David Culpin

Distributions of random binary sequences

Aplikace matematiky, Vol. 25 (1980), No. 6, 408–416

Persistent URL: http://dml.cz/dmlcz/103879

DISTRIBUTIONS OF RANDOM BINARY SEQUENCES

DAVID CULPIN

(Received August 21, 1978)

1. INTRODUCTION

We are concerned with infinite binary sequences, denoted by $\boldsymbol{X}$ or $\{X_1, X_2, \ldots\}$, where each X_i is random and takes values 0 or 1. The simplest such sequence has the X_i's independent and identically distributed, a Bernoulli sequence.

Infinite binary sequences can be mapped onto the interval $[0, 1]$ in many ways; for example

$$\boldsymbol{X} \to H\boldsymbol{X} = \sum_{n=1}^{\infty} X_n/2^n ,$$

which is almost a one-to-one mapping. It is easily seen that if $\boldsymbol{X}$ is Bernoulli then $H\boldsymbol{X}$ is uniformly distributed when $P[X_n = 1] = \frac{1}{2}$; and Víšek [1] has obtained the distribution of $H\boldsymbol{X}$ when $P[X_n = 1] \neq \frac{1}{2}$. It is of interest also to find the mapping G of $\boldsymbol{X}$ onto $[0, 1]$ for which $G\boldsymbol{X}$ is uniformly distributed when $P[X_n = 1]$ has any particular value; and we could take this further and seek such a mapping when $\boldsymbol{X}$ is not a Bernoulli sequence. That is what is done in this article.

2. THE BERNOULLI CASE

If $\boldsymbol{X}$ is Bernoulli, what is the distribution of $H\boldsymbol{X}$? One thing that H does to sequences of 0's and 1's is to order them in a lexicographical manner. Thus we can write $\boldsymbol{x} < \boldsymbol{y}$ when $H\boldsymbol{x} < H\boldsymbol{y}$, and it can be seen that $\boldsymbol{x} < \boldsymbol{y}$ means that if the first of the elements which differ between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the n^{th}, then $x_n < y_n$, except that it is not allowed that both $x_{n+1} = x_{n+2} = \ldots = 1$ and $y_{n+1} = y_{n+2} = \ldots = 0$. It is also convenient to write $\boldsymbol{x} = \boldsymbol{y}$ when $H\boldsymbol{x} = H\boldsymbol{y}$, in which case the corresponding elements of $\boldsymbol{x}$ and $\boldsymbol{y}$ need not be identical; when they are identical, we write $\boldsymbol{x} \equiv \boldsymbol{y}$.

Returning to the distribution of $H\boldsymbol{X}$,

$$P[H\boldsymbol{X} < H\boldsymbol{x}] = P[\boldsymbol{X} < \boldsymbol{x}] =$$

$$= P[\bigcup_{n=1}^{\infty} (X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n < x_n)] = \sum_{n=1}^{\infty} x_n q \prod_{i=1}^{n-1} p^{x_i} q^{1-x_i} ,$$

where $p = 1 - q = P[X_n = 1]$. It is easy to see that $P[\boldsymbol{X} = \boldsymbol{x}] = 0$ if $0 < p < 1$, in which case $P[\boldsymbol{X} \leqq \boldsymbol{x}] = P[\boldsymbol{X} < \boldsymbol{x}]$. We shall assume henceforth that $0 < p < 1$.

There are two things we can note about the above calculation which will assist in our further progress. Firstly, the above probability is a function of $\boldsymbol{x}$, so we can write $P[\boldsymbol{X} \leqq \boldsymbol{x}] = B\boldsymbol{x}$ (B for Bernoulli), and hence

$$P[H\boldsymbol{X} \leqq H\boldsymbol{x}] = B\boldsymbol{x} = \sum_{n=1}^{\infty} x_n q \prod_{i=1}^{n-1} p^{x_i} q^{1-x_i} .$$

