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FINITE ELEMENT ANALYSIS OF THE SIGNORINI PROBLEM IN SEMI – COERCIVE CASES

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INTRODUCTION

A numerical analysis of the Signorini problem in the plane elastostatics by finite elements has been studied in [1] for boundary conditions which guarantee the coerciveness of the strain energy functional over the whole energy space. It is the aim of the present paper to extend the results to some semi-coercive cases, i.e. for boundary conditions and external forces, which imply the coerciveness of the potential energy over the subset of admissible displacements or over a subspace of the energy space only.

In other words, we assume that if there exist admissible rigid displacements then the resultants of the body forces and surface tractions have proper directions.

Moreover, we restrict ourselves to the cases, when the subspace of rigid virtual displacements had the dimension one, in order to obtain uniqueness of the solution.

We prove a priori error estimates provided the solution is smooth enough. The convergence will be proven even in the case of non-regular solution.

1. FORMULATIONS OF THE SIGNORINI PROBLEM

Let $\Omega \subset R^2$ be a bounded plane domain with Lipschitz boundary, occupied by an elastic body. Let $\boldsymbol{u} = (u_1, u_2) \in [H^1(\Omega)]^2$ be displacement vectors. The strain tensor ε is defined by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2\,. \tag{1.1}$$

By means of the generalized Hooke's law we define the stress tensors

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl}\,, \quad i, j = 1, 2 \tag{1.2}$$

where the summation is implied over any repeated subscript over the range 1, 2, the coefficients $c_{ijkl} \in L_\infty(\Omega)$ satisfy the symmetry conditions

(1.3) $$c_{ijkl} = c_{jikl} = c_{klij}$$

and there exists a positive constant c_0 such that

(1.4) $$c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqq c_0\varepsilon_{ij}\varepsilon_{ij}$$

holds for any symmetric ε almost everywhere in Ω.

Under external loads (see Fig. 1) the body is in equilibrium and the stress tensor satisfies the equilibrium equations

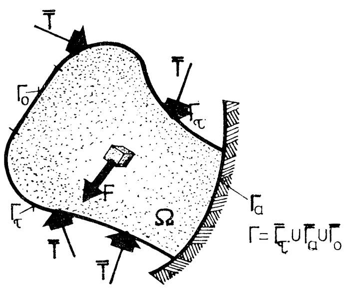

Fig. 1.

(1.5) $$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0\,, \quad i = 1, 2\,,$$

where F_i are components of the body force vector.

The stress vector $\boldsymbol{T}$ with the components

$$T_i = T_{ij}n_j$$

where $\boldsymbol{n} = (n_1, n_2)$ is the unit outward normal to the boundary $\partial\Omega \equiv \Gamma$, can be decomposed into the normal component

$$T_n = T_i n_i = \tau_{ij}n_i n_j$$

and the tangential component

$$T_t = T_i t_i = \tau_{ij}t_i n_j$$

where $\mathbf{t} = (t_1, t_2) = (-n_2, n_1)$ is the unit tangential vector. We write also

$$u_n = u_i n_i\,, \quad u_t = u_i t_i$$

for the normal and tangential displacement components.

Let the boundary Γ consist of three mutually disjoint parts Γ_a, Γ_τ and Γ_0, i.e.

(1.6) $$\bar{\Gamma} = \bar{\Gamma}_a \cup \bar{\Gamma}_\tau \cup \bar{\Gamma}_0\,,$$

where Γ_a contains a set open in Γ,

$$(1.7) \qquad \mathbf{T} = \bar{T} \quad \text{on} \quad \Gamma_\tau ,$$

$$(1.8) \qquad u_n = 0 , \quad T_t = 0 \quad \text{on} \quad \Gamma_0 ,$$

(Γ_0 may be e.g. an axis of symmetry of the problem) and

$$(1.9) \qquad u_n \leqq 0 , \quad T_n \leqq 0 , \quad u_n T_n = 0 , \quad T_t = 0 \quad \text{on} \quad \Gamma_a$$

(conditions of Signorini).

Assume that $\mathbf{F} \in [L_2(\Omega)]^2$ and $\bar{T} \in [L_2(\Gamma_\tau)]^2$ are prescribed body forces and surface loads, respectively.

