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EVOLUTIONARY VARIATIONAL INEQUALITIES AND APPLICATIONS IN PLASTICITY

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INTRODUCTION

This paper concerns an abstract mathematical theory of evolutionary variational inequalities of two different types motivated by the classical flow theory of plasticity. The quasi-statical traction boundary value problems in infinitesimal strain approach for elastic-plastic bodies both with and without strain hardening are presented as examples.

The mathematical theory of plasticity leads to variational inequalities. Duvaut - Lions [1] have proved existence of solutions of these inequalities for materials without hardening. However, in the quasi-statical case of the traction boundary value problem their assumptions are too strong, thus restricting essentially the applicability of the theorem.

To our knowledge the existence proof for the boundary value problem for elastic - plastic materials with hardening has not yet been accomplished. From the physical point of view this case has been discussed by Nguyen [2]. By means of the here introduced plastic potencial, Nguyen formulates the problem as an evolutionary variational inequality. In comparison with parabolic variational inequalities there is no coercive elliptic term in Nguyen's inequality.

In the course of the proof-reading we have got acquainted, before its publication, with the paper [3] by Dr. K. Gröger, who solves a similar problem by formulating it as a parabolic inequality. The main difference between our treatment and that of Dr. K. Gröger is his assumption of nondegenerative kinematic hardening.

In our paper we use the penalty method to prove existence theorems for inequalities of Nguyen's type. It is interesting that the equation with a penalty term has the physical meaning of artificial viscosity with internal variables. With some caution we can say that the above mentioned penalty method is also of constructive character. For related problems see the papers [4], [5], [8], [9] and the book [10]. For general questions, see the book [6].

In this section we collect the mathematical notation, definitions and assertions used later. The proofs can be found in [11].

Let H be a Hilbert space with a scalar product $[.,.]_H$ and a norm $\|.,.\|_H = [.,.]_H^{1/2}$, V a closed subspace of H, $V^\perp$ the orthogonal complement of V in H and $P_V : H \to V$ the orthogonal projection on V.

Denote by $C([0, T], H)$ the space of all continuous functions $\sigma : [0, T] \to H$ with the scalar product

$$(\sigma, \tau)_{L_2((0,T),H)} \equiv \int_0^T [\sigma(t), \tau(t)]_H \, dt$$

and the norm

$$\|\sigma\|_{L_2((0,T),H)} \equiv (\sigma, \sigma)_{L_2((0,T),H)}^{1/2} .$$

The closure of $C([0, T], H)$ with respect to the norm $\|.\|_{L_2((0,T),H)}$ is denoted by $L_2((0, T), H)$. It is a Hilbert space of Bochner strongly measurable and square integrable functions $\sigma : (0, T) \to H$. Let $\sigma \in C([0, T], H)$, $t \in [0, T]$. We define

$$\dot{\sigma}(t) \equiv \lim_{s \to t} \frac{\sigma(s) - \sigma(t)}{s - t} \quad (\text{in } H),$$

whenever this limit exists (if $t = 0$ or $t = T$, we consider $s \to t+$ or $s \to t-$, respectively). $C^1([0, T], H)$ denotes the subspace of $C([0, T], H)$ such that $\dot{\sigma} \in C([0, T], H)$ with the scalar product

$$(\sigma, \tau)_{H^1} \equiv \int_0^T \{[\sigma(t), \tau(t)]_H + [\dot{\sigma}(t), \dot{\tau}(t)]_H\} \, dt$$

and the norm

$$\|.\|_{H^1} \equiv (.,.)_{H^1}^{1/2} .$$

By $C_0^1([0, T], H)$ we denote the subspace of $C^1([0, T], H)$ such that $\sigma(0) = 0$ with the scalar product

$$(\sigma, \tau)_{H_0^1} \equiv \int_0^T [\dot{\sigma}(t), \dot{\tau}(t)] \, dt$$

and the norm

$$\|.\|_{H_0^1} \equiv (.,.)_{H_0^1}^{1/2} .$$

The closure of $C^1([0, T], H)$ or $C_0^1([0, T], H)$ with respect to the norm $\|.\|_{H^1}$ or $\|.\|_{H_0^1}$ is denoted by $H^1((0, T), H)$ or $H_0^1((0, T), H)$ (briefly H^1 or H_0^1), respectively. H^1 or H_0^1 is a Hilbert space with respect to the scalar product $(.,.)_{H^1}$ or $(.,.)_{H_0^1}$, respectively. $\|.\|_{H^1}$, $\|.\|_{H_0^1}$ represent equivalent norms on H_0^1. Let $\sigma \in H^1$ or H_0^1,

$\sigma_n \in C^1$ or C_0^1, $\sigma_n \to \sigma$ in H^1 or H_0^1, respectively. The time derivative $\dot{\sigma}$ of σ is defined by $\dot{\sigma} \equiv \lim \dot{\sigma}_n$ in $L_2((0, T), H)$. To $\sigma \in H^1$ there exists $\tilde{\sigma} \in C([0, T], H)$ such that $\sigma(t) = \tilde{\sigma}(t)$ a.e. on $[0, T]$. In this paper we use only these continuous representants $\tilde{\sigma}$ of the elements $\sigma \in H^1$. Let $\sigma, \tau \in H^1$, $t_1, t_2 \in [0, T]$. Then

$$\int_{t_1}^{t_2} [\dot{\sigma}, \tau]_H \, ds = [\sigma(t_2), \tau(t_2)]_H - [\sigma(t_1), \tau(t_1)]_H - \int_{t_1}^{t_2} [\sigma, \dot{\tau}]_H \, ds$$

(consequently

$$\int_0^t [\dot{\sigma}, \sigma]_H \, ds = \tfrac{1}{2}\|\sigma(t)\|_H^2 - \tfrac{1}{2}\|\sigma(0)\|_H^2) .$$

If a sequence $\sigma_n \in H^1$ or H_0^1 converges weakly to σ in H^1 or H_0^1, respectively: $\sigma_n \to \sigma$, then $\sigma_n(t) \to \sigma(t)$ weakly in H for all $t \in [0, T]$.

