+1]} = G(u_{i+k}^{[n]}, u_{i+k}^{[n-1]}),$$
$$u_{i+k}^{[0]} = (y_{i+k-1}, z_{i+k-1})^{\mathsf{T}},$$
$$u_{i+k}^{[1]} = (\psi(y_{i+k}^{[0]}, z_{i+k}^{[0]}), F(y_{i+k}^{[0]}, z_{i+k}^{[0]}))^{\mathsf{T}},$$

$n = 0, 1, \ldots$, which correspond to the simple iterations defined by (8), or by the simple iterations

$$\tilde{u}_{i+k}^{[n+1]} = G(\tilde{u}_{i+k}^{[n]}, \tilde{u}_{i+k}^{[n]}),$$
$$\tilde{u}_{i+k}^{[0]} = (y_{i+k-1}, z_{i+k-1})^{\mathsf{T}},$$

$n = 0, 1, \ldots$, which correspond to the modified iterations defined by (9). Now Lemma 2 follows from Lemma 1.

Remark 1. In view of the estimates given in Lemma 1 we can expect that in order to find a solution of (3) it is better to use the modified iterations defined by (9) (they should converge more rapidly).

Remark 2. If the functions ϕ_i, $i \in \mathcal{N}$, appearing in (3) do not depend on y_{i+k}, z_{i+k} (this corresponds to the explicit methods for ODEs), the system (3) becomes simpler. In this case y_{i+k} is given explicitly by the first equation and z_{i+k} is uniquely determined by the second equation.

3. CONVERGENCE, CONSISTENCY AND STABILITY

Similarly as in the case of ODEs (see [4]) we introduce

Definition. *The method* (3) *is convergent to a solution* Y *of* (1) *if* $\max\{\|Y(x_s) - y_s\| : 0 \leqq s \leqq N\} \to 0$ as $h \to 0$. The order of convergence is p if $\max\{\|Y(x_s) - y_s\| : 0 \leqq s \leqq N\} = 0(h^p)$ as $h \to 0$.

Definition. *The method* (3) *is consistent with the problem* (1) *on the solution* Y *if*

$$\sum_{s=0}^{k} \alpha_s(i)\, Y(x + sh) = h\phi_i(x + kh, \ldots, x, Y(x + kh), \ldots, Y(x),$$
$$Y'(x + kh), \ldots, Y'(x), h) + h\,\eta(x, h, i)$$

and $\eta(h) \to 0$ *as* $h \to 0$, *where* $\eta(h)$ *is defined by*

$$\eta(h) := \sup\{\|\eta(x, h, i)\| : x \in [x_0, x_0 + a - kh],\quad 0 \leqq i \leqq N - k\}.$$

The method (3) *is of an order* p *if* $\eta(h) = 0(h^p)$.

We have the following

Theorem 1. *Assume that* $Y \not\equiv 0$, (H_2) *and* (H_4) *hold. Then the method* (3) *is consistent with the problem* (1) *on the solution* Y *if and only if*

$$\sum_{s=0}^{k} \alpha_s(i) = 0\,, \quad i \in \mathscr{N}\,, \tag{12}$$

and

$$\sum_{s=0}^{k} s\,\alpha_s(i) f(x, Y(x), Y'(x)) = \phi_i(x, \ldots, x, Y(x), \ldots, Y(x), Y'(x), \ldots, Y'(x), 0)\,, \tag{13}$$

for $x \in I$, $i \in \mathscr{N}$.

The proof of this theorem is almost identical to that in the case of ODEs (see [4]) and is therefore omitted.

Remark 3. It is obvious that (13) holds if

$$\sum_{s=0}^{k} s\,\alpha_s(i) f(x, y, z) = \phi_i(x, \ldots, x, y, \ldots, y, z, \ldots, z, 0)$$

for $x \in I$, $i \in \mathscr{N}$ and for any (y, z) belonging to a set $\mathscr{D} \subset B \times B$ which contains the values of (Y, Y').

