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THE BENDERS METHOD AND PARAMETRIZATION OF THE RIGHT-HAND SIDES IN THE MIXED INTEGER LINEAR PROGRAMMING PROBLEM

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The parametrization in the mathematical programming is spoken about when elements of a problem are functions of one or more parameters. Then we study properties of such a problem, mainly its optimal or suboptimal solutions, in dependence on the parameters. From the point of view of a model the parametrization introduces a dynamical factor into a model thus making possible a postoptimal analysis of a solution. So far the most frequent application of the parametrization in practice has been in LP, mainly as one-dimensional linear parametrization of the objective function or the right-hand sides of the constraints.

Since recently, the interest in integer linear programming has grown continuously. Partly because it makes it possible to describe more adequately some real situations than the ordinary LP can do, partly because it serves an equivalent formulation of some nonstandard extreme problems. Therefore it is natural to consider the possibility of parametrization also in these problems.

In the field of pure integer programming there exist only few works concerning the subject, e.g. [22], [28], [30], [32], [33], [34], [35]. Probably it is because we still lack a suitable methodological basis for developing a parametric theory and numerical methods.

In the field of the mixed integer linear programming (MILP) it is possible – to a certain extent – to make use of the methodology of LP. Considering two basic cases of the parametrization, namely (1) one-dimensional linear parametrization of the objective function, and (2) one-dimensional linear parametrization of the right-hand sides, the situation is as follows: The optimal value of the objective function of the problem (1) is a continuous, convex, and piecewise linear function of the parameter. This property strongly facilitates the construction of this function or its approximation. The original work to the subject is [30], and also see [25].

The problem (2) is substantially more difficult due to a more complicated structure of the objective function optimal value, which is, in general, a discontinuous and non-concave function of the parameter, defined on a disconnected set. Under the assumption that the set Y of integer variables is bounded, the problem was analysed and algorithmically solved in [25].

In the present work we deal with the problem (2) with special emphasis on the numerical solution to be practically realizable. What we think to be essential in this respect is the algorithmic compactness of a numerical method in the sense that both point and parametric optimizations can be realized by the same or similar numerical procedures – in the way the simplex method is applied in the parametric LP. It has turned out that this is possible on the basis of the Benders method [12], but under certain restrictions and specializations, namely:

a) The problem is solved approximately in the sense of obtaining a suboptimal value of the objective function within a given tolerance ε from the optimal value. Apparently such restriction is negligible in practice.

b) A complete algorithmic solution is given only for a special case of the MILP problem, which can be roughly characterized as a parametrization of the right-hand sides of the type $\leqq b_i + \theta h_i$ where $h_i > 0$ and θ is a parameter.

c) The general MILP problem can be parametrically solved only unless the given suboptimal solution changes (this provides a sensitivity analysis of a suboptimal solution in a given direction) or unless the continuity from the right of the optimal value of the objective function is violated.

The procedure suggested for solving the problems described above consists in an iterative application of an algorithm which we call algorithm PB and which is, in fact, a formal variant of the Benders algorithm – here called algorithm B. We denote the whole combined procedure as algorithm ΠPB.

The algorithms B and PB are based on a dual decomposition of the linear part of the MILP problem. In [12] the Benders algorithm was described only for a bounded set Y of integer variables and for the constraint conditions in the form of inequalities. Here we weaken these restrictions. This is made possible by exploiting one deep property of the MILP problems, which we call regularity and which has been rarely employed so far. Concerning their solvability, regular MILP problems behave analogously to LP problems with the well-known trichotomy of optimality, unboundedness, and unsolvability. Regularity, roughly speaking, is a consequence of boundedness of the set Y or rationality of coefficients of the problem. These assumptions, at the same time, enable us to prove the finiteness of the algorithms B, PB, and ΠPB.

Remark. Because of a large extent of the article it was necessary to omit the proofs of some less important assertions from the original text. We shall no longer mention this circumstance.

1. PROBLEM P

We shall deal with a mixed integer linear programming (MILP) problem denoted by P and defined as

$$\bar{\varphi} = \max_{x,y} \{cx + dy \mid Ax + By = b,\ x \geqq 0,\ y \in Y\}\,.$$

The symbols used have the following meaning: x is an n_1-dimensional vector of real variables; y is an $n_2$dimensional vector of integer variables ($n_1 + n_2 = n$, $n_1 > 0$, $n_2 > 0$); c and d are constant vectors whose dimensions correspond to those of x and y so that cx and dy are scalar products; 0 is the zero n_1-dimensional vector; b is a constant m-dimensional vector; A and B are matrices of the dimensions $(m \times n_1)$, $(m \times n_2)$ respectively. All the numbers are real. The relation operators between the vectors relate to all their components. The set Y is a nonempty set (possibly unbounded) of integer-component vectors y's.

Neither the row nor the column structure of vectors will be explicitly indicated, since it can always be determined from the context, e.g. from the product of a vector by a vector or by a matrix. Components of a vector will be subscripted by positive integer numbers; the subscript 0 or superscripts refer, as a rule, to the whole vectors. A vector with integer components will be briefly called an integer vector. The symbol E^r means the r-dimensional real Euclidean space.

The function $cx + dy$ is an objective function of the problem P. The set

$$Z = \{(x, y) \mid Ax + By = b,\ x \geqq 0,\ y \in Y\}$$

is the feasible region of the problem and its elements are feasible solutions. The aim of solving the problem P is to find – if there exists any – an element maximizing the objective function over the set Z. Our first intention is to make the terminology more precise.

Definition 1.1. *We say that the problem P is regular if one of the following possibilities occurs:*

1. $Z = \emptyset$.
2. $Z \neq \emptyset$ *and there exists an element* $(\bar{x}, \bar{y}) \in Z$ *such that* $c\bar{x} + d\bar{y} = \max_{x,y} \{cx + dy \mid (x, y) \in Z\}$.
3. $Z \neq \emptyset$ *and the function* $cx + dy$ *is unboundedly increasing on* Z.

The value $\bar{\varphi}$ is called an optimal value of the problem P. In case P is regular, we can always define an optimal value for it if we admit – in accordance with Definition 1.1 – infinite values for $\bar{\varphi}$ according to the following convention:

1. If $Z = \emptyset$, then we put $\bar{\varphi} = -\infty$ and say that the problem P has no solution. (If $Z \neq \emptyset$, then the problem does have a solution.)

2. If $Z \neq \emptyset$ and $\bar{\varphi} = c\bar{x} + d\bar{y}$, $(\bar{x}, \bar{y}) \in Z$, then P has a bounded solution. The element $(\bar{x}, \bar{y})$ is an optimal solution of the problem.

3. If $Z \neq \emptyset$ and $cx + dy$ is unboundedly increasing on Z, then we put $\bar{\varphi} = +\infty$ and say that the problem P has an unbounded solution.

It follows from Definition 1.1 that if P is not regular, then $Z \neq \emptyset$, the function $cx + dy$ is bounded from above on Z, but there exists no maximizing element for the function in Z. The value $\bar{\varphi}$ is not defined in this case.