Secondly, it can be seen that the only property of the function H that is involved in the calculation of the above probabilities is that $H\boldsymbol{x} < H\boldsymbol{y}$ if and only if $\boldsymbol{x} < \boldsymbol{y}$, for any $\boldsymbol{x}$ and $\boldsymbol{y}$. Let us call any such function *strictly increasing*. Then, for *any* function F that is strictly increasing, $F\boldsymbol{x} = F\boldsymbol{y}$ if and only if $\boldsymbol{x} = \boldsymbol{y}$, and hence

$$P[F\boldsymbol{X} \leqq F\boldsymbol{x}] = B\boldsymbol{x} .$$

Notice, now, that the function B is itself strictly increasing. This is readily proved directly, but there is no need to do so, as Theorem 1 provides a proof. We can therefore write

$$P[B\boldsymbol{X} \leqq B\boldsymbol{x}] = B\boldsymbol{x} .$$

This suggests that $B\boldsymbol{X}$ is uniformly distributed on $[0, 1]$. To establish this it is necessary to know that B maps *onto* $[0, 1]$; for in that case for any t in $[0, 1]$ there is an $\boldsymbol{x}$ such that $B\boldsymbol{x} = t$, and then $P[B\boldsymbol{X} \leqq t] = t$ for all t, showing that $B\boldsymbol{X}$ is uniformly distributed. That B maps onto $[0, 1]$ may be proved directly, but the proof can be found in Theorem 2.

Conversely, if $B\boldsymbol{X}$ is uniformly distributed, then, for any n and $x_1, \ldots, x_n$,

$$\begin{aligned} P[X_1 &= x_1, \ldots, X_n = x_n] = \\ &= P[\{x_1, \ldots, x_n, \mathbf{0}\} \leqq \boldsymbol{X} \leqq \{x_1, \ldots, x_n, \mathbf{1}\}] = \\ &= P[B\{x_1, \ldots, x_n, \mathbf{0}\} \leqq B\boldsymbol{X} \leqq B\{x_1, \ldots, x_n, \mathbf{1}\}] = \\ &= B\{x_1, \ldots, x_n, \mathbf{1}\} - B\{x_1, \ldots, x_n, \mathbf{0}\} = \\ &= \prod_{i=1}^{n} p^{x_i} q^{1-x_i} , \end{aligned}$$

where $\mathbf{0} = \{0, 0, 0, \ldots\}$ and $\mathbf{1} = \{1, 1, 1, \ldots\}$, which shows that $\boldsymbol{X}$ is a Bernoulli sequence.

We can now conclude that $\boldsymbol{X}$ is a Bernoulli sequence if and only if $B\boldsymbol{X}$ is uniformly distributed, where B is given by $B\boldsymbol{x} = P[\boldsymbol{X} \leqq \boldsymbol{x}]$ for all $\boldsymbol{x}$, provided that $0 < P[X_n = 1] < 1$ for all n.

Notice the similarity between the above result and the corresponding result for a one-dimensional continuous random variable X, namely that FX is uniformly distributed if and only if X has distribution function F, provided that F is strictly increasing and continuous. This suggests that results like the above can be obtained whether or not $\boldsymbol{X}$ is a Bernoulli sequence. As a help in expressing more general results, we make some definitions and establish a couple of preliminary theorems.

Let F be any real valued function defined on sequences of 0's and 1's. F will be called *continuous* if, for any sequences $\boldsymbol{x}$ and $\boldsymbol{y}$, $F\{x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots\} \to F\boldsymbol{x}$ as $n \to \infty$. If F is an increasing function, that is, if $F\boldsymbol{x} \leqq F\boldsymbol{y}$ whenever $\boldsymbol{x} < \boldsymbol{y}$, then to know that F is continuous it is sufficient to know that both $F\{x_1, \ldots, x_n, \mathbf{0}\}$ and $F\{x_1, \ldots, x_n, \mathbf{1}\} \to F\boldsymbol{x}$ as $n \to \infty$. F will be called a *continuous distribution function* (abbreviated *cts. d.f.*) if F is increasing and continuous, with $F\mathbf{0} = 0$ and $F\mathbf{1} = 1$. It is assumed, of course, that F is *uniquely defined*, in the sense that $F\boldsymbol{x} = F\boldsymbol{y}$ when $\boldsymbol{x} = \boldsymbol{y}$ (not necessarily $\boldsymbol{x} \equiv \boldsymbol{y}$). The first theorem establishes a canonical form for cts. d.f.'s.