To introduce the definition of stability of the method (3) we need some notions from the theory of recurrent equations. All these notions can be found in [9] or in condensed form in [4].

Definition. *The method* (3) *is stable if the trivial solution of the scalar recurrent equation*

$$\sum_{s=0}^{k} \alpha_s(i)\, c_{i+s} = 0\,, \quad i \in \mathscr{N}\,, \tag{14}$$

is uniformly stable.

Remark 4. In the case when the method (3) is stationary, i.e. α_s, $s = 0, 1, \ldots, k$, do not depend on i, the stability defined in this way is equivalent to the well known root condition. This means that no root of the polynomial

$$p(\lambda) = \sum_{s=0}^{k} \alpha_s \lambda^s$$

has modulus greater than one and every root with modulus one is simple.

In the proof of the convergence theorem the lemma given below plays an important role (see [4]).

Lemma 3. *Assume that the trivial solution of* (14) *is uniformly stable. Then there exists a constant* $C > 1$ *such that every solution of the equation*

$$\sum_{s=0}^{k} \alpha_s(i)\, d_{i+s} = h_i\,, \quad i \in \mathscr{N}\,,$$

(*considered in the space* B) *satisfies the inequality*

$$\max_{0 \leqq s \leqq k-1} \|d_{i+s}\| \leqq C\Big[\max_{0 \leqq s \leqq k-1} \|d_s\| + \sum_{s=0}^{i-1} \|h_s\|\Big]\,, \quad i \in \mathscr{N}\,.$$

(*we set* $\sum_{s=0}^{-1} = 0$).

Let $(\tilde{y}_i, \tilde{z}_i)$ denote the solution of the system

$$(15)\qquad \begin{aligned} \sum_{s=0}^{k} \alpha_s(i)\, \tilde{y}_{i+s} &= h\phi_i(x_{i+k}, \dots, x_i, \tilde{y}_{i+k}, \dots, \tilde{y}_i, \tilde{z}_{i+k}, \dots, \tilde{z}_i, h) + p_{i+k}^{(1)}\,, \\ \tilde{z}_{i+k} &= f(x_{i+k}, \tilde{y}_{i+k}, \tilde{z}_{i+k}) + p_{i+k}^{(2)}\,, \end{aligned}$$

$i = 0, 1, \dots, N - k$, where $\tilde{y}_s = y_s$, $\tilde{z}_s = z_s$ for $s = 0, 1, \dots, k - 1$. Here $p_i^{(j)} = r_i^{(j)} + q_i^{(j)}$, $j = 1, 2$, $r_i^{(1)}, r_i^{(2)}$, are local round-off errors and $q_i^{(1)}$, $q_i^{(2)}$ arise as a result of the fact that the system (3) cannot be solved exactly. We have $r_s^{(1)} = r_s^{(2)} = q_s^{(1)} = q_s^{(2)} = 0$ for $s = 0, 1, \dots, k - 1$. Denote the global error of the method (3) at the point x_i by $\varepsilon_i := Y(x_i) - \tilde{y}_i$, $i = 0, 1, \dots, N$. Put

$$p^{(j)} := \max_{0 \leqq s \leqq N} \left\| p_s^{(j)} \right\|, \quad j = 1, 2\,,$$

$$\varepsilon'_i := Y'(x_i) - \tilde{z}_i\,, \quad i = 0, 1, \dots, N\,.$$

We have the following

Theorem 2. *Suppose that*

1° *the conditions* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold,*

2° *the method* (3) *is stable and consistent with* (1) *on the solution* Y,

3° $\max_{0 \leqq s \leqq k-1} \left\| Y(x_s) - y_s \right\| = o(1)$, $\max_{0 \leqq s \leqq k-1} \left\| Y'(x_s) - z_s \right\| = o(1)$, $p^{(1)} = o(h)$, $p^{(2)} = o(1)$, *as* $h \to 0$.