Let us introduce the following forms

$$A(\mathbf{u}, \mathbf{v}) = \int_\Omega c_{ijkl} \varepsilon_{ij}(\mathbf{u}) \, \varepsilon_{kl}(\mathbf{v}) \, \mathrm{d}\mathbf{x} ,$$

$$L(\mathbf{v}) = \int_\Omega F_i v_i \, \mathrm{d}\mathbf{x} + \int_{\Gamma_\tau} \bar{T}_i v_i \, \mathrm{d}s ,$$

and the functional of total potential energy

$$\mathscr{L}(\mathbf{v}) = \tfrac{1}{2} A(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) .$$

Denote

$$V = \{\mathbf{v} \in [H^1(\Omega)]^2 \mid v_n = 0 \quad \text{on} \quad \Gamma_0\}$$

the space of virtual displacements and define the set of admissible displacements

$$K = \{\mathbf{v} \in V \mid v_n \leqq 0 \quad \text{on} \quad \Gamma_a\}$$

Definition 1.1. *An element $\mathbf{u} \in K$ will be called a weak solution of the Signorini problem if*

$$(1.10) \qquad \mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K .$$

Lemma 1.1. *Any "classical" solution of the problem, i.e. a solution of* (1.1), (1.2), (1.5), (1.7), (1.8), (1.9), *is a weak solution. On the other hand, if the weak solution is smooth enough, it represents a classical solution.*

Proof is parallel to that of Lemma 1.1 in [1].

Let us discuss the existence and uniqueness of a weak solution. To this end, we introduce the set of rigid body displacements:

$$R = \{\varrho = (\varrho_1, \varrho_2) \mid \varrho_1 = a_1 - b x_2, \varrho_2 = a_2 + b x_1\} ,$$

where a_1, a_2, b are arbitrary real numbers.

Denote $R' = R \cap K$ and let R^* be the subset of R' of all "bilateral" vectors, i.e.

$$(1.11) \qquad R^* = \{\varrho \in R' \mid \varrho \in R^* \Rightarrow -\varrho \in R^*\} .$$

It is easy to see, that R^* is a linear manifold and

$$(1.12) \qquad R^* = \{\varrho \in R \mid \varrho_n = 0 \quad \text{on} \quad \Gamma_a \cup \Gamma_0\} .$$

Introduce also the space

$$R_v = R \cap V \quad \text{of virtual rigid displacements} .$$

Theorem 1.1. *Assume that*

$$(1.13) \qquad R_v = R^* = R' , \quad \dim R_v = 1$$

and let

$$(1.14) \qquad L(\varrho) = 0 \quad \forall \varrho \in R_v .$$

Denote by $V = H \oplus R_v$ the orthogonal decomposition of the space V.
Then

(i) the functional $\mathscr{L}$ is coercive on H;

(ii) there exists a unique solution $\hat{\boldsymbol{u}} \in \hat{K}$ of the problem

$$(1.15) \qquad \mathscr{L}(\hat{\boldsymbol{u}}) \leqq \mathscr{L}(\mathbf{z}) \quad \forall \mathbf{z} \in \hat{K} , \quad \hat{K} = K \cap H ;$$

(iii) any weak solution $\boldsymbol{u}$ of the Signorini problem (1.10) can be written in the form

$$\boldsymbol{u} = \hat{\boldsymbol{u}} + \varrho ,$$

where $\hat{\boldsymbol{u}} \in \hat{K}$ is the solution of the problem (1.15) and $\varrho \in R_v$;

(iv) if $\hat{\boldsymbol{u}} \in \hat{K}$ is the solution of (1.15), then $\boldsymbol{u} = \hat{\boldsymbol{u}} + \varrho$, where ϱ is any element of R_v, represents a weak solution of the Signorini problem (1.10).

Remark 1.1. An example, when the assumptions (1.13) are satisfied, is shown in Fig. 2.

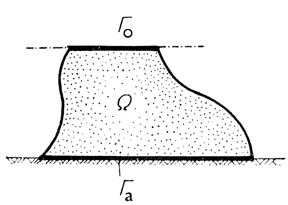

Fig. 2.

Remark 1.2. From the numerical point of view it is convenient to introduce the following scalar product in V (see [5] – I., Th. 2.3):

$$(\mathbf{u}, \mathbf{v})_V = \int_\Omega \varepsilon_{ij}(\mathbf{u})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} + p(\mathbf{u})\, p(\mathbf{v}),$$

where p is a linear continuous functional on V such that

$$\{\varrho \in R_v, p(\varrho) = 0\} \Rightarrow \varrho = 0.$$

For instance, if

$$R_v = \{\varrho \mid \varrho_1 = a_1 \in R^1, \varrho_2 = 0\}$$

(see Fig. 2), we can choose $p(\mathbf{v}) = \int_{\Gamma_1} v_1 \, \mathrm{d}s$, where $\Gamma_1 \subset \bar{\Omega}$, mes $\Gamma_1 > 0$.