Theorem 1. (a) *F is a cts. d.f. if and only if it can be written in the form*

$$F\boldsymbol{x} = \sum_{n=1}^{\infty} x_n f_n(x_1, \ldots, x_{n-1}, 0)$$

for any $\boldsymbol{x}$, where the f_n's are functions satisfying, for $n = 1, 2, \ldots$ and all $\boldsymbol{x}$,

(i) $f_n(x_1, \ldots, x_n) \geqq 0$,

(ii) $f_n(x_1, \ldots, x_{n-1}, 0) + f_n(x_1, \ldots, x_{n-1}, 1) = f_{n-1}(x_1, \ldots, x_{n-1})$ (*with* $f_0 = 1$),

(iii) $f_n(x_1, \ldots, x_n) \to 0$ *as* $n \to \infty$.

(b) *The functions f_n are uniquely determined by F according to the relation*

$$f_n(x_1, \ldots, x_n) = F\{x_1, \ldots, x_n, \mathbf{1}\} - F\{x_1, \ldots, x_n, \mathbf{0}\}.$$

(c) *F is strictly increasing if and only if $f_n(x_1, \ldots, x_n) > 0$ for all n and $\boldsymbol{x}$.*

The next theorem shows that for cts. d.f.'s the properties of continuity and mapping onto $[0, 1]$ are equivalent. Notice that this situation is analogous to that holding for corresponding functions of single variables.

Theorem 2. (a) *Any cts. d.f. maps onto $[0, 1]$.*

(b) *Any increasing function mapping onto $[0, 1]$ is a cts. d.f..*

Proof of Theorems 1 and 2 can be found in the Appendix at the end of this article.

Let us call a probability distribution P of a random sequence *continuous* if $P[\boldsymbol{X} = \boldsymbol{x}] = 0$ for all $\boldsymbol{x}$, and *positive* if $P[X_1 = x_1, \ldots, X_n = x_n] > 0$ for all n and $\boldsymbol{x}$. We now make use of the above results to establish a relationship between cts. d.f.'s and continuous probability distributions of random sequences.

Theorem 3. (a) *There is a one-to-one correspondence between cts. d.f.'s F and continuous probability distributions P of random sequences; the F corresponding to P is given by $F\boldsymbol{x} = P[\boldsymbol{X} \leqq \boldsymbol{x}]$, where $\boldsymbol{X}$ is a random sequence with probability distribution P.*

(b) *In this correspondence, strictly increasing F's correspond to positive P's.*

(c) *If $\boldsymbol{X}$ has positive continuous probability distribution P, then the F corresponding to P is such that $F\boldsymbol{X}$ is uniformly distributed on $[0, 1]$.*

(d) *If F is any strictly increasing function for which $F\boldsymbol{X}$ is uniformly distributed on $[0, 1]$, then F is a cts. d.f. and the P corresponding to F is the probability distribution of $\boldsymbol{X}$.*

Proof. (a) Let P be any continuous probability distribution and $\boldsymbol{X}$ a random sequence with P as its probability distribution. Define the functions f_n by

$$f_n(x_1, \ldots, x_n) = P[X_1 = x_1, \ldots, X_n = x_n]$$

for all n and $\boldsymbol{x}$, and let

$$F\boldsymbol{x} = \sum_{n=1}^{\infty} x_n f_n(x_1, \ldots, x_{n-1}, 0)$$

for all $\boldsymbol{x}$. To show that F is a cts. d.f., it is sufficient to show that the conditions of Theorem 1(a) on the f_n are satisfied. These follow easily from the definition of f_n; in particular, $f_n(x_1, \ldots, x_n) \to P[\boldsymbol{X} = \boldsymbol{x}] = 0$ as $n \to \infty$, for all $\boldsymbol{x}$.