Denote by $W^{1,2}(G)$ the Sobolev space

$$W^{1,2}(G) \equiv \left\{ v \in L_2(G) \,\middle|\, \frac{\partial v}{\partial x_i} \in L_2(G) , \quad i = 1, 2, 3 \right\} .$$

Here G is a bounded domain, $G \subset R^3$. Let S be the Hilbert space of all symmetric tensor functions $\sigma = \{\sigma_{ij}\}_{i,j=1,2,3}$, $\sigma_{ij} = \sigma_{ji}$, $\sigma_{ij} \in L_2(G)$ with the scalar product

$$(\sigma, \tau)_S \equiv \int_G \sigma_{ij}\tau_{ij} \, dx$$

and the norm $\|\cdot\|_S \equiv (.,.)_S^{1/2}$ (throughout the whole paper the summation convention is used). The formula

$$\varrho_{ij}(v) \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad i, j = 1, 2, 3$$

defines a tensor function $\varrho : [W^{1,2}(G)]^3 \to S$. Let $\mathscr{K}$ denote the range of ϱ and $\mathscr{H}$ the orthogonal complement of $\mathscr{K}$ in S. It is well known that $\mathscr{K}$ and $\mathscr{H}$ are closed subspaces of S ([4], [5]).

Let $\tilde{S}$ be the space of all symmetric tensors $\sigma = \{\sigma_{ij}\}_{i,j=1,2,3}$, $\sigma_{ij} = \sigma_{ji}$, $\sigma_{ij} \in R^1$ with the scalar product $(\sigma, \tau)_{\tilde{S}} = \sigma_{ij}\tau_{ij}$ and the norm

$$\|\cdot\|_{\tilde{S}} \equiv (.,.)_{\tilde{S}}^{1/2} .$$

We conclude this section with a trivial lemma.

Lemma. *Let* $u_n \in L_\infty((0, T) \times G)$, $|u_n| \leqq C < \infty$, $u_n \to u$ *pointwise a.e. on* $(0, T) \times \times G$, $v_n \in L_2((0, T), S)$, $v_n \to v$ *weakly in* $L_2((0, T), S)$; *then* $u_n . v_n \to u . v$ *weakly in* $L_2((0, T), S)$.

In this section we present in the abstract form a special type of evolutionary variational inequality. Its form is motivated by the boundary value problem for elastic-plastic material without hardening (this is shown in the section "Example I"). We first formulate a certain abstract problem and then prove uniqueness, existence and convergence theorems concerning the solution of this problem.

Let K be a convex closed subset of H and let $\theta \in C_0^1([0, T], H)$.

Problem I. *Find $\sigma \in H_0^1$ such that for all $t \in [0, T]$,*

$$\sigma(t) - \theta(t) \in V, \tag{1}$$

$$\sigma(t) \in K, \tag{2}$$

$$\int_0^t [\dot{\sigma}, \tau - \sigma]_H \, ds \geqq 0 \quad \textit{for all} \quad \tau \in H_0^1 \quad \textit{for which} \tag{3}$$

$$\tau(s) - \theta(s) \in V \quad \textit{for all} \quad s \in [0, T],$$

$$\tau(s) \in K \quad \textit{for all} \quad s \in [0, T].$$

Theorem 1. *There exists at most one solution of Problem* I.

Proof. Let σ^1, σ^2 be two solutions. Taking $\sigma = \sigma^1, \tau = \sigma^2$ or $\sigma = \sigma^2, \tau = \sigma^1$ in (3), we obtain

$$\int_0^t [\dot{\sigma}^2 - \dot{\sigma}^1, \sigma^1 - \sigma^2]_H \, ds = \tfrac{1}{2}\|\sigma^2(t) - \sigma^1(t)\|_H^2 \leqq 0, \quad \text{q.e.d.}$$

Theorem 2. *If there is a functional g defined on H such that*

(4) *the Fréchet derivative g' exists and it is monotone:*

$$[g'(\sigma) - g'(\tau), \sigma - \tau]_H \geqq 0, \quad \forall \sigma, \tau \in H$$

and Lipschitz continuous:

$$\|g'(\sigma) - g'(\tau)\|_H \leqq C\|\sigma - \tau\|_H \quad \textit{for some} \quad C > 0, \quad \forall \sigma, \tau \in H,$$

$$g(\sigma) \geqq 0 \quad \textit{on} \quad H, \tag{5}$$

$$g(\sigma) = 0 \quad \textit{iff} \quad \sigma \in K, \tag{6}$$

$$g'(\sigma) = 0 \quad \textit{if} \quad \sigma \in K, \tag{7}$$

$$g(0) = 0 \tag{8}$$

and there is $\gamma > 0$ such that

(9) $$\theta(t) + \gamma\dot{\theta}(t) \in K \quad \textit{for all} \quad t \in [0, T],$$

then there exists a solution of Problem I.