Then (cf. [5] – I. Remark 4)

$$H = V \ominus R_v = \{\mathbf{v} \in V \mid p(\mathbf{v}) = 0\}.$$

Proof of Theorem 1.1. (i) For any $\mathbf{v} \in H$ the following inequality of Korn's type is valid see [5] – I. Remarks 3 and 4)

(1.16) $$c_1 \|\mathbf{v}\| \leqq |\mathbf{v}|,$$

where $\|\cdot\|$ is the norm in $[H^1(\Omega)]^2$ and

(1.16′) $$|\mathbf{v}|^2 = \int_\Omega \varepsilon_{ij}(\mathbf{v})\, \varepsilon_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x}.$$

Then we have for any $\mathbf{v} \in H$

$$L(\mathbf{v}) \geqq \tfrac{1}{2} c_0 |\mathbf{v}|^2 - L(\mathbf{v}) \geqq C\|\mathbf{v}\|^2 - \|L\|\, \|\mathbf{v}\|,$$

and the coerciveness of $\mathscr{L}$ over H follows.

(ii) Since $\mathscr{L}$ is Gâteaux differentiable and convex, $\hat{K}$ being convex and closed, there exists a solution $\hat{\mathbf{u}}$ of the problem (1.15).

Let $\mathbf{u}^1 \in \hat{K}$ and $\mathbf{u}^2 \in \hat{K}$ be two solutions of (1.15). Then we may write

$$A(\mathbf{u}^1, \mathbf{u}^2 - \mathbf{u}^1) \geqq L(\mathbf{u}^2 - \mathbf{u}^1),$$
$$A(\mathbf{u}^2, \mathbf{u}^1 - \mathbf{u}^2) \geqq L(\mathbf{u}^1 - \mathbf{u}^2).$$

Adding these two inequalities, we obtain

$$A(\mathbf{u}^2 - \mathbf{u}^1, \mathbf{u}^1 - \mathbf{u}^2) \geqq 0$$

and consequently,

$$c_0 |\mathbf{u}^1 - \mathbf{u}^2|^2 \leqq A(\mathbf{u}^1 - \mathbf{u}^2, \mathbf{u}^1 - \mathbf{u}^2) \leqq 0 \Rightarrow \mathbf{u}^1 - \mathbf{u}^2 \in R_v \cap H = \{0\}.$$

Therefore the solution is unique.

(iii) By virtue of (1.14) we have

$$\mathscr{L}(\mathbf{v}) = \mathscr{L}(\mathbf{v} + \varrho) \quad \forall \varrho \in R_v\,, \quad \forall \mathbf{v} \in V\,. \tag{1.17}$$

Moreover, it holds

$$P_H(K) = K \cap H\,, \tag{1.18}$$

where P_H is the projection onto H.

In fact, let $\mathbf{v} \in K$. Then using (1.12), (1.13), we obtain

$$P_H\mathbf{v} = \mathbf{v} - P_{R_v}\mathbf{v}\,,$$

$$(P_H\mathbf{v})_n = v_n - (P_{R*}\mathbf{v})_n = v_n \leqq 0 \quad \text{on} \quad \Gamma_a \Rightarrow P_H\mathbf{v} \in K \cap H\,.$$

The inclusion $K \cap H = P_H(K \cap H) \subset P_H(K)$ is obvious.

Let $\mathbf{u}$ be a weak solution of (1.10). By virtue of (1.17) we may write

$$\mathscr{L}(P_H\mathbf{v}) = \mathscr{L}(P_H\mathbf{v} + P_{R_v}\mathbf{v}) = \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in V\,;$$

furthermore, $P_H\mathbf{u} \in K \cap H$,

$$\mathscr{L}(P_H\mathbf{u}) = \mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) = \mathscr{L}(P_H\mathbf{v}) \quad \forall \mathbf{v} \in K$$

and from (1.18) we conclude that $P_H\mathbf{u}$ is a solution of (1.15);

The uniqueness implies that $P_H\mathbf{u} = \hat{\mathbf{u}}$, $\mathbf{u} = \hat{\mathbf{u}} + \varrho$, $\varrho \in R_v$.

(iv) Let $\mathbf{u} = \hat{\mathbf{u}} + \varrho$ where $\varrho \in R_v$. Then we have $\mathbf{u} \in K$ (because $\varrho \in R^*$) and

$$\mathscr{L}(\mathbf{u}) = \mathscr{L}(\hat{\mathbf{u}}) \leqq \mathscr{L}(\mathbf{z}) \quad \forall \mathbf{z} \in \hat{K}\,. \tag{1.19}$$

Let $\mathbf{v} \in K$. Using (1.17) and the decomposition

$$\mathbf{v} = P_H\mathbf{v} + P_{R_v}\mathbf{v}\,,$$

we obtain for $\mathbf{z} = P_H\mathbf{v} \in P_H(K) = \hat{K}$

$$\mathscr{L}(\mathbf{z}) = \mathscr{L}(\mathbf{v})\,. \tag{1.20}$$

Finally (1.19) and (1.20) lead to the relation

$$\mathscr{L}(\mathbf{u}) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K\,.$$

Theorem 1.2. *Assume that*

$$R^* = \{0\}\,, \quad \dim R_v = 1\,, \tag{1.21}$$

$$L(\varrho) \neq 0 \quad \forall \varrho \in R_v \dot{-} \{0\} \tag{1.22}$$

and either $R' = K \cap R = \{0\}$ *or*

$$R' = K \cap R \neq \{0\}\,, \tag{1.23}$$

$$L(\varrho) < 0 \quad \forall \varrho \in K \cap R \dot{-} \{0\}\,. \tag{1.24}$$

Then $\mathscr{L}$ *is coercive on* K *and there exists a unique weak solution* $\mathbf{u} \in K$ *of the Signorini problem* (1.10).