Now let F be any cts. d.f.. By Theorem 1(a),

$$F\boldsymbol{x} = \sum_{n=1}^{\infty} x_n f_n(x_1, \ldots, x_{n-1}, 0)\,,$$

where the functions f_n possess the properties listed in that theorem. Now define the function P by

$$P[X_1 = x_1, \ldots, X_n = x_n] = f_n(x_1, \ldots, x_n)$$

for all n and $\boldsymbol{x}$. The properties possessed by the functions f_n ensure that P is a continuous probability distribution. This establishes the one-to-one correspondence,

$$F\boldsymbol{x} = \sum_{n=1}^{\infty} x_n P[X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = 0]\,,$$

between cts. d.f.'s F and continuous probability distributions P of random sequences. Further,

$$F\boldsymbol{x} = \sum_{n=1}^{\infty} P[X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n < x_n] =$$

$$= P[\bigcup_{n=1}^{\infty} (X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n < x_n)] =$$

$$= P[\boldsymbol{X} < \boldsymbol{x}] = P[\boldsymbol{X} \leqq \boldsymbol{x}]\,.$$

(b) In the correspondence of (a), by Theorem 1(c), F is strictly increasing if and only if $f_n(x_1, \ldots, x_n) > 0$, that is $P[X_1 = x_1, \ldots, X_n = x_n] > 0$, for all n and x, that is, if and only if P is positive.

(c) Suppose that $\boldsymbol{X}$ has positive continuous probability distribution P, which corresponds to the cts. d.f. F. By Theorem 2(a), for any $t \in [0, 1]$ there is an $\boldsymbol{x}$ such that $F\boldsymbol{x} = t$. Then

$$P[F\boldsymbol{X} \leqq t] = P[F\boldsymbol{X} \leqq F\boldsymbol{x}] = P[\boldsymbol{X} \leqq \boldsymbol{x}] = F\boldsymbol{x} = t\,,$$

which shows that $F\boldsymbol{X}$ is uniformly distributed.

(d) If $F\boldsymbol{X}$ is uniformly distributed on $[0, 1]$, then F must map onto $[0, 1]$. Therefore, by Theorem 2(b), F is a cts. d.f.. It is required to show that the P corresponding to F is the probability distribution of $\boldsymbol{X}$. For any $\boldsymbol{x}$ and n, the event $[X_1 = x_1, \ldots$ $\ldots, X_n = x_n]$ is equivalent to $[\{x_1, \ldots, x_n, \mathbf{0}\} \leqq \boldsymbol{X} \leqq \{x_1, \ldots, x_n, \mathbf{1}\}]$, which is equivalent to

$$[F\{x_1, \ldots, x_n, \mathbf{0}\} \leqq F\boldsymbol{X} \leqq F\{x_1, \ldots, x_n, \mathbf{1}\}]\,.$$

The probability of this last event is

$$F\{x_1, \ldots, x_n, \mathbf{1}\} - F\{x_1, \ldots, x_n, \mathbf{0}\}\,.$$

With $F\boldsymbol{x} = \sum_{n=1}^{\infty} x_n P[X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = 0]$, Theorem 1(b) tells us that

$$P[X_1 = x_1, \ldots, X_n = x_n] = F\{x_1, \ldots, x_n, \mathbf{1}\} - F\{x_1, \ldots, x_n, \mathbf{0}\}\,,$$

which is therefore the probability of the event $[X_1 = x_1, \ldots, X_n = x_n]$. This shows that P is the probability distribution of $\boldsymbol{X}$, and completes the proof of Theorem 3.

Theorem 3 answers the question which was posed at the beginning of this article. Suppose we have a random sequence $\boldsymbol{X}$ and want a criterion for its having a continuous probability distribution P. From Theorem 3(b) the cts. d.f. F corresponding to P can be found. In the case that P is a Bernoulli distribution, F is the B which was defined earlier. From Theorem 3(c) and (d) the desired criterion is that $F\boldsymbol{X}$ be uniformly distributed.

APPENDIX

Proof of Theorem 1. Suppose F is a cts. d.f.. Let

$$f_n(x_1, \ldots, x_n) = F\{x_1, \ldots, x_n, \mathbf{1}\} - F\{x_1, \ldots, x_n, \mathbf{0}\}\,.$$

Then

$$\sum_{n=1}^{N} x_n f_n(x_1, \ldots, x_{n-1}, 0) =$$

$$= \sum_{n=1}^{N} x_n [F\{x_1, \ldots, x_{n-1}, 0, \mathbf{1}\} - F\{x_1, \ldots, x_{n-1}, \mathbf{0}\}] =$$

$$= \sum_{n=1}^{N} x_n [F\{x_1, \ldots, x_{n-1}, 1, \mathbf{0}\} - F\{x_1, \ldots, x_{n-1}, \mathbf{0}\}] =$$

$$= \sum_{n=1}^{N} [F\{x_1, \ldots, x_n, \mathbf{0}\} - F\{x_1, \ldots, x_{n-1}, \mathbf{0}\}] =$$

$$= F\{x_1, \ldots, x_N, \mathbf{0}\} .$$

Letting $N \to \infty$ gives the required form for $F\boldsymbol{x}$. The properties required of f_n follow easily from properties of F. In particular, the f_n are positive if F is strictly increasing.