Note that the concept of regularity, introduced by Definition 1.1, transfers a characteristic property of the LP problems to the MILP problems. Each LP problem is regular in our sense. On the other hand, there exist nonregular MILP problems – see e.g. [8].

Now we shall express the problem P in the form in which the attention is more focused on the integer variables of the problem. Let us denote by $L(y)$ the linear programming problem

$$\lambda(y) = \max_x \{cx \mid Ax = q(y),\ x \geqq 0\}$$

where

$$q(y) = b - By\,.$$

We put $\lambda(y) = -\infty$ if $L(y)$ has no solution, and $\lambda(y) = +\infty$ if $L(y)$ has an unbounded solution.[1] If we denote the feasible region of the problem $L(y)$ by the symbol $X(y)$, i.e.

$$X(y) = \{x \mid Ax = q(y),\ x \geqq 0\}\,,$$

then

$$(x, y) \in Z \Leftrightarrow x \in X(y)\,, \quad y \in Y\,. \tag{1.1}$$

Denoting further

$$\varphi(y) = dy + \lambda(y)\,,$$

we can write on the basis of (1.1)

$$\begin{aligned} \bar{\varphi} &= \max_{x,y} \{cx + dy \mid (x, y) \in Z\} = \\ &= \max_y \{dy + \max_x \{cx \mid x \in X(y)\} \mid y \in Y\} = \\ &= \max_y \{dy + \lambda(y) \mid y \in Y\} = \\ &= \max_{y \in Y} \varphi(y)\,. \end{aligned} \tag{1.2}$$

The equivalence relations in (1.2) are to be understood in the sense that either all expressions involved have their values defined or none.

[1]) For LP problems we use the same terminology as for MILP problems.

In the rest of this section we shall deal with the dual problem to $L(y)$. We denote it by $\tilde{L}(y)$ and write

$$\tilde{\lambda}(y) = \min_u \{u\, q(y) \mid u \in U\}$$

where U is the feasible region

$$U = \{u \mid uA \geqq c,\ u \in E^m\}\,.$$

Now the values $\tilde{\lambda}(y) = +\infty$ and $\tilde{\lambda}(y) = -\infty$ imply that the problem $\tilde{L}(y)$ has no solution or an unbounded solution, respectively. From the theory of linear programming the following implications are known:

$$(1.3) \qquad -\infty < \tilde{\lambda}(y) < +\infty \Rightarrow \tilde{\lambda}(y) = \lambda(y)\,,$$

$$(1.4) \qquad \tilde{\lambda}(y) = +\infty \Rightarrow \lambda(y) = \begin{cases} +\infty & \text{if } X(y) \neq \emptyset \\ -\infty & \text{if } X(y) = \emptyset\,, \end{cases}$$

$$(1.5) \qquad \tilde{\lambda}(y) = -\infty \Rightarrow \lambda(y) = -\infty\,,$$

$$(1.6) \qquad \lambda(y) = +\infty \Rightarrow \tilde{\lambda}(y) = +\infty\,,$$

$$(1.7) \qquad \lambda(y) = -\infty \Rightarrow \tilde{\lambda}(y) = \begin{cases} -\infty & \text{if } U \neq \emptyset \\ +\infty & \text{if } U = \emptyset\,. \end{cases}$$

Lemma 1.1. *If $U \neq \emptyset$, then $\tilde{\lambda}(y) = \lambda(y)$ for all $y \in Y$.*

Theorem 1.1. *If $U = \emptyset$, then either $\bar{\varphi} = +\infty$ or $\bar{\varphi} = -\infty$.*

Apparently, the assumption $U \neq \emptyset$ will be useful in the sequel while it causes no substantial restriction and is easy to be verified. Now we are going to analyse the structure of the polyhedral set U.

Let A_i $(1 \leqq i \leqq m)$ stand for the i-th row and r for the rank of the matrix A. We can assume without loss of generality that the first r rows of the matrix are linearly independent.

Lemma 1.2. *Let $U \neq \emptyset$. If r is the rank of the matrix A and the vectors $A_1, \ldots, A_r$ are linearly independent, then*

$$(1.8) \qquad U = U_0 + E^{m-r}\ [2]$$

where

$$U_0 = \{u \mid uA \geqq c,\ u \in E^m,\ u_i = 0 \quad (i = r + 1, \ldots, m)\}\,.$$

[2]) The addition means the equivalence $u \in U \Leftrightarrow u = u' + u''$, $u' \in U_0$, $u'' \in E^{m-r}$.

For a proof see [7] – Chapter 2, Lemma 2.1. The proof is constructive. Under the assumption $r < m$ a basis of the space E^{m-r} is formed by the vectors

$$(1.9)\qquad w^{(j)} = (\lambda_{1,r+j}, \ldots, \lambda_{r,r+j}, 0, \ldots, -1, \ldots, 0) \quad (j = 1, \ldots, m - r)$$

where the coefficients $\lambda_{i,r+j}$ are given by the equations

$$(1.10)\qquad A_k = \sum_{i=1}^{r} \lambda_{ik} A_i \quad (k = r + 1, \ldots, m)$$

and -1 is the $(r + j)$-th component of the vector. If $r = m$, then the equivalence (1.8) holds trivially for $U_0 = U$, $E^0 = \{0\}$.

For convenience we denote the set of j's at $w^{(j)}$ by K, so we have

$$K = \{1, 2, \ldots, m - r\}$$

and $K = \emptyset$ for $r = m$.

The set U_0 is a convex polytope which is uniquely determined by its elements $u^{(i)}$ and $v^{(j)}$ where

$u^{(i)}$ $(i \in G)$ are all vertices of the set U_0,
$v^{(j)}$ $(j \in H)$ are all direction vectors of the unbounded edges of the set U_0.

If $U \neq \emptyset$, then it is also $U_0 \neq \emptyset$ and $G \neq \emptyset$. The sets G and H are finite. The following equivalence is well-known:

$$(1.11)\qquad U_0 = \{u \mid u = \sum_{i \in G} \alpha_i u^{(i)} + \sum_{j \in H} \beta_j v^{(j)} ,$$
$$\sum_{i \in G} \alpha_i = 1 , \quad \alpha_i \in E^1 , \quad \alpha_i \geqq 0 \ (i \in G) , \quad \beta_j \in E^1 , \quad \beta_j \geqq 0 \ (j \in H)\} .$$ [3])

Using the set U_0 as a feasible region instead of U, we obtain a linear programming problem $\tilde{L}_0(y)$ defined by

$$\tilde{\lambda}_0(y) = \min_u \{u\, q(y) \mid u \in U_0\} .$$

On the basis of (1.11) and Lemma 1.2, three lemmas can be proved, namely:

Lemma 1.3. *If $U \neq \emptyset$, the conditions*

$$v^{(j)}\, q(y) \geqq 0 \quad (j \in H) ,$$
$$w^{(j)}\, q(y) = 0 \quad (j \in K)$$

are necessary and sufficient in order that $\tilde{\lambda}(y) > -\infty$.

[3]) An empty sum, if it occurs, is assigned always the zero value.