Remark 1.3. An example, when the assuptions (1.21), (1.23) are satisfied, is shown in Fig. 3. An example satisfying the assumptions (1.21) and $R' = \{0\}$ is presented in Fig. 4.

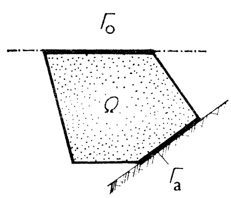

Fig. 3.

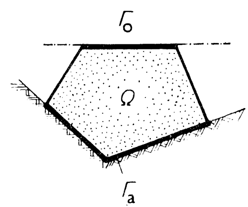

Fig. 4.

Proof of Theorem 1.2. (i) Let us comider the case $R' = \{0\}$. We shall need the following abstract result ([4] – Th. 2.2):

Proposition 1. *Let $|u|$ be a seminorm in a Hilbert space H with the norm $\|u\|$. Assume that if we introduce the subspace*

$$R = \{u \in H \mid |u| = 0\},$$

then $\dim R < \infty$ *and it holds*

$$(1.25) \qquad c_1\|u\| \leqq |u| + \|P_R u\| \leqq c_2\|u\| \quad \forall u \in H,$$

where P_R is the orthogonal projection onto R.

Let K be a convex closed subset of H, containing the origin, $K \cap R = \{0\}$, $\beta : H \to R^1$ a penalty functional with a differential, which is 1 – positively homogeneous[1]) *and such that*

$$\beta(u) = 0 \Leftrightarrow u \in K.$$

Then it holds

$$(1.26) \qquad |u|^2 + \beta(u) \geqq c\|u\|^2 \quad \forall u \in H.$$

The Proposition 1. can be applied with: $H = V$, $R = R_v$, $|v|$ defined as in (1.16'),

$$\beta(\mathbf{u}) = \frac{1}{2}\int_{\Gamma_a} ([u_n]^+)^2 \, \mathrm{d}s.$$

To verify (1.25), we make use of the inequality of Korn's type and of the decomposition $V = Q \oplus R_v$ to obtain

$$(1.27) \qquad \|\mathbf{u}\|^2 = \|P_Q\mathbf{u}\|^2 + \|P_{R_v}\mathbf{u}\|^2 \leqq c|P_Q\mathbf{u}|^2 + \|P_{R_v}\mathbf{u}\|^2 = = c|\mathbf{u}|^2 + \|P_{R_v}\mathbf{u}\|^2.$$

[1]) I.e., $D\beta(tu, v) = tD\beta(u, v)\ \forall t > 0,\ u, v \in H$.

From (1.26) it follows that

$$|\boldsymbol{u}|^2 \geqq c\|\boldsymbol{u}\|^2 \quad \forall \boldsymbol{u} \in K . \tag{1.28}$$

Then one can deduce easily that $\mathscr{L}$ is coercive on K and the existence of a weak solution $\boldsymbol{u}$ of the Signorini problem (1.10).

If $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ are two weak solutions of (1.10), using the same approach as in the proof of Theorem 1.1 (ii), we obtain

$$\varrho = \boldsymbol{u}^1 - \boldsymbol{u}^2 \in R_v .$$

Moreover

$$\mathscr{L}(\boldsymbol{u}^1) = \mathscr{L}(\boldsymbol{u}^2) \Rightarrow L(\boldsymbol{u}^1) = L(\boldsymbol{u}^2) \Rightarrow L(\varrho) = 0$$

and from the assumption (1.22) we conclude that $\varrho = 0$.

(ii) Let us consider the case (1.23), (1.24). We shall employ the following abstract result ([4] – Th. 2.3):

Proposition 2. *Let the assumptions of Proposition* 1 *be satisfied with the only exception that* $K \cap R \neq \{0\}$.

Moreover, let f *be a linear bounded functional on* H *such that*

$$f(v) < 0 \quad \forall v \in K \cap R \dot{-} \{0\} . \tag{1.29}$$

Then

$$|u|^2 + \beta(u) - f((u) \geqq c_1\|u\| - c_2 \quad \forall u \in H . \tag{1.30}$$

The Proposition 2 can be applied with the same H, R, $|\cdot|$, β as previously and with

$$f(v) = L(v) .$$

Then (1.30) implies that $\mathscr{L}$ is coercive over K. The existence and uniqueness of the weak solution can be obtained in the same way as in the previous case (i).