Suppose that

$$F\boldsymbol{x} = \sum_{n=1}^{\infty} x_n \, g_n(x_1, \ldots, x_n, 0) ,$$

where the functions g_n possess the same properties as the f_n. Then

$$F\{x_1, \ldots, x_N, \mathbf{1}\} - F\{x_1, \ldots, x_N, \mathbf{0}\} =$$

$$= \sum_{n=N+1}^{\infty} g_n(x_1, \ldots, x_N, 1, \ldots, 1, 0) =$$

$$= \lim_{M \to \infty} \sum_{n=N+1}^{M} g_n(x_1, \ldots, x_N, 1, \ldots, 1, 0) =$$

$$= \lim_{M \to \infty} \sum_{n=N+1}^{M} [g_{n-1}(x_1, \ldots, x_N, 1, \ldots, 1) - g_n(x_1, \ldots, x_N, 1, \ldots, 1)] =$$

$$= \lim_{M \to \infty} [g_N(x_1, \ldots, x_N) - g_M(x_1, \ldots, x_N, 1, \ldots, 1)] =$$

$$= g_N(x_1, \ldots, x_N)$$

for all N and $\boldsymbol{x}$.

This shows, firstly, that the functions f_n are uniquely determined by F. Secondly, it assists in the proof of the converse of part (a) of the theorem. Thus, if

$$F\boldsymbol{x} = \sum_{n=1}^{\infty} x_n f_n(x_1, \ldots, x_{n-1}, 0) ,$$

then $f_n(x_1, \ldots, x_n) = F\{x_1, \ldots, x_n, \mathbf{1}\} - F\{x_1, \ldots, x_n, \mathbf{0}\}$; in particular, $f_0 = F\mathbf{1} - F\mathbf{0}$. It is required to show that F is a cts. d.f.. Clearly $F\mathbf{0} = 0$. Hence $F\mathbf{1} = f_0 = 1$.

Observe now that

$$F\{x_1, \ldots, x_n, \mathbf{0}\} = \sum_{s=1}^{n} x_s f_s(x_1, \ldots, x_{s-1}, 0) \to$$

$$\to \sum_{s=1}^{\infty} x_s f_s(x_1, \ldots, x_{s-1}, 0) = F\boldsymbol{x} \quad \text{as} \quad n \to \infty ;$$

(1) $$F\{x_1, \ldots, x_n, \mathbf{1}\} = f_n(x_1, \ldots, x_n) + F\{x_1, \ldots, x_n, \mathbf{0}\} \to 0 + F\boldsymbol{x} = F\boldsymbol{x} \quad \text{as} \quad n \to \infty .$$

Once it is known that F is increasing, these two facts will be sufficient to establish the continuity of F.

It is now shown that F is increasing and is uniquely defined in the sense required for a cts. d.f.. For $n = 1, 2, \ldots$ let

$$F^{(n)}\boldsymbol{x} = \sum_{s=n}^{\infty} x_s f_s(x_1, \ldots, x_{s-1}, 0) .$$