Then the method (3) *is convergent to the solution* Y *of* (1), *i.e.* $\lim_{h \to 0} \max_{0 \leqq s \leqq N} \left\| Y(x_s) - y_s \right\| = 0$.

Proof. By consistency,

$$(16)\qquad \begin{aligned} \sum_{s=0}^{k} \alpha_s(i)\, Y(x_{i+s}) = h\phi_i(&x_{i+k}, \dots, x_i, Y(x_{i+k}), \dots, Y(x_i)\,, \\ &Y'(x_{i+k}), \dots, Y'(x_i), h) + h\eta(x_i, h, i)\,. \end{aligned}$$

Subtracting (15) from (16) we get

$$(17)\qquad \sum_{s=0}^{k} \alpha_s(i)\, \varepsilon_{i+s} = h\gamma_i + h\eta(x_i, h, i) - p_{i+k}^{(1)}\,,$$

$i = 0, 1, \dots, N - k$, where

$$\begin{aligned} \gamma_i := \phi_i(&x_{i+k}, \dots, x_i, Y(x_{i+k}), \dots, Y(x_i), Y'(x_{i+k}) \dots, Y'(x_i), h) - \\ &- \phi_i(x_{i+k}, \dots, x_i, \tilde{y}_{i+k}, \dots, \tilde{y}_i, \tilde{z}_{i+k}, \dots, \tilde{z}_i, h)\,. \end{aligned}$$

Define e_i, $i = 0, 1, \dots, N - k + 1$, by $e_i := \max_{0 \leqq s \leqq k-1} \left\| \varepsilon_{i+s} \right\|$. Then $\left\| \varepsilon_{s+j} \right\| \leqq e_s$ for $j = 0, 1, \dots, k - 1$, $s = 0, 1, \dots, N - k + 1$, and $\left\| \varepsilon_{s+k} \right\| \leqq e_{s+1}$ for $s = 0, 1, \dots$

$\ldots, N - k$. Applying Lemma 3 and the condition (H_3) to (17) we obtain

$$e_i \leqq C\Big[e_0 + hL\sum_{s=0}^{i-1}\sum_{j=0}^{k}\big(\|\varepsilon_{s+j}\| + \|\varepsilon'_{s+j}\|\big) + \sum_{s=0}^{i-1}\big(h\|\eta(x_s, h, s)\| + \|p^{(1)}_{s+k}\|\big)\Big]$$

for $i = 0, 1, \ldots, N - k + 1$. Denote the right hand side of the last inequality by w_i, Then $e_i \leqq w_i$, $i = 0, 1, \ldots, N - k + 1$, and

$$w_{i+1} - w_i = C\Big[hL\sum_{j=0}^{k}\big(\|\varepsilon_{i+j}\| + \|\varepsilon'_{i+j}\|\big) + h\|\eta(x_i, h, i)\| + \|p^{(1)}_{i+k}\|\Big] \tag{18}$$

for $i = 0, 1, \ldots, N - k$. We now estimate $\|\varepsilon'_j\|$. Since

$$Y'(x_j) = f(x_j, Y(x_j), Y'(x_j)),$$
$$\tilde{z}_j = f(x_j, \tilde{y}_j, \tilde{z}_j) + p^{(2)}_j$$

for $j = k, k + 1, \ldots, N$, subtracting these equations and using (H_1) we arrive at the inequality

$$\|\varepsilon'_j\| \leqq \frac{L_1}{1 - L_2}\|\varepsilon_j\| + \frac{1}{1 - L_2}\|p^{(2)}_j\|, \tag{19}$$