Remark 1.4. We avoid the cases when the subspace R_v of virtual rigid displacements has greater dimension than 1.

In such cases the solution is not unique even in the subspaces of the type $V \ominus R^*$ (cf. [3], [4]).

2. FINITE ELEMENT APPROXIMATIONS

Let the assumptions of Theorem 1.1 or Theorem 1.2 be satisfied. Henceforth let Ω be a *polygonal* bounded domain. Let us carve Ω into triangles, creating a triangulation $\mathscr{T}_h$.

Let the points $\bar{\Gamma}_\tau \cap \bar{\Gamma}_a$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_0$ and $\bar{\Gamma}_a \cap \bar{\Gamma}_0$ coincide with some vertices of $\mathscr{T}_h$.

A family $\{\mathcal{T}_h\}$, $0 < h \leqq 1$, of triangulations will be called *regular*, if a positive constant α exists independent of h and such that no interior angle in $\mathcal{T}_h$ is less than α. Let V_h be the space of linear finite elements, i.e. the space of continuous functions in $\bar{\Omega}$, piecewise linear over $\mathcal{T}_h$. We define:

$$K_h = K \cap [V_h]^2 = \{\mathbf{v} \in [V_h]^2 \mid v_n = 0 \text{ on } \Gamma_0, v_n \leqq 0 \text{ on } \Gamma_a\}$$

in case of Theorem 1.2 and

$$K_h = \hat{K} \cap [V_h]^2 = \{\mathbf{v} \in [V_h]^2 \mid p(\mathbf{v}) = 0,\ v_n = 0 \text{ on } \Gamma_0,\ v_n \leqq 0 \text{ on } \Gamma_a\}$$

in case of Theorem 1.1 (cf. Remark 1.2).

A function $\boldsymbol{u}_h \in K_h$ will be called a *finite element approximation* of the Signorini problem, if

$$\mathscr{L}(\boldsymbol{u}_h) \leqq \mathscr{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K_h\,. \tag{2.1}$$

Lemma 2.1. *There exists a unique solution of the problem* (2.1).

Proof. The set K_h is closed and convex subset of K and of H, respectively. Theorems 1.2 and 1.1 imply that the functional $\mathscr{L}$ is coercive over K_h. Hence the existence of $\boldsymbol{u}_h$ follows. The uniqueness can be proved in the same way as in Theorems 1.1. and 1.2.

Let us derive an apriori estimate for the error $\boldsymbol{u}_h - \boldsymbol{U}$, where $\boldsymbol{U} = \hat{\boldsymbol{u}} \in \hat{K}$ in the case of Theorem 1.1 and $\boldsymbol{U} = \boldsymbol{u}$ in the case of Theorem 1.2. We employ the method proposed by Falk [2], which is based on the following lemma.

Lemma 2.2. *Let* $|\cdot|$ *be the seminorm defined in* (1.16′). *Then it holds*

$$C_0|\boldsymbol{U} - \boldsymbol{u}_h|^2 \leqq L(\boldsymbol{U} - \mathbf{v}_h) + A(\boldsymbol{U}, \mathbf{v}_h - \boldsymbol{U}) + A(\boldsymbol{u}_h - \boldsymbol{U}, \mathbf{v}_h - \boldsymbol{U}) \quad \forall \mathbf{v}_h \in K_h\,. \tag{2.3}$$

Proof. Since

$$A(\boldsymbol{U}, \mathbf{v} - \boldsymbol{U}) \geqq L(\mathbf{v} - \boldsymbol{U})$$

holds for any $\mathbf{v} \in \hat{K}$ (any $\mathbf{v} \in K$, respectively), we may write

$$A(\boldsymbol{U}, \boldsymbol{U}) \leqq A(\boldsymbol{U}, \mathbf{v}_h) + L(\boldsymbol{U} - \boldsymbol{u}_h)\,. \tag{2.4}$$

From the definition (2.1) it follows that

$$A(\boldsymbol{u}_h, \boldsymbol{u}_h) \leqq A(\boldsymbol{u}_h, \mathbf{v}_h) + L(\boldsymbol{u}_h - \mathbf{v}_h) \quad \forall \mathbf{v}_h \in K_h\,. \tag{2.5}$$