Lemma 1.4. *If $U \neq \emptyset$, the conditions*

$$w^{(j)}\, q(y) = 0 \quad (j \in K)$$

are sufficient in order that $\tilde{\lambda}_0(y) = \tilde{\lambda}(y)$.

Lemma 1.5.[4] *If $U \neq \emptyset$ and $\tilde{\lambda}_0(y) > -\infty$, then*

$$\tilde{\lambda}_0(y) = \min_{i \in G} u^{(i)}\, q(y)\,.$$

If $\tilde{\lambda}_0(y) = -\infty$, then it follows from (1.11) that there exists a vector $v^{(j)}$ such that $v^{(j)}\, q(y) < 0$. For any $u^{(i)} \in U_0$ we have then $u(\tau) \equiv u^{(i)} + \tau v^{(j)} \in U_0$ $(\tau \in E^1, \tau \geqq 0)$ and $u(\tau)\, q(y) \to -\infty$ for $\tau \to +\infty$. This justifies

Definition 1.2. *We say that the problem $\tilde{L}_0(y)$ has an unbounded solution along the direction $v^{(j)}$ if $v^{(j)}\, q(y) < 0$.*

2. OPTIMALITY, ε-SUBOPTIMALITY, REGULARITY

In (1.2) the problem P was expressed in the form

$$\bar{\varphi} = \max_{y \in Y} \varphi(y) \tag{2.1}$$

from which it can be seen that an optimal solution of the problem P is, in fact, determined by the y-component. It is because the solution of the corresponding LP problem can be considered trivial in this context.

Definition 2.1. *A vector $\bar{y}$ is called an optimal vector of the problem P if there exists a vector $\bar{x}$ such that $(\bar{x}, \bar{y})$ is an optimal solutions of the problem P.*

Notice the significant distinction between the notions *optimal vector* and *optimal solution*. Analogously we define the pair of notions feasible vector and feasible solution.

Lemma 2.1. *A vector $\bar{y}$ is an optimal vector of the problem P if and only if*

$$\bar{y} \in Y\,, \tag{2.2}$$

$$-\infty < \lambda(\bar{y}) < +\infty\,, \tag{2.3}$$

$$\bar{\varphi} = \varphi(\bar{y})\,. \tag{2.4}$$

In this work we shall not be aimed at obtaining an optimal solution of the MILP

[4]) See [7] — Chapter 2, Theorem 4.2.

problem, but shall be contented with a suboptimal solution for which an error bound ε is guaranteed. Such a restriction, quite acceptable in practice, leads to a saving of computational work.

Definition 2.2. *Let $\varepsilon > 0$. a) A vector $\bar{y}$ is called an ε-suboptimal vector of the problem P if it satisfies $\bar{y} \in Y$, $-\infty < \lambda(\bar{y}) < +\infty$, and*

$$\varphi(y) < \varphi(\bar{y}) + \varepsilon$$

for all $y \in Y$. b) A vector $(\bar{x}, \bar{y})$ is called an ε-suboptimal solution of the problem P if $\bar{y}$ is an ε-suboptimal vector of the problem P and $\bar{x}$ is an optimal solution of the problem $L(\bar{y})$.

To make the foregoing definition clearer let us formulate one of its consequences: Let $\bar{y}$ be an ε-suboptimal vector of the problem P. If a feasible solution (x, y) of the problem P gives the value of the objective function greater than $\varphi(\bar{y})$, then the difference is less than ε. Indeed, according to (1.1) and Definition 2.2 we have

$$\varphi(\bar{y}) < cx + dy \leqq \lambda(y) + dy = \varphi(y) < \varphi(\bar{y}) + \varepsilon\,.$$

The following assertion relates the notion of an optimal vector of the problem P to that of an ε-suboptimal vector.

Lemma 2.2. *A vector $\bar{y}$ is optimal if and only if it is ε-suboptimal for all sufficiently small $\varepsilon > 0$.*

Notice that the ε-suboptimality is defined independently of the regularity of the problem P. In the next theorem we shall formulate a necessary and sufficient condition of the ε-suboptimality – a suitable theoretical basis for the construction of computational algorithms. Let us first define a set

$$\Omega(G^*, H^*) = \{(y, \mu) \mid u^{(i)}\, q(y) \geqq \mu \quad (i \in G^*)\,,$$
$$v^{(j)}\, q(y) \geqq 0 \quad (j \in H^*)\,, \quad w^{(j)}\, q(y) = 0 \quad (j \in K)\,, \quad y \in Y\,, \quad \mu \in E^1\}$$

where $G^* \subseteq G$, $H^* \subseteq H$. Then we consider an auxiliary maximization problem – we denote it by $M(G^*, H^*)$ and define as

$$\bar{\psi}(G^*, H^*) = \max_{y,\mu} \{dy + \mu \mid (y, \mu) \in \Omega(G^*, H^*)\}\,.$$

Since it is a MILP problem, we can use for it the terminology and theory originally worked out for the problem P. Adding still another symbol,

$$\bar{\psi}_S(G^*, H^*) = \sup \{dy + \mu \mid (y, \mu) \in \Omega(G^*, H^*)\}\,,$$

we are able to formulate

Theorem 2.1. (*ε-suboptimality criterion*).

a) *If $\bar{y}$ is an ε-suboptimal vector of the problem P, then $U \neq \emptyset$, $\bar{y}$ is a feasible vector of the problem $M(G, H)$, and*

$$\bar{\psi}_S(G, H) \leqq \varphi(\bar{y}) + \varepsilon \,. \tag{2.5}$$

b) *If $U \neq \emptyset$, $\bar{y} \in Y$, and*

$$\bar{\psi}_S(G^*, H^*) < \varphi(\bar{y}) + \varepsilon \tag{2.6}$$

for some $G^* \subseteq G$, $H^* \subseteq H$, then $\bar{y}$ is an ε-suboptimal vector of the problem P.

Proof. a) Necessity: According to Definition 2.2 it is $-\infty < \lambda(\bar{y}) < +\infty$. Hence $U \neq \emptyset$ because otherwise $\lambda(\bar{y}) = \pm\infty$ in virtue of (1.4). Then according to Lemma 1.1 it is $\lambda(\bar{y}) = \tilde{\lambda}(\bar{y})$, so that Lemma 1.3 implies

$$v^{(j)}\, q(\bar{y}) \geqq 0 \quad (j \in H)\,, \tag{2.7}$$

$$w^{(j)}\, q(\bar{y}) = 0 \quad (j \in K) \tag{2.8}$$

and on the basis of Lemmas 1.4 and 1.5 it is

$$u^{(i)}\, q(\bar{y}) \geqq \min_{i \in G} u^{(i)}\, q(\bar{y}) = \tilde{\lambda}_0(\bar{y}) = \tilde{\lambda}(\bar{y}) = \lambda(\bar{y})\,. \tag{2.9}$$