Proof. Let ε_n tend to zero through some sequence of positive numbers: $\varepsilon_n \to 0+$, $\varepsilon_n > 0$. For each ε_n, (4) implies the existence of a unique solution $\mu_n \in C_0^1([0, T], V)$ of the equation

$$\dot{\mu}_n + \frac{1}{\varepsilon_n} P_V(g'(\mu_n + \theta)) = -P_V(\dot{\theta}).$$

Put $\sigma_n \equiv \mu_n + \theta$. Then

(10) $$\sigma_n(t) - \theta(t) \in V \quad \text{for all} \quad t \in [0, T],$$

(11) $$[\dot{\sigma}_n(t), \tilde{\tau}]_H + (1/\varepsilon_n)[g'(\sigma_n(t), \tilde{\tau})]_H = \quad \text{for all} \quad t \in [0, T] \quad \text{and all} \quad \tilde{\tau} \in V.$$

Let us remark that by (11) we have

(12) $$\dot{\sigma}_n(t) + \frac{1}{\varepsilon_n} g'(\sigma_n(t)) \in V^{\perp} \quad \text{for all} \quad t \in [0, T].$$

Taking $\tilde{\tau} \equiv \gamma(\dot{\sigma}_n - \dot{\theta}) + (\sigma_n - \theta) \in C([0, T], V)$ in (11), then integrating (11) over $(0, t)$, using (7), the formula

$$\frac{d}{dt} g(\sigma) = [g'(\sigma), \dot{\sigma}]_H$$

and the fact that $\theta + \gamma\dot{\theta} \in K$, we get

$$\gamma \int_0^t [\dot{\sigma}_n, \dot{\sigma}_n]_H \, ds + \frac{1}{2}[\sigma_n(t), \sigma_n(t)]_H + \frac{1}{\varepsilon_n} \gamma g(\sigma_n(t)) + \\ + \frac{1}{\varepsilon_n} \int_0^t [g'(\theta + \gamma\dot{\theta}), \theta + \gamma\dot{\theta} - \sigma_n]_H \, ds - \\ - \frac{1}{\varepsilon_n} \int_0^t [g'(\sigma_n), \theta + \gamma\dot{\theta} - \sigma_n]_H \, ds = \int_0^t [\dot{\sigma}_n, \theta + \gamma\dot{\theta}]_H \, ds.$$

Using monotonicity and the inequality $2ab \leqq (1/\xi) a^2 + \xi b^2$, $\forall \xi > 0$, we get for some constant $C > 0$ the estimate

(13) $$\int_0^t [\dot{\sigma}_n, \dot{\sigma}_n]_H \, ds + [\sigma_n(t), \sigma_n(t)]_H + \frac{1}{\varepsilon_n} g(\sigma_n(t)) \leqq \\ \leqq C \int_0^T [\gamma\dot{\theta} + \theta, \gamma\dot{\theta} + \theta]_H \, ds.$$

This estimate implies the existence of an element $\sigma \in H_0^1$ and a subsequence (again denoted by σ_n) convergent to σ weakly in H_0^1: $\sigma_n \rightharpoonup \sigma$. It follows that $\sigma_n(t) \rightharpoonup \sigma(t)$ weakly in H for all $t \in [0, T]$. Further, since g is weakly lower semicontinuous due to its monotonicity, (13) implies $g(\sigma(t)) = 0$ for all $t \in [0, T]$. Hence we see that (2) is satisfied. Since V is closed, we obtain also (1). It remains to show (3). From (11) and (7) it follows that

$$(14) \qquad \int_0^t [\dot{\sigma}_n, \tau - \sigma_n]_H \, ds + \frac{1}{\varepsilon_n} \int_0^t [g'(\sigma_n), \tau - \sigma_n]_H \, ds - \\ - \frac{1}{\varepsilon_n} \int_0^t [g'(\tau), \tau - \sigma_n]_H \, ds = 0$$

for all $\tau \in H_0^1$ such that $\tau(s) - \theta(s) \in V$, $\tau(s) \in K$ for all $s \in [0, T]$. By monotonicity and (14) we have

$$(15) \qquad 0 \leqq \int_0^t [\dot{\sigma}_n, \tau - \sigma_n]_H \, ds = \int_0^t [\dot{\sigma}_n, \tau]_H \, ds - \tfrac{1}{2}\|\sigma_n(t)\|_H^2 .$$

Using (15) we see that

$$0 \leqq \limsup \int_0^t [\dot{\sigma}_n, \tau]_H \, ds - \tfrac{1}{2} \liminf \|\sigma_n(t)\|_H^2 \leqq \\ \leqq \int_0^t [\dot{\sigma}, \tau]_H \, ds - \tfrac{1}{2}\|\sigma(t)\|_H^2 = \int_0^t [\dot{\sigma}, \tau - \sigma]_H \, ds .$$

This completes the proof of Theorem 2.

Theorem 3. *Let σ_n, σ denote the approximate solutions from the proof of Theorem 2 and their weak limit, respecively. Then we have $\sigma_n \to \sigma$ strongly in $C([0, T], H)$.*

Proof. Taking $\tau = \sigma$ in (15), we get

$$\int_0^t [\dot{\sigma}_n, \sigma - \sigma_n]_H \, ds \geqq 0 .$$

It follows

$$(16) \qquad \tfrac{1}{2}[\sigma_n(t) - \sigma(t), \sigma_n(t) - \sigma(t)]_H \leqq \int_0^t [\dot{\sigma}, \sigma - \sigma_n]_H \, ds .$$

Since $\sigma_n \rightharpoonup \sigma$ weakly in H_0^1, we have $\sigma_n \rightharpoonup \sigma$ weakly in $L_2((0, T), H)$. Put

$$\psi_n(t) \equiv \int_0^t [\dot{\sigma}, \sigma - \sigma_n]_H \, ds .$$

The sequence of the functions ψ_n is pointwise convergent to zero and uniformly bounded on $[0, T]$. It is

$$\frac{1}{2}\int_0^T \|\sigma_n(t) - \sigma(t)\|_H^2 \, dt \leqq \int_0^T \psi_n(t) \, dt$$

and by the Lebesgue Theorem we get $\sigma_n \to \sigma$ strongly in $L_2((0, T), H)$. It follows from (16) that

$$\tfrac{1}{2}\|\sigma_n(t) - \sigma(t)\|_H^2 \leqq \int_0^T \{\|\dot{\sigma}(s)\|_H \cdot \|\sigma(s) - \sigma_n(s)\|_H\} \, ds \leqq$$

$$\leqq \|\sigma\|_{H_0^1} \cdot \|\sigma - \sigma_n\|_{L_2((0,T),H)} \, .$$

This implies that $\sigma_n \to \sigma$ strongly in $C([0, T], H)$.