Then (2.4), (2.5) and (1.4) imply

$$\begin{aligned} C_0|\boldsymbol{U} - \boldsymbol{u}_h|^2 &\leqq A(\boldsymbol{U} - \boldsymbol{u}_h, \boldsymbol{U} - \boldsymbol{u}_h) = A(\boldsymbol{U}, \boldsymbol{U}) + A(\boldsymbol{u}_h, \boldsymbol{u}_h) - \\ &- 2A(\boldsymbol{U}, \boldsymbol{u}_h) \leqq L(\boldsymbol{U} - \mathbf{v}_h) + A(\boldsymbol{U}, \boldsymbol{u}_h) + A(\boldsymbol{u}_h, \mathbf{v}_h) - 2A(\boldsymbol{U}, \boldsymbol{u}_h) = \\ &= L(\boldsymbol{U} - \mathbf{v}_h) + A(\boldsymbol{U}, \mathbf{v}_h - \boldsymbol{u}) + A(\boldsymbol{u}_h - \boldsymbol{U}, \mathbf{v}_h - \boldsymbol{U})\,. \end{aligned}$$

Theorem 2.1. *Let the solution* $\mathbf{U}$ *be such that the stress components* $\tau_{ij}(\mathbf{U}) \in H^1(\Omega)$, $i, j = 1, 2$, $\mathbf{U} \in [H^2(\Omega)]^2$ *and* $U_n \in H^2(\Gamma_a \cap \Gamma_m)$ *holds for any side* Γ_m *of the polygonal boundary* Γ. *Then we have the estimate*

$$|\mathbf{U} - \mathbf{u}_h| \leqq Ch \,, \tag{2.6}$$

where the constant C *depends on* $\mathbf{U}$ *and not on* h.

Proof. Integrating by parts and using the boundary conditions we obtain

$$A(\mathbf{U}, \mathbf{v}_h - \mathbf{U}) + L(\mathbf{U} - \mathbf{v}_h) = \int_\Omega (B\mathbf{U})_j (v_h - U)_j \, dx +$$

$$+ \int_\Gamma \tau_{ij}(\mathbf{U}) \, n_j (v_h - U)_i \, ds - \int_\Omega F_j (v_h - U)_j \, dx - \int_{\Gamma_\tau} \overline{T}_j (v_h - U)_j \, ds =$$

$$= \int_{\Gamma_a} \tau_{ij}(\mathbf{U}) \, n_j \, (v_h - U)_i \, ds = \int_{\Gamma_a} T_n(\mathbf{U}) \, (v_{hn} - U_n) \, ds \,,$$

where

$$(Bu)_j = - \frac{\partial}{\partial x_i} \left(c_{ijkm} \varepsilon_{km}(\mathbf{U}) \right) = - \frac{\partial}{\partial x_i} \tau_{ij}(\mathbf{U}) \,, \quad j = 1, 2 \,.$$

Thus the right – hand side in (2.3) can be estimated as follows

$$A(\mathbf{u}_h - \mathbf{U}, \mathbf{v}_h - \mathbf{U}) + \int_{\Gamma_a} T_n(\mathbf{U}) \, (v_{hn} - U_n) \, ds \leqq \tag{2.7}$$

$$\leqq \tfrac{1}{2} c_1 \varepsilon |\mathbf{u}_h - \mathbf{U}|^2 + \tfrac{1}{2} c_1 \varepsilon^{-1} |\mathbf{v}_h - \mathbf{U}|^2 + c_2(\mathbf{U}) \, \|v_{hn} - U_n\|_{L_2(\Gamma_a)} \,.$$

with an arbitrary positive ε.

First let us consider the case of Theorem 1.1, i.e. $\mathbf{U} = \hat{\mathbf{u}}$. Choosing $\mathbf{v}_h = P_H \, \hat{\mathbf{u}}_I$, i.e. the orthogonal projection of the Lagrange linear interpolate of $\hat{\mathbf{u}}$ on the triangulation $\mathcal{T}_h$, we can easily verify that $\mathbf{v}_h \in K_h = H \cap K \cap [V_h]^2$. In fact, $P_H \hat{\mathbf{u}}_I = \hat{\mathbf{u}}_I - \varrho$, $\varrho \in R^*$, consequently

$$(P_H \hat{\mathbf{u}}_I)_n = (\hat{\mathbf{u}}_I)_n - \varrho_n = (\hat{\mathbf{u}}_I)_n \quad \text{on} \quad \Gamma_0 \cup \Gamma_a \,. \tag{2.8}$$

It is readily seen that $(\hat{\mathbf{u}}_I)_n = 0$ on Γ_0 and $(\hat{\mathbf{u}}_I)_n \leqq 0$ on Γ_a, so that $P_H \hat{\mathbf{u}}_I \in K$.