The relations (2.9), (2.7) and (2.8) imply the inclusion $(\bar{y}, \lambda(\bar{y})) \in \Omega(G, H)$, which means that $\bar{y}$ is a feasible vector of the problem $M(G, H)$. Finally, (2.5) must be true, since otherwise there would exist a vector $(\hat{y}, \hat{\mu}) \in \Omega(G, H)$ such that

$$d\hat{y} + \hat{\mu} \geqq \varphi(\bar{y}) + \varepsilon\,.$$

But then on the basis of Lemmas 1.1, 1.3 and 1.5 we could write

$$\lambda(\hat{y}) = \tilde{\lambda}(\hat{y}) = \tilde{\lambda}_0(\hat{y}) = \min_{i \in G} u^{(i)}\, q(\hat{y}) \geqq \hat{\mu}$$

and hence

$$\varphi(\hat{y}) = d\hat{y} + \lambda(\hat{y}) \geqq d\hat{y} + \hat{\mu} \geqq \varphi(\bar{y}) + \varepsilon\,,$$

which contradicts the ε-suboptimality of the vector $\bar{y}$.

b) Sufficiency: The assumptions $U \neq \emptyset$ and (2.6) imply $-\infty < \lambda(\bar{y}) < +\infty$. Let us have an arbitrary vector $\hat{y} \in Y$ such that $\tilde{\lambda}(\hat{y}) > -\infty$. On the basis of Lemmas 1.1, 1.3, 1.4, 1.5 and the inequality (2.6) we can write

$$\varphi(\hat{y}) = d\hat{y} + \lambda(\hat{y}) = d\hat{y} + \tilde{\lambda}(\hat{y}) = d\hat{y} + \tilde{\lambda}_0(\hat{y}) =$$
$$= d\hat{y} + \min_{i \in G} \{u^{(i)}\, q(\hat{y}) \mid v^{(j)}\, q(\hat{y}) \geqq 0 \ (j \in H)\,, \quad w^{(j)}\, q(\hat{y}) = 0 \ (j \in K)\} =$$
$$= \max_{\mu \in E^1} \{d\hat{y} + \mu \mid u^{(i)}\, q(\hat{y}) \geqq \mu \ (i \in G)\,, \quad v^{(j)}\, q(\hat{y}) \geqq 0 \ (j \in H)\,,$$

$$w^{(j)} q(\hat{y}) = 0 \ (j \in K)\} \leqq \sup \{dy + \mu \mid (y, \mu) \in \Omega(G, H)\} =$$
$$= \bar{\psi}_S(G, H) \leqq \bar{\psi}_S(G^*, H^*) < \varphi(\bar{y}) + \varepsilon .$$

For $\hat{y}$ such that $\tilde{\lambda}(\hat{y}) = -\infty$ we obtain immediately

$$\varphi(\hat{y}) = d\hat{y} + \lambda(\hat{y}) = d\hat{y} + \tilde{\lambda}(\hat{y}) = -\infty < \varphi(\bar{y}) + \varepsilon .$$

Thus we have indicated all which is needed to prove the ε-suboptimality of the vector $\bar{y}$. Q.E.D.

By means of Lemma 2.2 and Theorem 2.1 we can easily obtain the following

Theorem 2.2. (*Optimality criterion*). *A vector $\bar{y}$ is an optimal vector of the problem P if and only if $U \neq \emptyset$ and*

$$\lambda(\bar{y}) = \bar{\mu} \tag{2.10}$$

where $(\bar{y}, \bar{\mu})$ is an optimal solution of the problem $M(G^*, H^*)$ for some $G^* \subseteq G$ and $H^* \subseteq H$.

In the rest of this section we shall deal with the regularity property of the MILP problem. If the set Y is of the form

$$Y = \{y \in E^{n_2} \mid y \geqq 0, \ y_j \text{ integer } (j = 1, \ldots, n_2)\} , \tag{2.11}$$

we can assign it a convex set

$$\bar{Y} = \{y \in E^{n_2} \mid y \geqq 0\} .$$

Denoting for a moment the problem P by P_Y, we have immediately defined a problem $\bar{P} = P_{\bar{Y}}$. This is an LP problem. Later we shall use the notation $\bar{Z}$ and $\bar{\varphi}$ for the feasible region and the optimal value of the problem $\bar{P}$, respectively. Now let us return to the problem P.

Theorem 2.3. *The problem P is regular if one of the following assumptions is true*:

(a) *The set Y is bounded.*

(b) *The set Y is of the form* (2.11) *and all elements of the matrices A, B are rational numbers.*

For a proof of this important assertion see [8]. In the same paper another useful theorem can be found:

Theorem 2.4. *If the assumption* (b) *from Theorem* 2.3 *is satisfied and the problem P has a solution, then both the problems P and $\bar{P}$ either have bounded solutions or unbounded solutions.*

In this connection, we should mention also

Theorem 2.5. *If $U \neq \emptyset$ and Y is bounded, then $\bar{\varphi} < +\infty$.*

The problem of regularity of the MILP and also, in the same sense, ILP problems has been investigated in detail by several authors, apparently independently of one another. To our knowledge, the most original works on the subject are [3], [30] – Chapter 4, and [1]. More recent works are [6], [2], [9], [5], [8]. Naturally, the terminology has not yet been unified. The reader should be especially aware of a rather narrower meaning of the concept "regularity" used in [1], [2], [9] than that defined here. Noltemeier in [30] uses the pair of terms *Stützparameter* – *Berührparameter* in correspondence to ours, regularity – nonregularity.

3. ALGORITHM B

In this section we shall briefly describe an algorithm for the solution of the problem P, the basic idea of which is as follows: A sequence of approximations of the feasible region of the auxiliary problem $M(G, H)$ is constructed so that the approximations become better step by step until the ε-suboptimality criterion is satisfied.

An algorithm of this type was originally invented by Benders in 1962 [12] for the MILP problems under the assumption that the constraint conditions are inequalities $Ax + By \leqq b$ and the set Y is bounded. Since that time Benders' algorithm has been not only many times described and modified (see e.g. [10], [11], [13]–[21]), but also successfully applied (e.g. [18]). Here we reformulate the Benders algorithm under more general assumptions and call it algorithm B.

Step (O). Let G_0 and H_0 be arbitrary subsets of the sets G and H respectively, e.g. $G_0 = H_0 = \emptyset$. Put $k = 1$, $G^1 = G_0$, $H^1 = H_0$, and go to the step (A^1).

Step (A^k). Given sets G^k and H^k, solve an auxiliary problem $M^k \equiv M(G^k, H^k)$ as follows:

$$\max_{y,\mu} \{dy + \mu \mid u^{(i)} q(y) \geqq \mu \ (i \in G^k),$$
$$v^{(j)} q(y) \geqq 0 \ (j \in H^k), \quad w^{(j)} q(y) = 0 \ (j \in K), \quad y \in Y, \quad \mu \in E^1\}.$$

If the problem M^k has no solution, then go to (E^k). If the problem M^k has an unbounded solution, then choose its feasible solution (y^k, μ^k) satisfying $dy^k + \mu^k \geqq \omega$ where ω is a given constant, and go to (B^k). Otherwise find an optimal solution (y^k, μ^k) of the problem M^k and go to (B^k).