EXAMPLE I.

As an example of application of Abstract theory I we study in this section the traction boundary value problem for an elastic-plastic material without hardening in infinitesimal strain approach. First we show a motivation of the definition of a weak solution of this problem. In this consideration we suppose the functions ε, e, p, σ, λ introduced below to belong to or to have components from $C^1([0, T], L_2(G))$. Let us consider a body in the time interval $[0, T]$ occupying at time $t = 0$ a bounded domain $G \subset R^3$ with lipschitz boundary ∂G. Let u denote the vector of the displacement field at a fixed time $t \in [0, T]$. The strain tensor ε is defined by

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \quad i, j = 1, 2, 3 \, ,$$

$$u_k \in W^{1,2}(G) \, , \quad k = 1, 2, 3 \, .$$

Denote by e, p the elastic and plastic parts of ε, respectively, and suppose that

$$\varepsilon_{ij} = e_{ij} + p_{ij} \quad \text{for all} \quad t \in [0, T] \, , \quad \text{a.e. on} \quad G \, . \tag{17}$$

The constitutive equations for e and p are supposed to be of the form

$$e_{ij} = M_{ijkl}\sigma_{kl} \, , \tag{18}$$

$$\dot{p}_{ij} = \hat{\lambda} \frac{\partial f}{\partial \sigma_{ij}}(\sigma) \, ; \tag{19}$$

here $\hat{\lambda}$ and σ are such that $\hat{\lambda} = 0$ for $f(\sigma) < k$, $\hat{\lambda} \geqq 0$ for $f(\sigma) = k$, $f(\sigma) \leqq k$ in the course of deformation (k is a constant), σ denotes the stress tensor, $\hat{\lambda}$ is an undetermined nonnegative function and f is the so called plasticity function. We suppose f to be a convex function on $\tilde{S}$ with bounded first and second partial derivatives, such that $f(0) < k$ and if $f(\tau) \geqq k$, then

$$\frac{\partial f}{\partial \sigma_{ij}}(\tau)\frac{\partial f}{\partial \sigma_{ij}}(\tau) > 0 .$$

Let M_{ijkl} denote elasticity coefficients of Hook's law such that

(20) $$M_{ijkl} = M_{jikl} = M_{klij} \in L_\infty(G) ; \quad i, j, k, l = 1, 2, 3 .$$

(21) $$C_1\|\tau\|^2_{\tilde{S}} \leqq \tau_{ij}M_{ijkl}(x)\,\tau_{kl} \leqq C_2\|\tau\|^2_{\tilde{S}}$$

$$\text{a.e. on } G \text{ for all } \tau \in \tilde{S} , \quad C_2 > C_1 > 0 .$$

Now (21) implies existence of a matrix L_{ijkl} inverse to M_{ijkl} in $\tilde{S}$ a.e. on G such that

(22) $$\sigma_{ij} = L_{ijkl}e_{kl} ,$$

(23) $$L_{ijkl} = L_{jikl} = L_{klij} \in L_\infty(G) , \quad i, j, k, l = 1, 2, 3 ,$$

(24) $$C_3\|\tau\|^2_{\tilde{S}} \leqq \tau_{ij}L_{ijkl}(x)\,\tau_{kl} \leqq C_4\|\tau\|^2_{\tilde{S}} ,$$

$$\text{a.e. on } G \text{ for all } \tau \in \tilde{S} ; \quad C_4 > C_3 > 0 .$$

We introduce another scalar product on S by

(25) $$[\omega, \tau]_S \equiv \int_G \omega_{ij}M_{ijkl}\tau_{kl}\,dx ; \quad \omega, \tau \in S .$$

The norms $\|\tau\|_S$ and $[\tau, \tau]_S^{1/2}$ are equivalent.
Let body forces

$$F \in [C_0^1([0, T], L_2(G))]^3$$

and surface tractions

$$g \in [C_0^1([0, T], L_2(\partial G))]^3$$

be given and satisfy the global equilibrium conditions. We want to prove uniqueness and existence of a symmetric tensor field σ such that (18), (19) are in some sense satisfied together with the equilibrium and boundary conditions

(26) $$\int_G \sigma_{ij}\varrho_{ij}(v)\,dx = \int_G F_iv_i\,dx + \int_{\partial G} g_iv_i\,dS ,$$

$$\forall v \in [W_2^1(G)]^3 , \quad \forall t \in [0, T]$$

and the condition $\varepsilon = e + p \in \mathscr{K}$ for all $t \in [0, T]$. To this end we first formulate precisely what we mean by a solution.