Since ϱ belongs to $[V_h]^2$, $P_H \hat{\mathbf{u}}_I \in [V_h]^2$. Therefore $P_H \hat{\mathbf{u}}_I \in K_h$. Further we may write

$$|P_H \hat{\mathbf{u}}_I - \hat{\mathbf{u}}| = |\hat{\mathbf{u}}_I - \hat{\mathbf{u}}| \leqq Ch |\hat{\mathbf{u}}|_2 \,, \tag{2.9}$$

$$\|(P_H \hat{\mathbf{u}}_I)_n - \hat{\mathbf{u}}_n\|_{L_2(\Gamma_a)} = \|(\hat{\mathbf{u}}_I)_n - \hat{\mathbf{u}}_n\|_{L_2(\Gamma_a)} \leqq Ch^2 \sum_m \|\hat{\mathbf{u}}_n\|_{H^2(\Gamma_a \cap \Gamma_m)} \,, \tag{2.10}$$

where we have used the relation (2.8).

From (2.3), (2.7) and (2.9), (2.10) we obtain the estimate (2.6), choosing ε sufficiently small.

Finally, let us consider the case of Theorem 1.2, i.e. $\boldsymbol{U} = \boldsymbol{u}$. With the choice $\boldsymbol{v}_h = \boldsymbol{u}_{\mathrm{I}}$, we obtain $\boldsymbol{v}_h \in K_h$ and the estimates of the form (2.9) and (2.10) for $|\boldsymbol{u}_{\mathrm{I}} - \boldsymbol{u}|$ and $\|u_{\mathrm{I}n} - u_n\|_{L_2(\Gamma_a)}$, respectively. Then (2.6) follows as previously.

3. CONVERGENCE OF THE FINITE ELEMENT APPROXIMATIONS TO A NON – REGULAR SOLUTION

The a priori error estimate (2.6) has been deduced under strict regularity assumptions. In general, however, such regularity of the solution cannot be expected for domains with polygonal boundary (see [3], [4]). Therefore we shall study the convergence of the finite element approximations in a general case, i.e. without any regularity requirement imposed on the solution. The proof will be based on the following theorems.

Theorem 3.1. *Let W be a Hilbert space with the norm $\|\cdot\|$ and a semi – norm $|\cdot|$. Let K be a closed convex subset of W, $0 < h \leqq 1$ a real parameter, $K_h \subset K$ a closed convex subset for any h.*

(i) *Let a differentiable functional $\mathscr{J}$ be defined on W such that $\mathscr{J}$* has a second *Gâteaux differential, satisfying the following condition: positive constants α_0, C exist such that*

$$(3.1) \qquad \alpha_0\|z\|^2 \leqq D^2\mathscr{J}(u, z, z) \leqq C\|z\|^2 \quad \forall u \in K\,, \quad \forall z \in W\,.$$

Let $u(u_h)$ denote the element minimizing $\mathscr{J}$ over the set $K(K_h)$.
Assume that for any h an element $v_h \in K_h$ exists such that

$$(3.2) \qquad \|u - v_h\| \to 0 \quad \text{for} \quad h \to 0\,.$$

(3.3) Then it holds $\|u - u_h\| \to 0$ for $h \to 0$.

(ii) *Let the functional $\mathscr{J}$ be coercive on K and satisfies instead of (3.1) the inequalities*

$$(3.4) \qquad \alpha_0|z|^2 \leqq D^2J(u, z, z) \leqq C\|z\|^2 \quad \forall u \in K\,, \quad \forall z \in W\,.$$

Let the unique minimizing element $u(u_h)$ exist and let the assumption (3.2) hold. Then

$$u_h \rightharpoonup u \quad (\textit{weakly}) \textit{ in } \quad W\,,$$

$$|u - u_h| \to 0 \quad \textit{for} \quad h \to 0\,.$$

Proof of the part (i) is given in [1] – Th. 3.1. The part (ii) can be proven by a parallel approach.

Theorem 3.2. *Assume that the number of points $\bar{\Gamma}_0 \cap \bar{\Gamma}_a$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_0$ and $\bar{\Gamma}_a \cap \bar{\Gamma}_\tau$ is finite. Then the set $K \cap [C^\infty(\bar{\Omega})]^2$ is dense in K.*

Proof is analogous to that of Theorem 3.2 in [1]. The main results of the present Section is contained in the following Theorem.

Theorem 3.3. *Let the assumptions of Theorem* 3.2, *Theorem* 1.1 *and of Theorem* 1.2, *respectively, be satisfied. Let* $\mathbf{U}$ *denote the solution* $\hat{\mathbf{u}}$ *of the problem* (1.15) *and the solution* $\mathbf{u}$ *of the problem* (1.10) *respectively. Then*

$$\mathbf{u}_h \to \mathbf{U} \quad in \quad [H^1(\Omega)]^2 \tag{3.5}$$

holds for any regular family of triangulations and $h \to 0$.

Proof. (i) Consider first the case of Theorem 1.1 and apply the part (i) of Theorem 3.1, setting $\mathbf{u} = \hat{\mathbf{u}}$,

$$W = H\,, \quad K = \hat{K}\,, \quad \mathscr{J} = \mathscr{L}\,, \quad \|\cdot\| = \|\cdot\|_{[H^1(\Omega)]^2}\,.$$

Then it is easy to verify, that (3.1) holds, making use of (1.4) and (1.16).