Step (B^k). Solve the problem $\tilde{L}_0(y^k)$. Let $\lambda^k \equiv \tilde{\lambda}_0(y^k)$ be its optimal value. Go to (C^k).

Step (C^k). If the problem $\tilde{L}_0(y^k)$ has an unbounded solution (i.e. $\lambda^k = -\infty$) along the direction $v^{(j)}$, put

$$G^{k+1} = G^k, \quad H^{k+1} = H^k \cup \{j\}$$

and go to (A^{k+1}). If $\lambda^k > -\infty$, go to (D^k).

Step (D^k). If it holds

$$\lambda^k > \mu^k - \varepsilon, \tag{3.1}$$

then go to (E^k). Otherwise let $u^{(i)}$ be an optimal basic solution of the problem $\tilde{L}_0(y^k)$. Put

$$G^{k+1} = G^k \cup \{i\}, \quad H^{k+1} = H^k$$

and go to (A^{k+1}).

Step (E^k). End.

Before we formulate a convergence theorem for the algorithm we shall present two easy-to-verify auxiliary assertions.

Lemma 3.1. *If the elements of the matrix A and the components of the vector c are rational, then also the elements of the vectors $u^{(i)}$ $(i \in G)$, $v^{(j)}$ $(j \in H)$ and $w^{(j)}$ $(j \in K)$ are rational.*

For the sake of brevity we summarize the assumptions from Theorem 2.3 and Lemma 3.1 as

Assumption 3.1. *One of the following conditions is fulfilled*:

(a) *The set Y is bounded.*

(b) *The set Y is of the form* (2.11). *The elements of the matrices A, B and the components of the vector c are rational numbers.*

Lemma 3.2. *If Assumption* 3.1 *holds, then each problem M^k $(k \geqq 1)$ is regular.*

Theorem 3.1. *Let $U \neq \emptyset$ and let Assumption* 3.1 *be satisfied. If the problem P has no unbounded solution and if ω is a sufficiently large number, then one of the following two possibilities occurs in the algorithm* B *for some $k = \bar{k}$*:

1. *The problem M^k has no solution. Then also P has no solution.*
2. *The problem M^k has an optimal solution (y^k, μ^k) and the inequality* (3.1) *holds. Then y^k is an ε-suboptimal vector of the problem P.*

Proof. The problem P is regular according to Theorem 2.3. According to Lemma 3.2 also each problem M^k is regular. Let us choose ω satisfying

$$\omega \geqq \bar{\varphi} + \varepsilon. \tag{3.2}$$

If M^k has no solution, then also P has none. Indeed, if it were not so, then it would exist an optimal vector of the problem P, which means – according to Theorem 2.1 – that the problem $M(G, H)$ and consequently also $M^k \equiv M(G^k, H^k)$ would have a solution.

If (y^k, μ^k) is an optimal solution of the problem M^k and if the inequality (3.1) holds, then using Lemmas 1.4 and 1.1 we can obtain

$$\bar{\psi}_S(G^k, H^k) = dy^k + \mu^k < dy^k + \tilde{\lambda}_0(y^k) + \varepsilon = dy^k + \lambda(y^k) + \varepsilon = \varphi(y^k) + \varepsilon ,$$

which implies the ε-suboptimality of the vector y^k in virtue of Theorem 2.1.

Now we are going to prove the finiteness of the algorithm. In Step (C^k) we have $v^{(j)} q(y^k) < 0$ so that $j \notin H^k$. In Step (D^k), if (3.1) does not hold, we have

$$u^{(i)} q(y^k) = \tilde{\lambda}_0(y^k) \leqq \mu^k - \varepsilon < \mu^k \leqq u^{(\iota)} q(y^k) \quad (\iota \in G^k)$$

so that $i \notin G^k$. Thus, altogether, we have one of the sharp inclusions

$$G^{k+1} \supset G^k \quad \text{or} \quad H^{k+1} \supset H^k . \tag{3.3}$$

If the problem P has no solution, then $X(y^k) = \emptyset$, i.e. $\tilde{\lambda}_0(y^k) = -\infty$ for all y^k. Thus $G^k = \emptyset$ for all k. With respect to (3.3) and to the finiteness of the set H, it must occur that the control passes from Step (A^k) to (E^k) for some $k = \bar{k}$, so that the problem M^k has no solution.

If the problem P has a solution, then under our assumptions it has an optimal one. Again, (3.3) together with the finiteness of the sets G and H implies that for some $k = \bar{k}$ in Step (D^k) the control passes to (E^k), because anyone of the problems M^k has necessarily a solution. Thus for $k = \bar{k}$ the inequality (3.1) holds. Then with regard to (2.1) and (3.2) we have

$$dy^k + \mu^k < dy^k + \lambda^k + \varepsilon = \varphi(y^k) + \varepsilon \leqq \bar{\varphi} + \varepsilon \leqq \omega ,$$

which says that the vector (y^k, μ^k) is an optimal solution of the problem M^k. Q.E.D.

A lower bound for the constant ω, as follows from the proof of Theorem 3.1, is given by the relation (3.2). Therefore we can choose it as follows:

a) If the condition (a) of Assumption 3.1 holds, the problem M^k can have an unbounded solution only when $G^k = \emptyset$. If this occurs and y^k is a feasible vector of the problem M^k, then (y^k, μ) is a feasible solution for an arbitrary $\mu \in E^1$. Thus we can take $\omega = +\infty$, which implies $\mu^k = +\infty$.

b) If the condition (b) of Assumption 3.1 holds, we first compute the optimal value $\tilde{\varphi}$ of the problem $\bar{P}$ and then put $\omega = \tilde{\varphi} + \varepsilon$. Obviously, $\omega \geqq \bar{\varphi} + \varepsilon$. Let us mention that if $\tilde{\varphi} = +\infty$, then there are two possibilities: 1. The problem P has a solution. In this case, according to Theorem 2.4, it is also $\bar{\varphi} = +\infty$ and the algorithm B cannot be applied. 2. The problem P has no solution. In this case, according to the proof of Theorem 3.1, it must be $G^k = \emptyset$ for all k and therefore, for the same reason as in a), we can take $\omega = +\infty$ and $\mu^k = +\infty$ for an arbitrary feasible vector y^k of the problem M^k.

Altogether, we cannot guarantee the algorithm B to give results in anyone of the following cases:

(1) $U = \emptyset$,

(2) $U \neq \emptyset$, the condition (b) of Assumption 3.1 holds, and $\tilde{\varphi} = +\infty$,

(3) Assumption 3.1 is not satisfied.

However, if the problem P satisfying (1) or (2) is known to have a solution, then it has an unbouded one (see Theorems 1.1 and 2.4).

Remark 3.1. If the inequality (3.1) in the algorithm B is replaced by the equality $\lambda^k = \mu^k$, then Theorem 3.1 will remain valid but with the vector $y^{\bar{k}}$ being now an optimal vector of the problem P. The proof works well if we take $\omega > \bar{\varphi}$ and make use of Theorem 2.2.