For $N > n$,

$$\begin{aligned}
&\sum_{s=n+1}^{N} (1 - x_s) f_s(x_1, \ldots, x_{s-1}, 1) = \\
&= \sum_{s=n+1}^{N} (1 - x_s) \left[F\{x_1, \ldots, x_{s-1}, \mathbf{1}\} - F\{x_1, \ldots, x_{s-1}, 0, \mathbf{1}\}\right] = \\
&= \sum_{s=n+1}^{N} \left[F\{x_1, \ldots, x_{s-1}, \mathbf{1}\} - F\{x_1, \ldots, x_s, \mathbf{1}\}\right] = \\
&= F\{x_1, \ldots, x_n, \mathbf{1}\} - F\{x_1, \ldots, x_N, \mathbf{1}\} = \\
&= F^{(n+1)}\{x_1, \ldots, x_n, \mathbf{1}\} - F^{(n+1)}\{x_1, \ldots, x_N, \mathbf{1}\} = \\
&= F\{x_1, \ldots, x_n, \mathbf{1}\} - F\{x_1, \ldots, x_n, \mathbf{0}\} - F^{(n+1)}\{x_1, \ldots, x_N, \mathbf{1}\} = \\
&= f_n(x_1, \ldots, x_n) - F^{(n+1)}\{x_1, \ldots, x_N, \mathbf{1}\} .
\end{aligned}$$

Letting $N \to \infty$ yields, in view of (1),

(2) $$f_n(x_1, \ldots, x_n) - F^{(n+1)}\boldsymbol{x} = \sum_{s=n+1}^{\infty} (1 - x_s) f_s(x_1, \ldots, x_{s-1}, 1) \geqq 0$$

for all n. Now take any $\boldsymbol{x}$ and $\boldsymbol{y}$ for which $\boldsymbol{x} \leqq \boldsymbol{y}$ and $\boldsymbol{x} \not\equiv \boldsymbol{y}$. For some n,

$$\boldsymbol{x} = \{x_1, \ldots, x_{n-1}, 0, \boldsymbol{x}^{(n+1)}\}$$

and

$$\boldsymbol{y} = \{x_1, \ldots, x_{n-1}, 1, \boldsymbol{y}^{(n+1)}\} ,$$

where

$$\boldsymbol{x}^{(n+1)} = \{x_{n+1}, x_{n+2}, \ldots\}$$

and

$$\boldsymbol{y}^{(n+1)} = \{y_{n+1}, y_{n+2}, \ldots\}$$

(or else, when $\boldsymbol{x} = \boldsymbol{y}$, their rôles may be interchanged). Then

$$F\boldsymbol{y} - F\boldsymbol{x} = f_n(x_1, \ldots, x_{n-1}, 0) + F^{(n+1)}\boldsymbol{y} - F^{(n+1)}\boldsymbol{x} \geqq 0 ,$$

by (2), with $F\boldsymbol{y} - F\boldsymbol{x} = 0$ when $\boldsymbol{x}^{(n+1)} = \mathbf{1}$ and $\boldsymbol{y}^{(n+1)} = \mathbf{0}$, that is $\boldsymbol{x} = \boldsymbol{y}$. This shows that F is increasing and is uniquely defined.

Now suppose that $f_n(x_1, \ldots, x_n) > 0$ for all n and $\boldsymbol{x}$, and let $\boldsymbol{x} < \boldsymbol{y}$. Then

$$F\boldsymbol{y} = F\boldsymbol{x} \Rightarrow f_n(x_1, \ldots, x_{n-1}, 0) = F^{(n+1)}\boldsymbol{x} \,\&\, F^{(n+1)}\boldsymbol{y} = 0 \Rightarrow$$
$$\Rightarrow \text{(using (2))}\ \boldsymbol{x}^{(n+1)} = \mathbf{1} \,\&\, \boldsymbol{y}^{(n+1)} = \mathbf{0} \Rightarrow \boldsymbol{x} = \boldsymbol{y}\,.$$

As $\boldsymbol{x} < \boldsymbol{y}$, $F\boldsymbol{y}$ cannot equal $F\boldsymbol{x}$; therefore $F\boldsymbol{y} > F\boldsymbol{x}$. This shows that F is strictly increasing when $f_n(x_1, \ldots, x_n) > 0$ for all n and $\boldsymbol{x}$.