$j = k, k + 1, \ldots, N$. Assume that $i \geqq k$. Using (19) in (18) we get

$$w_{i+1} \leqq w_i + C\Bigg[hL\Big(1 + \frac{L_1}{1 - L_2}\Big)\sum_{j=0}^{k}\|\varepsilon_{i+j}\| + h\frac{L}{1 - L_2}\sum_{j=0}^{k}\|p^{(2)}_{i+j}\| +$$
$$+ h\|\eta(x_i, h, i)\| + \|p^{(1)}_{i+k}\|\Bigg].$$

Consequently

$$w_{i+1} \leqq w_i + hA_1(w_{i+1} + w_i) + h\,\tilde{p}^{(2)} + h\,\tilde{\eta}(h) + \tilde{p}^{(1)},$$

where

$$A_1 = kCL\Big(1 + \frac{L_1}{1 - L_2}\Big),$$
$$\tilde{p}^{(2)} = \frac{(k + 1)\,CL}{1 - L_2}\,p^{(2)},$$
$$\tilde{\eta}(h) = C\,\eta(h),$$
$$\tilde{p}^{(1)} = Cp^{(1)}.$$

For $0 \leqq i < k$ we obtain

$$w_{i+1} \leqq w_i + hA_1(w_{i+1} + w_i) + hkCLf_0 + h\tilde{p}^{(2)} + h\,\tilde{\eta}(h) + \tilde{p}^{(1)},$$

where $f_0 := \max\limits_{0 \leqq s \leqq k-1}\|\varepsilon'_s\|$. This can be summarized in

$$w_{i+1} \leqq w_i + hA_1(w_{i+1} + w_i) + hkCLf_0 + h\tilde{p}^{(2)} + h\,\tilde{\eta}(h) + \tilde{p}^{(1)}$$

for all $i = 0, 1, \ldots, N - k$. Let $\bar{h} \in I_{h_0}$ be such that $A_2 := 1 - \bar{h}A_1 < 0$. Provided

$0 \leqq h \leqq \bar{h}$, routine manipulations yield

$$w_{i+1} \leqq (1 + hA_3)\, w_i + h\bar{f}_0 + h\bar{p}^{(2)} + h\,\bar{\eta}(h) + \bar{p}^{(1)}\,,$$

$i = 0, 1, \ldots, N - k$, where $A_3 = 2A_1A_2^{-1}$, $\bar{f}_0 = kCLA_2^{-1}f_0$, $\bar{p}^{(2)} = A_2^{-1}\bar{p}^{(2)}$, and so on. Using Lemma 1.2 of [3] we conclude

$$e_i \leqq w_i \leqq w_0 \exp\left(A_3(x_i - x_0)\right) + \frac{1}{A_3}\left(\exp\left(A_3(x_i - x_0)\right) - 1\right) . \tag{20}$$

$$\cdot \left(\bar{f}_0 + \bar{p}^{(2)} + \bar{\eta}(h) + \frac{1}{h}\,\bar{p}^{(1)}\right),$$

$i = 0, 1, \ldots, N - k + 1$. Now in view of the assumption 3° the theorem follows.

Corollary. *Instead of* 3° *assume that* $e_0 = O(h^p)$, $f_0 = O(h^p)$, $p^{(1)} = O(h^{p+1})$, $\eta(h) = O(h^p)$, $p^{(2)} = O(h^p)$. *Then the order of convergence is* p.

5. THE SPECIAL CASES OF THE METHOD (3)

Linear multistep methods and Runge-Kutta methods can be easily adapted to the case of IODEs. We now present these adaptations:

1. *Linear multistep methods*:

$$\sum_{s=0}^{k} \alpha_s y_{i+s} = h \sum_{s=0}^{k} \beta_s z_{i+s}\,, \tag{21}$$

$$z_{i+k} = f(x_{i+k}, y_{i+k}, z_{i+k})\,,$$

$i = 0, 1, \ldots, N - k$. For these methods the consistency and order conditions are the same as in the case of ODEs (see [6]).