Denote

$$\tilde{K} \equiv \{\tau \in \tilde{S} \mid f(\tau) \leqq k\} .$$

It follows that (19) can be written for all $t \in [0, T]$, a.e. on G in the equivalent form

(27) $$\dot{p}_{ij}(\tau_{ij} - \sigma_{ij}) \leqq 0\,, \quad \forall \tau \in \tilde{K}\,;$$

(28) $$\sigma \in \tilde{K}\,.$$

Denote $K \equiv \{\tau \in S \mid f(\tau) \leqq k \text{ a.e. on } G\}$. The relations (27) and (28) imply for all $t \in [0, T]$

(29) $$\int_G \dot{p}_{ij}(\tau_{ij} - \sigma_{ij})\,\mathrm{d}x \leqq 0\,, \quad \forall \tau \in K\,;$$

(30) $$\sigma \in K\,.$$

Moreover, let us now suppose τ to be time dependent and such that

(31) $$\int_G \tau_{ij}\varrho_{ij}(v)\,\mathrm{d}x = \int_G F_i v_i\,\mathrm{d}x + \int_{\partial G} g_i v_i\,\mathrm{d}S$$

for all $v \in [W_2^1(G)]^3$ and $\tau \in K$ for all $t \in [0, T]$. From (31) and (26) it follows that $\tau - \sigma \in \mathscr{H}$ for all $t \in [0, T]$. Further, $\varepsilon \in \mathscr{K}$ for all $t \in [0, T]$ implies $\dot{\varepsilon} \in \mathscr{K}$, hence

(32) $$\int_G \dot{\varepsilon}_{ij}(t)\,(\tau_{ij}(t) - \sigma_{ij}(t))\,\mathrm{d}x = 0\,, \quad \forall t \in [0, T]\,.$$

The relations (18), (29) and (32) yield

$$\int_G \dot{\sigma}_{kl}(t)\,M_{klij}(\tau_{ij}(t) - \sigma_{ij}(t))\,\mathrm{d}x \geqq 0\,, \quad \forall t \in [0, T]\,.$$

It is well known ([10]) that there exists $\theta \in C_0^1([0, T], S)$ such that

(33) $$\int_G \theta_{ij}\varrho_{ij}(v)\,\mathrm{d}x = \int_G F_i v_i\,\mathrm{d}x + \int_{\partial G} g_i v_i\,\mathrm{d}S\,,$$
$$\forall v \in [W_0^1(G)]^3\,, \quad \forall t \in [0, T]$$

(e.g. the "elastic solution" of our problem).
Now we find that τ satisties (31) if and only if

$$\tau - \theta \in \mathscr{H}\,, \quad \forall t \in [0, T]\,.$$

The foregoing considerations motivate the following definition of a solution.

Definition 1. *A stress field $\sigma \in H_0^1((0, T), S)$ satisfying for all $t \in [0, T]$ the conditions*

(34) $$\sigma(t) - \theta(t) \in \mathscr{H}\,,$$

(35) $$\sigma(t) \in K\,,$$

(36) $$\int_0^t [\dot{\sigma}, \tau - \sigma]_S\,\mathrm{d}s \geqq 0\,, \quad \forall \tau \in H_0^1((0, T), S)$$

such that $\tau(s) - \theta(s) \in \mathscr{H}$, $\tau(s) \in K$, $\forall s \in [0, T]$ *will be called a weak solution of the traction boundary value problem for elastic-plastic material without hardening.*

Theorem 4. *Let there be* $\theta \in C_0^1([0, T], S)$ *satisfying* (33) *and* $\gamma > 0$ *such that* $f(\theta(t) + \gamma\dot{\theta}(t)) \leqq k$ *for all* $t \in [0, T]$. *Then there exists a unique solution in the sense of definition* 1.

Proof. Put $H \equiv S$, $\mathscr{H} \equiv V$, $[\sigma, \tau]_S \equiv [\sigma, \tau]_H$. Denote by $\hat{M}: \tilde{S} \to \tilde{S}$ and $\hat{L}: \tilde{S} \to \tilde{S}$ the mutually inverse mappings represented by the matrices M_{ijkl} and L_{ijkl}. We find that $V^\perp = \hat{L}(\mathscr{K})$ and $\mathscr{K} = \hat{M}(V^\perp)$. Let us define a functional g on S by

$$g(\sigma) \equiv \int_G \{(([f(\sigma) - k]^+)^2 + 1)^{1/2} - 1\} \, dx \tag{37}$$

(here $y^+ = \frac{1}{2}(y + |y|)$). Then the Fréchet derivative

$$g'_{kl}(\sigma) = \frac{[f(\sigma) - k]^+}{(([f(\sigma) - k]^+)^2 + 1)^{1/2}} \frac{\partial f}{\partial \sigma_{ij}}(\sigma) \, L_{ijkl} \tag{38}$$

exists. (Here g' is identified with an element of H with the scalar product $[.,.]_H$.)
We can write also

$$Dg(\sigma, \tau) \equiv \int_G \left\{ \frac{[f(\sigma) - k]^+}{(([f(\sigma) - k]^+)^2 + 1)^{1/2}} \frac{\partial f}{\partial \sigma_{ij}}(\sigma) \, \tau_{ij} \right\} dx \, . \tag{39}$$

Then

$$Dg(\sigma, \tau) = [g'(\sigma), \tau]_S \, .$$

The assumptions (4)–(8) can be easily verfied by a routine calculation and so we can use Theorems 1 and 2.

ABSTRACT THEORY II.