Proof of Theorem 2

(a) Let F be a cts. d.f.. It has the form given by Theorem 1(a). To show that F maps onto $[0, 1]$, for any $t \in [0, 1]$ we find an $\boldsymbol{x}$ such that $F\boldsymbol{x} = t$. Define the sequence $v_0, x_1, v_1, x_2, \ldots$ as follows: $v_0 = t$; for $n = 1, 2, \ldots, x_n = 0$ or 1 according as v_{n-1} is $\leqq$ or $> f_n(x_1, \ldots, x_{n-1}, 0)$, and $v_n = v_{n-1} - x_n f_n(x_1, \ldots, x_{n-1}, 0)$. Then

$$t = \sum_{s=1}^{n} x_s f_s(x_1, \ldots, x_{s-1}, 0) + v_n \tag{3}$$

for $n = 1, 2, \ldots$. We show that $0 \leqq v_n \leqq f_n(x_1, \ldots, x_n)$ for all n. The inequality is true for $n = 0$ (with $f_0 = 1$). Suppose it to be true for $n - 1$. If $x_n = 0$ then $v_{n-1} \leqq f_n(x_1, \ldots, x_{n-1}, 0)$ and $v_n = v_{n-1}$, so $0 \leqq v_n \leqq f_n(x_1, \ldots, x_n)$. If $x_n = 1$ then $v_{n-1} > f_n(x_1, \ldots, x_{n-1}, 0)$ and $v_n = v_{n-1} - f_n(x_1, \ldots, x_{n-1}, 0)$, so $0 < v_n \leqq f_{n-1}(x_1, \ldots, x_{n-1}) - f_n(x_1, \ldots, x_{n-1}, 0) = f_n(x_1, \ldots, x_n)$. Thus in either case $0 \leqq v_n \leqq f_n(x_1, \ldots, x_n)$. This induction argument shows that $0 \leqq v_n \leqq f_n(x_1, \ldots, x_n)$ for all n, so $v_n \to 0$ as $n \to \infty$. Letting $n \to \infty$ in (3) gives $t = F\boldsymbol{x}$.

(b) Conversely, let F be an increasing function mapping onto $[0, 1]$. For any $\boldsymbol{x}$,

$$F\{x_1, \ldots, x_n, \mathbf{0}\} \leqq F\boldsymbol{x} \leqq F\{x_1, \ldots, x_n, \mathbf{1}\}\,,$$

and the two outer quantities are respectively non-decreasing and non-increasing functions of n. Therefore there exist

$$F_0\boldsymbol{x} = \lim_{n\to\infty} F\{x_1, \ldots, x_n, \mathbf{0}\}$$

and

$$F_1\boldsymbol{x} = \lim_{n\to\infty} F\{x_1, \ldots, x_n, \mathbf{1}\}\,,$$

and $F_0\boldsymbol{x} \leqq F\boldsymbol{x} \leqq F_1\boldsymbol{x}$. Suppose that, for some $\boldsymbol{x}$, $F_0\boldsymbol{x} < F\boldsymbol{x}$. As F maps onto $[0, 1]$, there is a $\boldsymbol{y}$ such that $F_0\boldsymbol{x} < F\boldsymbol{y} < F\boldsymbol{x}$. But then $\boldsymbol{y} < \boldsymbol{x}$ and so $F\boldsymbol{y} \leqq F_0\boldsymbol{x}$, which is false. Therefore $F_0\boldsymbol{x} = F\boldsymbol{x}$ for all $\boldsymbol{x}$. Similarly $F_1\boldsymbol{x} = F\boldsymbol{x}$ for all $\boldsymbol{x}$. Therefore F is continuous. Clearly $F\mathbf{0} = 0$ and $F\mathbf{1} = 1$. Thus F is a cts. d.f..

Acknowledgement. I am grateful to Richard Cowan for introducing me to the problem and for his interest in seeing it clarified.

Reference

[1] *J. A. Víšek:* On properties of binary random numbers. Apl. mat. 19 (1974), 375.

Souhrn

DISTRIBUCE NÁHODNÝCH BINÁRNÍCH POSLOUPNOSTÍ

DAVID CULPIN

Posloupnost $\{X_1, X_2, \ldots\}$ je Bernoulliova posloupnost s $P[X_n = 1] = p = 1 - q$, právě když

$$\sum_{n=1}^{\infty} X_n q \prod_{i=1}^{n-1} p^{X_i} q^{1-X_i}$$

má stejnoměrné rozložení. Tento výsledek je v článku dokázán a zobecněn na posloupnosti, které nejsou Bernoulliovy.

Author's address: Dr. *David Culpin,* CSIRO, Division of Mathematics and Statistics, PO Box 218, Lindfield, 2070, Australia.