Allah-Bakhsh Thaheem

A note on states of von Neumann algebras

Aplikace matematiky, Vol. 24 (1979), No. 3, 199–200

Persistent URL: http://dml.cz/dmlcz/103796

A NOTE ON STATES OF VON NEUMANN ALGEBRAS

A. B. THAHEEM

(Received April 22, 1977)

1. INTRODUCTION

In this note we essentially prove that on a von Neumann algebra (possibly of uncountable cardinality) there exists a family of states having mutually orthogonal supports (projections) converging to the identity operator. The projections thus obtained yield a direct sum decomposition of the von Neumann algebra into subalgebras which can be very useful in the quantum field theory.

Here M denotes a von Neumann algebra acting on the Hilbert space H. Let ϕ be a positive linear functional on M such that $\|\phi\| = 1$; then ϕ is called a state on M. If p is the greatest of all projections q such that $\phi(q) = 0$ then the projection $1 - p$ is called the support of ϕ (see for example [1; p. 31]).

I am grateful to Professor A. Van Daele for many useful discussions.

2. THE MAIN RESULTS

Theorem. *Let M be a von Neumann algebra. Then there exists a family $\{\phi_\alpha : \alpha \in \Omega\}$ of normal states whose supports e_α are mutually orthogonal and $\sum_{\alpha\in\Omega} e_\alpha = 1$.*

Proof. Let J be a collection of all families $\{\psi_\alpha : \alpha \in \Omega\}$ where ψ_α are normal states whose supports are mutually orthogonal. J can be ordered by inclusion. Let J_0 be a chain in J. Put

$$A = \bigcup_{\beta\in J_0} \beta\,.$$

Then every element in A is an element of some β in J_0 and therefore it is a normal state on M. Let ψ_1 and ψ_2 be two distinct elements in A. Then there exist β_1 and β_2 such that $\psi_1 \in \beta_1$ and $\psi_2 \in \beta_2$. Since J_0 is a chain, hence either $\beta_1 \subseteq \beta_2$ or $\beta_2 \subseteq \beta_1$. In the first case $\psi_1, \psi_2 \in \beta_2$ and hence ψ_1 and ψ_2 have mutually orthogonal supports.

Similarly if $\beta_2 \subseteq \beta_1$. It follows that A is a family of normal states with mutually orthogonal supports. Therefore $A \in J$ and A is an upper bound for J_0. Hence using Zorn's lemma we obtain a family $\{\phi_\alpha : \alpha \in \Omega\}$ as a maximal element in J with mutually orthogonal supports e_α. Put

$$e = \sum_{\alpha \in \Omega} e_\alpha .$$

The sum is well-defined because e_α are mutually orthogonal. If $e \neq 1$ then choose a vector $\xi \neq 0$ in the Hilbert space H such that $(1 - e)\,\xi = \xi$ or in other words, $\xi \in (1 - e)\,H$. Put $\phi(x) = \langle x\xi, \xi \rangle$, $x \in M$. Then ϕ is a normal state on M. As $\phi(e) = \langle e\xi, \xi \rangle = 0$, the support of ϕ is orthogonal to e and hence to all e_α. Thus $\{\phi_\alpha : \alpha \in \Omega\} \cup \{\phi\}$ is again in J. This contradicts the maximality and so $e = 1$. This completes the proof of the theorem.

References

[1] *S. Sakai:* C^*-algebras and W^*-algebras. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60. Springer-Verlag, Berlin, 1971.

Souhrn

POZNÁMKA O STAVECH NA VON NEUMANNOVÝCH ALGEBRÁCH

A. B. THAHEEM

Nechť M je von Neumannova algebra na Hilbertově prostoru H. Kladný lineární funkcionál ϕ na M se nazývá stav na M, je-li $\|\phi\| = 1$. Je-li p největší z projekcí q takových, že $\phi(q) = 0$, pak projekce $1 - p$ se nazývá nosič ϕ.

Věta. *Nechť M je von Neumannova algebra. Pak existuje množina $\{\phi_\alpha : \alpha \in \Omega\}$ normálních stavů, jejichž nosiče jsou navzájem ortogonální a platí* $\sum_{\alpha \in \Omega} e = 1$.

Author's address: Dr. *A. B. Thaheem*, Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.