2. *Runge-Kutta methods*:

$$y_{i+1} = y_i + h \sum_{r=1}^{R} w_r k_r\,, \quad i = 0.\ 1, \ldots, N - 1\,, \tag{22}$$

$$k_r = f(x_i + ha_i, y_i + h \sum_{s=1}^{R} b_{rs} k_s, k_r)\,, \quad r = 1, 2, \ldots, R\,,$$

where

$$a_r = \sum_{s=1}^{R} b_{rs}\,, \quad r = 1, 2, \ldots, R\,.$$

Also for these methods the consistency and order conditions are the same as in the case of ODEs (see [6]).

A more complicated situation arises in the case of Rosenbrock methods and second derivative methods.

3. *Rosenbrock methods.*

For the equation $y' = g(y)$ these methods assume the form (see [7] [10])

$$(23)\qquad \begin{aligned} y_{i+1} &= y_i + h\sum_{r=1}^{R} w_r k_r\,, \quad i = 0, 1, \ldots, N-1\,,\\ k_r &= g(y_i + h\sum_{s=1}^{r-1} b_s k_s) + h a_r \frac{\partial g}{\partial y}(y_i + h\sum_{s=1}^{r-1} c_s k_s)\, k_r\,, \end{aligned}$$

$r = 1, 2, \ldots, R$. Here $\partial g/\partial y$ is the Frechet derivative. The parameters w_r, a_r, b_r, c_r, can be evaluated via the Taylor series comparison. For the equation $y' = g(x, y)$ the method (23) can be rewritten in the form

$$(24)\qquad \begin{aligned} y_{i+1} &= y_i + h\sum_{r=1}^{R} w_r k_r\,, \quad i = 0, 1, \ldots, N-1\,,\\ k_r &= g(x_i + h\sum_{s=1}^{r-1} b_s, y_i + h\sum_{s=1}^{r-1} b_s k_s) + h a_r \Big(\frac{\partial g}{\partial x}(x_i + h\sum_{s=1}^{r-1} c_s\,,\\ &\quad y_i + h\sum_{s=1}^{r-1} c_s k_s) + \frac{\partial g}{\partial y}(x_i + h\sum_{s=1}^{r-1} c_s, y_i + h\sum_{s=1}^{r-1} c_s k_s)\, k_r\Big)\,, \end{aligned}$$

$r = 1, 2, \ldots, R$.

Consider now the problem (1) or equivalently the problem (2). From the relation $y' = g(x, y) = f(x, y, y')$ it follows that

$$(25)\qquad y'' = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} y' = \left(I - \frac{\partial f}{\partial y'}\right)^{-1}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'\right).$$

Here I is the identity operator. On the basis of this relation and (24) we propose the method

$$(26)\qquad \begin{aligned} y_{i+1} &= y_i + h\sum_{r=1}^{R} w_r k_r\,, \quad i = 0, 1, \ldots, N-1\,,\\ y_{i,r} &= y_i + h\sum_{s=1}^{r-1} b_s k_s\,,\\ z_{i,r} &= f(x_i + h\sum_{s=1}^{r-1} b_s, y_{i,r}, z_{i,r})\,,\\ \bar{y}_{i,r} &= y_i + h\sum_{s=1}^{r-1} c_s k_s\,,\\ \bar{z}_{i,r} &= f(x_i + h\sum_{s=1}^{r-1} c_s, \bar{y}_{i,r}, \bar{z}_{i,r})\,,\\ k_r &= \bar{z}_{i,r} + h a_r\Big(\Big(I - \frac{\partial f}{\partial y'}\Big)^{-1} \frac{\partial f}{\partial x}(x_i + h\sum_{s=1}^{r-1} c_s, \bar{y}_{i,r}, \bar{z}_{i,r}) +\\ &\quad + \Big(I - \frac{\partial f}{\partial y'}\Big)^{-1} \frac{\partial f}{\partial y}\Big(x_i + h\sum_{s=1}^{r-1} c_s, \bar{y}_{i,r}, \bar{z}_{i,r}\Big) k_r\Big)\,, \end{aligned}$$

$r = 1, 2, \ldots, R$. The convergence of the method (26) follows from Theorem 2 if we assume in addition that the functions $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial y'$, are bounded and satisfy the Lipschitz condition with respect to the second and third arguments.