4. PROBLEM $P(\theta)$

In this section we proceed to the subject proper of our study — the parametrization of the right-hand sides in the MILP problem. We formulate the problem $P(\theta)$:

$$\bar{\varphi}(\theta) = \max_{x,y} \{cx + dy \mid Ax + By = b + \theta h,\ x \geqq 0,\ y \in Y\}$$

where h is an m-dimensional vector and θ is a real parameter ($\theta \in E^1$). The other symbols have the same meaning as in the problem P. All concepts, terms and symbols introduced in the previous sections for the problem P apply to the problem $P(\theta)$ with the only change that their dependence upon the variable θ is formally expressed. Namely, we shall use symbols: $Z(\theta)$, $q(y, \theta)$, $\varphi(y, \theta)$, $X(y, \theta)$, $L(y, \theta)$, $\tilde{L}(y, \theta)$, $\tilde{L}_0(y, \theta)$, $\lambda(y, \theta)$, $\tilde{\lambda}(y, \theta)$, $\tilde{\lambda}_0(y, \theta)$. So, for instance, it is

$$(4.1)\qquad \begin{aligned} q(y, \theta) &= b + \theta h - By\,, \\ \lambda(y, \theta) &= \max_x \{cx \mid Ax = q(y, \theta),\ x \geqq 0\}\,, \\ \varphi(y, \theta) &= dy + \lambda(y, \theta)\,. \end{aligned}$$

Now, we introduce some concepts and assertions known from the parametric linear programming (see e.g. [4]).

Lemma 4.1. *Let $U \neq \emptyset$. Then for a given $y \in Y$ and θ_0 the problem $\tilde{L}(y, \theta)$ either has an unbounded solution for all $\theta \in (\theta_0, +\infty)$ or there exists an optimal solution $\bar{u}$ of the problem $\tilde{L}(y, \theta_0)$ and a number $\theta_1 > \theta_0$ such that $\bar{u}$ is an optimal solution of the problem $\tilde{L}(y, \theta)$ for all $\theta \in \langle\theta_0, \theta_1\rangle$, i.e. $\tilde{\lambda}(y, \theta) = \bar{u}\, q(y, \theta)$ for all $\theta \in \langle\theta_0, \theta_1\rangle$. An analogous assertion is true for the interval $(-\infty, \theta_0)$.*

Definition 4.1. *We say that the function $f(\theta)$ has a corner at a point θ_0 if it is continuous in θ_0 and linear in some intervals $(\theta_0 - \tau, \theta_0\rangle$ and $\langle\theta_0, \theta_0 + \tau)$ with $\tau > 0$, but is not linear in $(\theta_0 - \tau, \theta_0 + \tau)$.*

Lemma 4.2. *Let $U \neq \emptyset$. Then for a given $y \in Y$ there exist numbers $-\infty \leqq \theta_-(y) \leqq \theta^-(y) \leqq +\infty$ such that*

$$\tilde{\lambda}(y, \theta) = -\infty \quad \text{if} \quad \theta \notin D(y),$$
$$\tilde{\lambda}(y, \theta) > -\infty \quad \text{if} \quad \theta \in D(y)$$

for the interval $D(y) = \langle \theta_-(y), \theta^-(y) \rangle \cap (-\infty, +\infty)$. In case the interval $D(y)$ contains more than one point, the function $\tilde{\lambda}(y, \theta)$ is continuous, concave, and piecewise linear with a finite number of corners in $D(y)$.

Definition 4.2. *The points $\theta_-(y)$, $\theta^-(y)$ from Lemma 4.2 and the points θ at which $\tilde{\lambda}(y, \theta)$ has corners are called critical points of the function $\tilde{\lambda}(y, \theta)$, or of the problem $\tilde{L}(y, \theta)$.*

5. ELEMENTARY PARAMETRIC PROBLEM T_ε

The main purpose of this work is to study ε-suboptimal solutions of the problem $P(\theta)$, if there exist any, relative to the parameter θ. Let us start with

Assumption 5.1. *There are given numbers $\theta_0 < \theta_1$ and a vector $y_0 \in Y$ such that the optimal value $\tilde{\lambda}(y_0, \theta)$ of the problem $\tilde{L}(y_0, \theta)$ is a linear function for $\theta \in \in \langle \theta_0, \theta_1 \rangle$.*

The possibility of satisfying this assumption is guaranteed by Lemma 4.1. First we shall deal with a problem which we call an *elementary parametric problem* and denote by $T_\varepsilon(\theta_0, \theta_1, y_0)$ or briefly T_ε. It consists in the following: Let Assumption 5.1 hold. Defining for a moment the function

$$\hat{\varphi}(y_0, \theta) = \begin{cases} \varphi(y_0, \theta) + \varepsilon & \text{if there exists } y \in Y \text{ for which} \\ & \varphi(y, \theta) \geqq \varphi(y_0, \theta) + \varepsilon \\ \varphi(y_0, \theta) & \text{otherwise}, \end{cases}$$

we can define the set

$$\Theta_\varepsilon = \{\theta \mid \hat{\varphi}(y_0, \theta) = \varphi(y_0, \theta) + \varepsilon,\ \theta \in \langle \theta_0, \theta_1 \rangle\}.$$

Now we want to know the number

$$\bar{\theta}_\varepsilon = \begin{cases} \inf \Theta_\varepsilon & \text{if} \quad \Theta_\varepsilon \neq \emptyset \\ \theta_1 & \text{if} \quad \Theta_\varepsilon = \emptyset \end{cases}$$

which we call the solution of the problem $T_\varepsilon(\theta_0, \theta_1, y_0)$.

With respect to Definition 2.2 we can formulate the above said still in other words: The solution of the problem $T_\varepsilon(\theta_0, \theta_1, y_0)$ defines an interval $\langle \theta_0, \bar{\theta}_\varepsilon \rangle$ of the maximal length such that $\langle\theta_0, \bar{\theta}_\varepsilon\rangle \subseteq \langle\theta_0, \theta_1\rangle$ and y_0 is an ε-suboptimal vector of the problem $P(\theta)$ for all $\theta \in \langle\theta_0, \bar{\theta}_\varepsilon)$, if $\bar{\theta}_\varepsilon > \theta_0$.