In this section we consider a problem similar to Problem 1. It is motivated by the boundary value problem for elastic-plastic material with isotropic hardening. We use the same notation as in Abstract theory I except the symbols g and K. Let L be a Hilbert space with the scalar product $(.,.)_L$ and the norm

$$\|.\|_L \equiv (.,.)_L^{1/2} \, .$$

Let K *be a convex closed subset* of the cartesian product $H \times L$. In $H \times L$ we define the scalar product by

$$(\{\sigma, \alpha\}, \{\tau, \beta\})_{H \times L} \equiv [\sigma, \tau]_H + (\alpha, \beta)_L \, ,$$

where

$$\{\sigma, \alpha\} \in H \times L, \quad \{\tau, \beta\} \in H \times L \, .$$

Let us denote

$$H^1 = H^1((0, T), L).$$

Let $\alpha_0 \in L$ *be a fixed element such that* $\{0, \alpha_0\} \in K$. Let us suppose that *for every* $\sigma \in C([0, T], H)$ *there exists* $\gamma \in C([0, T], L)$ *such that* $\{\sigma(t), \gamma(t)\} \in K$ *for all* $t \in [0, T]$.

Problem II. *Find* $\sigma \in H_0^1((0, T),$ $\alpha \in H^1((0, T), L)$ *with* $\alpha(0) = \alpha_0$ *satisfying for all* $t \in [0, T]$ *the conditions*

(40) $$\sigma(t) - \theta(t) \in V,$$

(41) $$\{\sigma(t), \alpha(t)\} \in K,$$

(42) $$\int_0^t ([\dot{\sigma}, \tau - \sigma]_H + (\dot{\alpha}, \beta - \alpha)_L)\, ds \geqq 0$$

$$\forall \tau \in H_0^1((0, T), H), \quad \beta \in H^1((0, T), L), \quad \beta(0) = \alpha_0,$$

$$\tau(s) - \theta(s) \in V, \quad \forall s \in [0, T],$$

$$\{\tau(s), \beta(s)\} \in K, \quad \forall s \in [0, T].$$

Theorem 5. *There exists at most one solution of Problem II.*

Proof. The proof is almost the same as that of Theorem 1.

Theorem 6. *If there is a functional g defined on* $H \times L$ *such that*

(43) *the Frèchet derivative*

$$g'(\sigma, \alpha) = \{g_1'(\sigma, \alpha), g_2'(\sigma, \alpha)\} \in H \times L$$

exists and is monotone and Lipschitz continuous (*see* (4)),

(44) $$g(\sigma, \alpha) \geqq 0 \quad on \quad H \times L,$$

(45) $$g(\sigma, \alpha) = 0 \quad iff \quad \{\sigma, \alpha\} \in K,$$

(46) $$g'(\sigma, \alpha) = 0 \quad if \quad \{\sigma, \alpha\} \in K,$$

(47) $$g(0, \alpha_0) = 0,$$

then there exists a solution of Problem II.

Proof. We use a similar method as in the proof of Theorem 2. The condition (43) implies the existence of a unique solution

$$\{\mu_n, \lambda_n\} \in C_0^1((0, T), V) \times C_0^1((0, T), L)$$

of the system

(48) $$\dot{\mu}_n + \frac{1}{\varepsilon_n} P_V(g'_1(\mu_n + \theta, \lambda_n + \alpha_0)) = -P_V(\dot{\theta}),$$

(49) $$\dot{\lambda}_n + \frac{1}{\varepsilon_n} g'_2(\mu_n + \theta, \lambda_n + \alpha_0) = 0.$$

Put $\sigma_n \equiv \mu_n + \theta$, $\alpha_n \equiv \lambda_n + \alpha_0$. Then

(50) $$\sigma_n(t) - \theta(t) \in V, \quad \forall t \in [0, T],$$

(51) $$[\dot{\sigma}_n(t), \tilde{\tau}]_H + (\dot{\alpha}_n(t), \tilde{\beta})_L + \frac{1}{\varepsilon_n} [g'_1(\sigma_n(t), \alpha_n(t)), \tilde{\tau}]_H +$$

$$+ \frac{1}{\varepsilon_n} (g'_2(\sigma_n(t), \alpha_n(t)), \tilde{\beta})_L = 0, \quad \forall \tilde{\tau} \in V, \quad \forall \tilde{\beta} \in L, \quad \forall t \in [0, T].$$

Let us note that

(52) $$\dot{\sigma}_n + \frac{1}{\varepsilon_n} g'_1(\sigma_n, \alpha_n) \in V^{\perp}.$$

In (51), put

$$\tilde{\tau} \equiv (\dot{\sigma}_n - \dot{\theta}) + (\sigma_n - \theta), \quad \tilde{\beta} \equiv \dot{\alpha}_n + \alpha_n - \gamma,$$

where $\gamma \in C([0, T], L)$ is such that $\{\theta(t) + \dot{\theta}(t), \gamma(t)\} \in K$ for all $t \in [0, T]$. An estimate can be established in a similar way as in the proof of Theorem 2. It is of the form

(53) $$\int_0^t [\dot{\sigma}_n, \dot{\sigma}_n]_H \, ds + \int_0^t (\dot{\alpha}_n, \dot{\alpha}_n)_L \, ds + [\sigma_n(t), \sigma_n(t)]_H +$$

$$+ (\alpha_n(t), \alpha_n(t))_L + \frac{1}{\varepsilon_n} g(\sigma_n(t), \alpha_n(t)) \leqq$$

$$\leqq \int_0^T [\theta + \dot{\theta}, \theta + \dot{\theta}]_H \, ds + \int_0^T (\gamma, \gamma)_L \, ds, \quad \forall t \in [0, T].$$

From (53) we obtain the existence of elements $\sigma \in H^1_0$, $\alpha \in H^1$, $\alpha(0) = \alpha_0$ and convergent subsequences (again denoted by σ_n and α_n): $\sigma_n \to \sigma$ weakly in H^1_0 and $\alpha_n \to \alpha$ weakly in H^1. We also obtain $\{\sigma(t), \alpha(t)\} \in K$ for all $t \in [0, T]$. Instead of (15) we get from (51) and (46) by monotonicity

(54) $$\int_0^t \{[\dot{\sigma}_n, \tau - \sigma_n]_H + (\dot{\alpha}_n, \beta - \alpha_n)_L\} \, ds \geqq 0.$$

Now (42) follows from (54) in the same way as (3) in Theorem 2.