4. *Second derivative methods.*

For the problem $y' = g(x, y)$ these methods assume the form

$$\sum_{s=0}^{k} \alpha_s y_{i+s} = h \sum_{s=0}^{k} \beta_s y'_{i+s} + h^2 \sum_{s=0}^{k} \gamma_s y''_{i+s}, \tag{27}$$

$i = 0, 1, \ldots, N - k$ (see [1]). On the basis of (25) and (27) we propose the method

$$\sum_{s=0}^{k} \alpha_s y_{i+s} = h \sum_{s=0}^{k} \beta_s z_{i+s} +$$

$$+ h^2 \sum_{s=0}^{k} \gamma_s \left(I - \left(\frac{\partial f}{\partial z}\right)_{i+s}\right)^{-1} \left(\left(\frac{\partial f}{\partial x}\right)_{i+s} + \left(\frac{\partial f}{\partial y}\right)_{i+s} z_{i+s}\right), \tag{28}$$

$$z_{i+k} = f(x_{i+k}, y_{i+k}, z_{i+k}),$$

$i = 0, 1, \ldots, N - k$. Here $(\partial f/\partial x)_i$, $(\partial f/\partial y)_i$, $(\partial f/\partial z)_i$ are abbreviations of $(\partial f/\partial x)$. (x_i, y_i, z_i), $(\partial f/\partial y)(x_i, y_i, z_i)$ and $(\partial f/\partial z)(x_i, y_i, z_i)$, respectively. The convergence of these methods follows from Theorem 2 under the same conditions as in the case of Rosenbrock methods.

Note that all the methods considered in this section are of the same order as the corresponding methods for ODEs.

6. NUMERICAL RESULTS

Consider the problem

$$y'(x) = \frac{1}{16}(\sin(x^2 y') - \sin(\exp(y))) + \frac{1}{x}, \quad 1 \leqq x \leqq 4, \tag{29}$$

$$y(1) = 0.$$

The theoretical solution of this problem is $Y(x) = \ln(x)$.

This problem has been solved by the following methods:

1. *Adams-Bashforth method* $(A - B)$:

$$y_{i+3} = y_{i+2} + \frac{h}{12}(23z_{i+2} - 16z_{i+1} + 5z_i),$$

$$z_{i+3} = f(x_{i+3}, y_{i+3}, z_{i+3}),$$

$i = 0, 1, \ldots, N - 3$.

2. *Adams-Moulton method* (A – M):

$$y_{i+2} = y_{i+1} + \frac{h}{12}(5z_{i+2} + 8z_{i+1} - z_i),$$

$$z_{i+2} = f(x_{i+2}, y_{i+2}, z_{i+2}),$$

$i = 0, 1, \ldots, N - 2.$

3. *Explicit Runge-Kutta method* (ERK):

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 4k_2 + k_3), \quad i = 2, 3, \ldots, N - 1,$$

$$k_1 = f(x_i, y_i, k_1),$$

$$k_2 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hk_1, k_2),$$

$$k_3 = f(x_{i+1}, y_i - hk_1 + 2hk_2, k_3).$$

4. *Implicit Runge-Kutta method* (IRK):

$$y_{i+1} = y_i + \frac{h}{4}(k_1 + 3k_2), \quad i = 2, 3, \ldots, N - 1,$$

$$k_1 = f(x_i, y_i, k_1),$$

$$k_2 = f(x_i + \tfrac{2}{3}h, y_i + \tfrac{1}{3}hk_1 + \tfrac{1}{3}hk_2, k_2).$$