The problem T_ε is obviously not much suitable for numerical solution. That is why we are going to define another problem, more suitable for numerical treatment, which we shall further show to be closely related to the original one. This auxiliary problem, denoted by $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$ – briefly $\tilde{T}_\varepsilon$, is the following:

$$\tilde{\theta}_\varepsilon = \min_{y,\theta} \{\theta \mid (y, \theta) \in \tilde{\Phi}_\varepsilon\}$$

where

$$\tilde{\Phi}_\varepsilon = \{(y, \theta) \mid dy + u^{(i)} q(y, \theta) \geqq \varphi(y_0, \theta) + \varepsilon \ (i \in G),$$
$$v^{(j)} q(y, \theta) \geqq 0 \ (j \in H), \quad w^{(j)} q(y, \theta) = 0 \ (j \in K), \quad y \in Y, \quad \theta \in \langle\theta_0, \theta_1)\} .$$

If Assumption 5.1 is satisfied, then the problem $\tilde{T}_\varepsilon$ is a MILP problem in the sense that it is defined by means of a finite number of linear equalities and inequalities over the set $Y \times E^1$. Indeed, the function $q(y, \theta)$ is linear according to (4.1) and the function $\varphi(y_0, \theta)$ is linear for $\theta \in \langle\theta_0, \theta_1)$ in virtue of Assumption 5.1, since in this case it is $-\infty < \tilde{\lambda}(y_0, \theta) < +\infty$ so that $U \neq \emptyset$ and $\varphi(y_0, \theta) = dy_0 + \tilde{\lambda}(y_0, \theta)$ for $\theta \in \langle\theta_0, \theta_1)$. Therefore we can use for the problem $\tilde{T}_\varepsilon$ the terminology introduced in Sections 1 and 4. If the problem $\tilde{T}_\varepsilon$ is regular, then it has either no solution or an optimal one. In case $\tilde{T}_\varepsilon$ has no solution, we extend the definition of the symbol $\tilde{\theta}_\varepsilon$ putting

$$\tilde{\theta}_\varepsilon = \theta_1 \quad \text{if} \quad \tilde{\Phi}_\varepsilon = \emptyset . \tag{5.1}$$

In order to guarantee the regularity of the problem $\tilde{T}_\varepsilon$, we formulate

Assumption 5.2. *One of the following conditions holds:*

(a) *The set Y is bounded.*

(b) *The set Y is of the form* (2.11). *The elements of the matrices A, B and the components of the vectors c, d and h are rational numbers.*

Lemma 5.1. *If Assumptions* 5.1 *and* 5.2 *hold, then the problem $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$ is regular.*

Now we present the main result of this section.

Theorem 5.1. *If Assumptions* 5.1 *and* 5.2 *hold, then*

$$\bar{\theta}_\varepsilon = \tilde{\theta}_\varepsilon .$$

Proof. First we show that

$$\bar{\theta}_\varepsilon \leqq \tilde{\theta}_\varepsilon . \tag{5.2}$$

If $\tilde{\Phi}_\varepsilon = \emptyset$, then according to (5.1) it is $\tilde{\theta}_\varepsilon = \theta_1$ so that (5.2) is obviously true. Let $\tilde{\Phi}_\varepsilon \neq \emptyset$. By virtue of Lemma 5.1 the problem $\tilde{T}_\varepsilon$ is regular so that it has an optimal solution $(\tilde{y}, \tilde{\theta}_\varepsilon) \in \tilde{\Phi}_\varepsilon$. Hence $\tilde{y} \in Y$, $\tilde{\theta}_\varepsilon \in \langle \theta_0, \theta_1)$ and

$$d\tilde{y} + u^{(i)}\, q(\tilde{y}, \tilde{\theta}_\varepsilon) \geqq \varphi(y_0, \tilde{\theta}_\varepsilon) + \varepsilon \ (i \in G)\,,$$
$$v^{(j)}\, q(\tilde{y}, \tilde{\theta}_\varepsilon) \geqq 0 \ (j \in H)\,, \quad w^{(j)}\, q(\tilde{y}, \tilde{\theta}_\varepsilon) = 0 \ (j \in K)\,.$$

At least one of these relations is always present; it is because Assumption 5.1 implies $U \neq \emptyset$ and thus $G \neq \emptyset$. By multiplying the relations by real coefficients and then summing them we can obtain

$$\Big(\sum_{i\in G} \alpha_i\Big)\, d\tilde{y} + \Big[\sum_{i\in G} \alpha_i u^{(i)} + \sum_{j\in H} \beta_j v^{(j)} + \sum_{j\in K} \gamma_j w^{(j)}\Big]\, q(\tilde{y}, \tilde{\theta}_\varepsilon) \geqq$$
$$\geqq \Big(\sum_{i\in G} \alpha_i\Big)\, [\varphi(y_0, \tilde{\theta}_\varepsilon) + \varepsilon]$$

for any $\alpha_i \geqq 0$ ($\sum \alpha_i = 1$, $i \in G$), $\beta_j \geqq 0$ ($j \in H$) and $\gamma_j \in E^1$ ($j \in K$), which yields, according to (1.11) and Lemma 1.2, the inequality

$$d\tilde{y} + u\, q(\tilde{y}, \tilde{\theta}_\varepsilon) \geqq \varphi(y_0, \tilde{\theta}_\varepsilon) + \varepsilon$$

for all $u \in U$. Then with respect to Lemma 1.1 we can write

$$\varphi(\tilde{y}, \tilde{\theta}_\varepsilon) = d\tilde{y} + \lambda(\tilde{y}, \tilde{\theta}_\varepsilon) = d\tilde{y} + \tilde{\lambda}(\tilde{y}, \tilde{\theta}_\varepsilon) =$$
$$= d\tilde{y} + \min_{u\in U} u\, q(\tilde{y}, \tilde{\theta}_\varepsilon) \geqq \varphi(y_0, \tilde{\theta}_\varepsilon) + \varepsilon\,,$$

which means $\tilde{\theta}_\varepsilon \in \Theta_\varepsilon$; thus (5.2) is true.

Now we shall demonstrate that the inequality in (5.2) is not fulfilled. Let us admit the contrary, i.e. $\bar{\theta}_\varepsilon < \tilde{\theta}_\varepsilon$. Then there exists a point $\hat{\theta} \in \Theta_\varepsilon \cap \langle \bar{\theta}_\varepsilon, \tilde{\theta}_\varepsilon)$ and a vector $\hat{y} \in Y$ such that

$$\varphi(\hat{y}, \hat{\theta}) \geqq \varphi(y_0, \hat{\theta}) + \varepsilon\,.$$

But this means that for all $u \in U$

$$(5.3) \qquad d\hat{y} + u\, q(\hat{y}, \hat{\theta}) \geqq \varphi(y_0, \hat{\theta}) + \varepsilon\,.$$

Representing again the set U by means of (1.11) and Lemma 1.2, we can write the inequality (5.3) equivalently as

$$d\hat{y} + \sum_{i\in G} \alpha_i u^{(i)}\, q(\hat{y}, \hat{\theta}) + \sum_{j\in H} \beta_j v^{(j)}\, q(\hat{y}, \hat{\theta}) + \sum_{j\in K} \gamma_j w^{(j)}\, q(\hat{y}, \hat{\theta}) \geqq \varphi(y_0, \hat{\theta}) + \varepsilon$$

for all $\alpha_i \geqq 0$ ($\sum \alpha_i = 1$, $i \in G$), $\beta_j \geqq 0$ ($j \in H$), $\gamma_j \in E^1$ ($j \in K$). Since $\hat{\theta} \in \langle \theta_0, \theta_1)$, it is $\varphi(y_0, \hat{\theta}) > -\infty$, so the preceding inequality implies

$$(5.4) \qquad v^{(j)}\, q(\hat{y}, \hat{\theta}) \geqq 0 \quad (j \in H)\,,$$

$$(5.5) \qquad w^{(j)}\, q(\hat{y}, \hat{\theta}) = 0 \quad (j \in K)\,.$$

Finally, taking into consideration the relations (5.3) for $u = u^{(i)}$ $(i \in G)$, (5.4) and (5.5), we come to the inclusion $(\hat{y}, \hat{\theta}) \in \tilde{\Phi}_\varepsilon$ which contradicts the optimality of $\tilde{\theta}_\varepsilon$, because of $\hat{\theta} < \tilde{\theta}_\varepsilon$. Therefore $\bar{\theta}_\varepsilon \geqq \tilde{\theta}_\varepsilon$ and this, together with (5.2), proves our assertion. Q.E.D.