To verify also (3.2), we employ Theorem 3.2. There exists

$$\mathbf{w} \in K \cap [C^\infty(\bar{\Omega})]^2 \quad \text{such that} \quad \|\mathbf{w} - \hat{\mathbf{u}}\| < \varepsilon_1 \quad \forall \varepsilon_1 > 0\,.$$

Then

$$P_H \mathbf{w} = \mathbf{w} - \varrho \in [C^\infty(\bar{\Omega})]^2\,, \quad (\varrho \in R^*)$$

$P_H \mathbf{w} \in K$ (cf. a similar argument in (2.6)), consequently

$$P_H \mathbf{w} \in \hat{K} \cap [C^\infty(\bar{\Omega})]^2\,.$$

Let us set

$$\mathbf{v}_h = P_H(P_H \mathbf{w})_\mathrm{I}\,,$$

where $(\ )_\mathrm{I}$ denotes the Lagrange linear interpolate over $\mathscr{T}_h$. Then the equivalence of the norm $\|\cdot\|$ and the seminorm (1.16′) in H (cf. (1.16)) yields that

$$\begin{aligned} \|\mathbf{v}_h - P_H \mathbf{w}\| &\leqq C|P_H(P_H \mathbf{w})_\mathrm{I} - P_H \mathbf{w}| = \\ &= C|(P_H \mathbf{w})_\mathrm{I} - P_H \mathbf{w}| \leqq C_1 h |P_H \mathbf{w}|_2 \end{aligned}$$

holds for any regular family of triangulations.

Moreover, we have

$$\|P_H \mathbf{w} - \hat{\mathbf{u}}\| \leqq C|P_H \mathbf{w} - \hat{\mathbf{u}}| = C|\mathbf{w} - \hat{\mathbf{u}}| \leqq C\|\mathbf{w} - \hat{\mathbf{u}}\| < C\varepsilon_1\,.$$

Therefore we may write

$$\|\mathbf{v}_h - \hat{\mathbf{u}}\| \leqq C_1 h |P_H \mathbf{w}|_2 + C\varepsilon_1$$

which results in (3.2).

Finaly, the convergence $\mathbf{u}_h \to \mathbf{u}$ in H follows from (3.3).

(ii) Consider the case of Theorem 1.2. We may apply the part (ii) of Theorem 3.1, setting

$$W = [H^1(\Omega)]^2 \, , \quad \mathscr{J} = \mathscr{L} \, , \quad (K = K) \, .$$

Then (3.4) holds and the solutions are unique, $\mathscr{L}$ is coercive on K. The assumption (3.2) can be verified on the basis of the density theorem 3.2. In fact, we choose $\boldsymbol{w} \in K \cap [C^\infty(\bar{\Omega})]^2$ sufficiently close to $\boldsymbol{u}$ and set $\boldsymbol{v}_h = \boldsymbol{w}_\mathrm{I}$. It is easy to see that $\boldsymbol{w}_\mathrm{I} \in K_h$ and that $\boldsymbol{v}_h$ converges to $\boldsymbol{w}$ for $h \to 0$ (cf. the proof of Theorem 3.3 in [1]).

Theorem 3.1 (ii) implies that $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ in W, $|\boldsymbol{u}_h - \boldsymbol{u}| \to 0$. Moreover, it holds (see e.g. [5] – I, Theorem 3.2)

$$|\boldsymbol{v}|^2 + \|\boldsymbol{v}\|_0^2 \geqq C\|\boldsymbol{v}\|^2 \quad \forall \boldsymbol{v} \in W \, , \tag{3.6}$$

where $\|\cdot\|_0$ denotes the norm in $[L_2(\Omega)]^2$.

Since $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ (weakly) in $[H^1(\Omega)]^2$, $\boldsymbol{u}_h \to \boldsymbol{u}$ in $[L_2(\Omega)]^2$ follows and the assertion (3.5) is a consequence of (3.6).

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Souhrn

ANALÝZA SIGNORINIHO ÚLOHY V SEMI-KOERCITIVNÍCH PŘÍPADECH METODOU KONEČNÝCH PRVKŮ.

IVAN HLAVÁČEK, JÁN LOVÍŠEK

Výsledky předchozího článku [1] jsou rozšířeny na úlohy, kdy existují netriviální přípustná posunutí tělesa jako tuhého celku a výslednice zatížení má správný směr, takže existuje řešení úlohy. Když prostor virtuálních posunutí tuhého tělesa má dimenzi jedna, lze dokázat i jednoznačnost řešení a koercivitu potenciální energie na množině přípustných funkcí.

Odvozují se odhady chyb v případě dostatečně regulárního řešení, resp. samotná konvergence aproximací k neregulárnímu řešení.

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