5. *Rosenbrock method* (ROS):

$$y_{i+1} = y_i + h(w_1k_1 + w_2k_2), \quad i = 2, 3, \ldots, N - 1,$$

$$z_{i+1} = f(x_{i+1}, y_{i+1}, z_{i+1}),$$

$$k_1 = z_i + ha_1\left(\frac{\dfrac{\partial f}{\partial x}}{1 - \dfrac{\partial f}{\partial z}}(x_i, y_i, z_i) + \frac{\dfrac{\partial f}{\partial y}}{1 - \dfrac{\partial f}{\partial z}}(x_i, y_i, z_i)\, k_1\right),$$

$$y_2 = y_i + b_1hk_1,$$

$$z_2 = f(x_i + b_1h, y_2, z_2),$$

$$\bar{y}_2 = y_i + c_1hk_1,$$

$$\bar{z}_2 = f(x_i + c_1h, \bar{y}_2, \bar{z}_2),$$

$$k_2 = \bar{z}_2 + ha_2\left(\frac{\dfrac{\partial f}{\partial x}}{1 - \dfrac{\partial f}{\partial z}}(x_i + c_1h, \tilde{y}_2, \tilde{z}_2) + \frac{\dfrac{\partial f}{\partial y}}{1 - \dfrac{\partial f}{\partial z}}(x_i + c_1h, \tilde{y}_2, \tilde{z}_2)\, k_2\right),$$

where $w_1 = -0.41315432$, $w_2 = 1.41315432$, $a_1 = 1.40824829$, $a_2 = 0.59175171$, $b_1 = c_1 = 0.17378667$.

All these methods have the order 3. Starting values are given by $y_0 = 1$, $y_1 = \ln(x_1)$, $y_2 = \ln(x_2)$, $z_0 = 1$, $z_1 = 1/x_1$, $z_2 = 1/x_2$. The iterations were running until two successive approximations differed by less than h^3. The computations were carried out on the Polish computer ODRA 1204. The errors at the point $x = 4$ and the computational time are given in the table below.

Table 1. Computational time and errors at the point $x = 4$.

h		A—B	A—M	IRK	ERK	ROS
0·1000	T	6·03	7·61	12·05	10·77	15·75
	error	$-6{\cdot}206\ 10^{-4}$	$6{\cdot}436\ 10^{-5}$	$-6{\cdot}377\ 10^{-5}$	$3{\cdot}706\ 10^{-5}$	$2{\cdot}256\ 10^{-4}$
0·0500	T	15·86	20·28	33·86	28·64	43·59
	error	$-1{\cdot}018\ 10^{-4}$	$8{\cdot}547\ 10^{-6}$	$-8{\cdot}481\ 10^{-6}$	$-3{\cdot}028\ 10^{-6}$	$8{\cdot}758\ 10^{-6}$
0·0250	T	39·10	49·99	86·63	70·84	108·66
	error	$-1{\cdot}458\ 10^{-5}$	$3{\cdot}640\ 10^{-7}$	$3{\cdot}300\ 10^{-7}$	$-1{\cdot}149\ 10^{-6}$	$-7{\cdot}630\ 10^{-7}$
0·0125	T	93·08	119·11	210·96	173·70	258·89
	error	$-1{\cdot}683\ 10^{-6}$	$1{\cdot}060\ 10^{-7}$	$0{\cdot}880\ 10^{-7}$	$-1{\cdot}550\ 10^{-7}$	$1{\cdot}050\ 10^{-7}$

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Souhrn

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O NUMERICKÉ INTEGRACI NEROZŘEŠENÝCH OBYČEJNÝCH DIFERENCIÁLNÍCH ROVNIC

V článku se ukazuje, jak je možno numerické metody řešení obyčejných diferenciálních rovnic upravit pro řešení nerozřešených obyčejných diferenciálních rovnic. Upravené metody jsou téhož řádu jako odpovídající metody pro obyčejné diferenciální rovnice. Je dokázána věta o konvergenci a uvedeny numerické příklady.

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