EXAMPLE II.

As an example of application of Abstract theory II we discuss in this section the same boundary value problem as in Example I, but we consider the material with isotropic hardening. The constitutive equation (19) is replaced by

$$\dot{p}_{ij} = \delta(f(\sigma) - n)\, h \left[\frac{\partial f}{\partial \sigma_{kl}}(\sigma)\, \dot{\sigma}_{kl} \right]^{+} \frac{\partial f}{\partial \sigma_{ij}}(\sigma)\,, \tag{55}$$

where $\delta(x) \equiv 0$ if $x \neq 0$, $\delta(0) \equiv 1$, $h \in L_\infty(G)$, $0 < h_1 \leqq h(x) \leqq h_2$ a.e. on G ($h_2 > > h_1 > 0$ are positive constants) and

$$n(t, x) \equiv \max\,(k, \max_{0 \leqq s \leqq t} f(\sigma(s, x)))$$

is the so called experience.
Put

$$\alpha \equiv h^{1/2} n\,, \quad F(\sigma, \alpha) \equiv h^{1/2} f(\sigma) - \alpha\,.$$

Then we can write (55) in the form

$$\dot{p}_{ij} = \dot{\alpha}\, \frac{\partial F}{\partial \sigma_{ij}}(\sigma, \alpha)\,, \tag{56}$$

$$-\dot{\alpha} = \dot{\alpha}\, \frac{\partial F}{\partial \alpha}(\sigma, \alpha)\,, \tag{57}$$

where (57) is a formal identity. Put

$$L \equiv L_2(g) \quad \text{and} \quad K \equiv \{\{\tau, \beta\} \in S \times L \mid F(\tau, \beta) \leqq 0 \ \text{a.e. on } G\}\,.$$

The following definition is derived from (56) and (57) similarly as Definition 1 from (19).

Definition 2. *A pair of functions*

$$\{\sigma, \alpha\} \in H_0^1((0, T), S) \times H^1((0, T), L)$$

with $\alpha(0) = h^{1/2} k$ *satisfying for all* $t \in [0, T]$ *the conditions*

$$\sigma(t) - \theta(t) \in \mathscr{H}\,, \tag{59}$$

$$\{\sigma(t), \alpha(t)\} \in K\,, \tag{60}$$

$$\alpha(t) \geqq h^{1/2} k\,, \tag{61}$$

$$\int_0^t \{[\dot{\sigma}, \tau - \sigma]_S + (\dot{\alpha}, \beta - \alpha)_L\}\, \mathrm{d}s \geqq 0 \tag{62}$$

$\forall\{\tau, \beta\} \in H_0^1 \times H^1$ *such that* $\beta(0) = h^{1/2}k$ *and* $\forall s \in [0, T] : \tau(s) - \theta(s) \in \mathscr{H}$, $\{\tau(s), \beta(s)\} \in K$ *will be called a weak solution of the traction boundary value problem for elastic-plastic material with isotropic hardening.*

Theorem 7. *There exists a unique solution in the sense of Definition* 2.

Proof. Let us define a functional g on $S \times L$ by

$$g(\sigma, \alpha) \equiv \int_G \{((F(\sigma, \alpha)^+)^2 + 1)^{1/2} - 1\} \, dx \,. \tag{63}$$

Then the Fréchet derivative $\{g'_1, g'_2\}$ exists, where

$$g'_{1,kl}(\sigma, \alpha) = \frac{F(\sigma, \alpha)^+}{((F(\sigma, \alpha)^+)^2 + 1)^{1/2}} \frac{\partial F}{\partial \sigma_{ij}} (\sigma, \alpha) \, L_{ijkl} \,, \tag{64}$$

$$g'_2(\sigma, \alpha) = - \frac{F(\sigma, \alpha)^+}{((F(\sigma, \alpha)^+)^2 + 1)^{1/2}} \,. \tag{65}$$

Denote

$$-\dot{\alpha}_n \equiv \frac{1}{\varepsilon_n} g'_2(\sigma_n, \alpha_n) \quad (\text{it is } \dot{\alpha}_n \geqq 0) \,, \tag{66}$$

$$\dot{p}_{n,kl} \equiv \frac{1}{\varepsilon_n} g'_{1,ij}(\sigma_n, \alpha_n) \, M_{ijkl} = \dot{\alpha}_n \frac{\partial F}{\partial \sigma_{kl}} (\sigma_n, \alpha_n) \tag{67}$$

and put $S \equiv H$, $\mathscr{H} \equiv V$, $[\sigma, \tau]_S \equiv [\sigma, \tau]_H$. We can use Theorems 5 and 6 to prove the uniqueness, (59), (60) and (62). Since $\dot{\alpha}_n \to \dot{\alpha}$ weakly also in $L_2((0, T), L)$, we have $\dot{\alpha} \geqq 0$. This and $\alpha(0) = h^{1/2}k$ implies (61). Let us observe that

$$\dot{\sigma}_n + \frac{1}{\varepsilon_n} g'_1(\sigma_n, \alpha_n) \in V^{\perp} \,,$$

therefore we have

$$\dot{e}_n(t) + \dot{p}_n(t) \in \mathscr{K} \quad \text{for all} \quad t \in [0, T] \tag{68}$$

for e_n defined by $e_n = \hat{M}\sigma_n$

Theorem 8. *Let σ_n, α_n denote the approximate solutions from the proof of Theorem 7 and σ, α their weak limits ($\sigma_n \to \sigma$ in H_0^1, $\alpha_n \to \alpha$ in H^1 as $\varepsilon_n \to 0+$). Let*

$$e_n \equiv \hat{M}\sigma_n \,, \quad e \equiv \hat{M}\sigma \,, \quad \dot{p}_n \equiv \dot{\alpha}_n \frac{\partial F}{\partial \sigma} (\sigma_n, \alpha_n) \,,$$

$$\dot{p} \equiv \dot{\alpha} \frac{\partial F}{\partial \sigma} (\sigma, \alpha) \,, \quad p_n(t) \equiv \int_0^t \dot{p}_n(s) \, ds \,, \quad p(t) \equiv \int_0^t \dot{p}(s) \, ds \,,$$

$$\varepsilon_n \equiv e_n + p_n \,, \quad \varepsilon \equiv e + p \,.$$

Then the sequences σ_n, e_n, α_n and a suitable subsequence again denoted by $\dot{p}_n$ fulfil

a) $\sigma_n \to \sigma$ *strongly in* $C([0, T], S)$,

b) $e_n \to e$ *strongly in* $C([0, T], S)$,

c) $\alpha_n \to \alpha$ *strongly in* $C([0, T], L)$,

d) $\dot{p}_n \to \dot{p}$ *weakly in* $L_2((0, T), S)$,

e) $\varepsilon(t) = e(t) + p(t)$ *is a compatible tensor for all* $t \in [0, T]$,

f) $\alpha(t, x) = \max\,(h^{1/2}(x)\,k, \max_{0 \leqq s \leqq t} h^{1/2}(x) f(\sigma(s, x)))$ *a.e. on* G, *for all* $t \in [0, T]$.

Proof. a) and b) can be proved in the same way as Theorem 3. The linearity of $e = \hat{M}\sigma$ implies c). Using Lemma from the end of Preliminaries and putting

$$v_n \equiv \dot{\alpha}_n\,, \quad u_n \equiv \frac{\partial F}{\partial \sigma}(\sigma_n, \alpha_n)\,,$$

we find d); note that the first derivatives of f are bounded and $\sigma_n \to \sigma$, $\alpha_n \to \alpha$ pointwise on $(0, T) \times G$ by the just proved assertions a), b). We can see immediately that $\varepsilon_n(t) \to \varepsilon(t)$ weakly in S and (68) implies e). It remains to prove f). Put

$$\tilde{f}(\sigma) \equiv h^{1/2} f(\sigma)\,, \quad \tilde{k} \equiv h^{1/2} k\,,$$

$$\tilde{n}(t, x) \equiv \max\,(\tilde{k}(x)\,, \max_{0 \leqq s \leqq t} \tilde{f}(\sigma(s, x)))\,.$$

Since $\tilde{f}$, $\tilde{n}$, $\alpha \in H^1((0, T), L_2(G))$, we find that the functions

$$f_x(t) \equiv \tilde{f}(\sigma(t, x))\,, \quad n_x(t) \equiv \tilde{n}(t, x)\,,$$

$$\alpha_x(t) \equiv \alpha(t, x)$$

are absolutely continuous on $[0, T]$ a.e. on G. It is

$$n_x(0) = \alpha_x(0) = \tilde{k}(x)\,, \quad \dot{\alpha}_x \geqq 0\,, \quad \dot{n}_x \geqq 0\,, \quad f_x \leqq \alpha_x \quad \text{on} \quad [0, T]\,.$$

Take a fixed $t \in [0, T]$.

It is either $f_x(t) = n_x(t)$ and consequently $n_x(t) \leqq \alpha_x(t)$, or $f_x(t) < n_x(t)$. In the latter case we have $\dot{n}_x(t) = 0$, $\dot{\alpha}_x(t) \geqq 0$. Put $\bar{t} \equiv \sup\,(s \in [0, t] \mid f_x(s) = n_x(s))$. Since $\dot{n}_x = 0$, $\dot{\alpha}_x \geqq 0$ a.e. on $(\bar{t}, t)$ and $n_x(\bar{t}) \leqq \alpha_x(\bar{t})$, we obtain $n_x(t) \leqq \alpha_x(t)$.

Now we show that $n_x(t) \geqq \alpha_x(t)$ on $[0, T]$. In (62) put $\tau = \sigma$, $\beta = \tilde{n}$. It follows that: $n_x(s) < \alpha_x(s)$ implies $\dot{\alpha}_x(s) = 0$. Put $\hat{t} \equiv \sup\,(s \in [0, t] \mid n_x(s) = \alpha_x(s))$. Let $\hat{t} < t$. Then $0 \leqq \hat{t} < t$, $n_x(\hat{t}) = \alpha_x(\hat{t})$ and on $(\hat{t}, t)$ it is $n_x(s) < \alpha_x(s)$. Hence $\dot{\alpha}_x(s) = 0$ a.e. on $(\hat{t}, t)$. This implies that $\alpha_x(\hat{t}) = n_x(\hat{t}) \leqq n_x(t) < \alpha_x(t) = \alpha_x(\hat{t})$, which is contradiction. Since this consideration is correct a.e. on G, we conclude that f) is proved.

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Souhrn

EVOLUČNÍ VARIAČNÍ NEROVNICE A JEJICH APLIKACE V PLASTICITĚ.

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V práci je studována abstraktní teorie evolučních variačních nerovnic motivovaných klasickou přírustkovou teorií plasticity. V existenčních důkazech se používá metoda penalisace. Jako příklady aplikace teorie jsou diskutovány hraniční úlohy pro elasticko-plastická tělesa bez zpevnění i se zpevněním deformací.

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