The significance of Theorem 5.1 lies in the fact that it enables us – under certain assumptions – to obtain a solution of the elementary parametric problem T_ε by means of an auxiliary MILP problem $\tilde{T}_\varepsilon$. This is, of course, still a theory because of a possibly large number of constraints in the problem $\tilde{T}_\varepsilon$. However, in the next section we shall describe an algorithm by means of which the solution of the problem $\tilde{T}_\varepsilon$ will be realized as an iteration process, during which suitably chosen constraints are added step by step to approximate the feasible region $\tilde{\Phi}_\varepsilon$ – analogously as in the algorithm B. Before this we present an assertion which represents a suitable necessary and sufficient condition making it possible to obtain a non-trivial optimal value $\tilde{\theta}_\varepsilon > \theta_0$.

Theorem 5.2. *Let Assumptions* 5.1 *and* 5.2 *hold. Then* $\tilde{\theta}_\varepsilon > \theta_0$ *if and only if* y_0 *is an* ε*-suboptimal vector of the problem* $P(\theta_0)$.

6. ALGORITHM PB

In this section we describe, under the name PB, an algorithm for the solution of the problem $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$. The purpose of this algorithm was mentioned earlier – so that we can immediately proceed to the formulation of the algorithm.

Step (O). Let G_0 and H_0 be arbitrary subsets of the sets G and H respectively, e.g. $G_0 = H_0 = \emptyset$. Put $G^1 = G_0$, $H^1 = H_0$, $\theta^0 = \theta_0$, $k = 1$, and go to the step (A^1).

Step (A^k). Given G^k, H^k, and θ^{k-1}. Solve an auxiliary problem $\tilde{T}^k_\varepsilon$ as follows:

$$\min_{y,\theta} \{\theta \mid dy + u^{(i)} q(y, \theta) \geqq \varphi(y_0, \theta) + \varepsilon \ (i \in G^k),$$
$$v^{(j)} q(y, \theta) \geqq 0 \ (j \in H^k), \quad w^{(j)} q(y, \theta) = 0 \ (j \in K), \quad y \in Y, \quad \theta \in \langle \theta^{k-1}, \theta_1)\} .$$

If the problem $\tilde{T}^k_\varepsilon$ has no solution, then put $\theta^k = \theta_1$ and go to (E^k). If the problem $\tilde{T}^k_\varepsilon$ has a solution, then find an optimal one (y^k, θ^k) and go to (B^k).

Step (B^k). Solve the problem $\tilde{L}_0(y^k, \theta^k)$ and denote by $\lambda^k = \tilde{\lambda}_0(y^k, \theta^k)$ its optimal value. Go to (C^k).

Step (C^k). If $\lambda^k = -\infty$ along the direction $v^{(j)}$, put

$$G^{k+1} = G^k , \quad H^{k+1} = H^k \cup \{j\} \tag{6.1}$$

and go to (A^{k+1}). If $\lambda^k > -\infty$, go to (D^k).

Step (D^k). If

$$dy^k + \lambda^k \geqq \varphi(y_0, \theta^k) + \varepsilon , \tag{6.2}$$

then go to (E^k). Otherwise let $u^{(i)}$ be an optimal basic solution of the problem $\tilde{L}_0(y^k, \theta^k)$. Put

$$G^{k+1} = G^k \cup \{i\}, \quad H^{k+1} = H^k \tag{6.3}$$

and go to (A^{k+1}).

Step (E^k). End.

We shall call Step (O) the 0-*th iteration* and Steps $(A^k), \ldots, (E^k)$ the *k-th iteration* $(k \geqq 1)$.

The following assertions guarantee the realizability of the algorithm PB and give an interpretation of the results.

Lemma 6.1. *If Assumptions* 5.1 *and* 5.2 *hold, then each problem* $\tilde{T}_\varepsilon^k$ $(k \geqq 1)$ *is regular.*

Theorem 6.1. *If Assumptions* 5.1 *and* 5.2 *hold, then*

$$\theta_0 \leqq \theta^1 \leqq \ldots \leqq \theta^k \leqq \theta_1 \quad (k \geqq 1) \tag{6.4}$$

and for some $k = \bar{k}$ *one of the following two possibilities occurs:*

1. *The problem* $\tilde{T}_\varepsilon^k$ *has no solution. Then also* $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$ *has none.*

2. *The problem* $\tilde{T}_\varepsilon^k$ *has an optimal solution* (y^k, θ^k) *and the inequality* (6.2) *is satisfied. Then the vector* (y^k, θ^k) *is an optimal solution of the problem* $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$.

Proof. Assumption 5.1 implies $U \neq \emptyset$. The regularity of the problems $\tilde{T}_\varepsilon^k$ for each $k \geqq 1$ is guaranteed by Lemma 6.1. We shall prove our theorem by mathematical induction. The symbol $\mathscr{I}^k$ $(k \geqq 1)$ will stand for the inductive assumption, namely: The inclusions

$$G^k \subseteq G, \quad H^k \subseteq H \tag{6.5}$$

hold and, if the problem $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$ has an optimal solution $(\tilde{y}_\varepsilon, \tilde{\theta}_\varepsilon)$, it is

$$\theta_0 \leqq \theta^{k-1} \leqq \tilde{\theta}_\varepsilon . \tag{6.6}$$

Now, if in the k-th iteration the algorithm does not end, we shall prove the assumption $\mathscr{I}^{k+1}$ to be valid. (6.5), (6.1) and (6.3) imply

$$G^{k+1} \subseteq G, \quad H^{k+1} \subseteq H . \tag{6.7}$$

If $(\tilde{y}_\varepsilon, \tilde{\theta}_\varepsilon)$ is an optimal solution of the problem $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$, then it follows from (6.5) and (6.6) that the vector $(\tilde{y}_\varepsilon, \tilde{\theta}_\varepsilon)$ is a feasible solution of the problem $\tilde{T}_\varepsilon^k$ and thus

$$\theta^k \leqq \tilde{\theta}_\varepsilon . \tag{6.8}$$

On the other hand, from the problem $\tilde{T}_\varepsilon^k$ we have

$$\theta^{k-1} \leqq \theta^k, \tag{6.9}$$

so that the relations (6.9), (6.6) and (6.8) result in

$$\theta_0 \leqq \theta^k \leqq \tilde{\theta}_\varepsilon. \tag{6.10}$$

The validity of the assumption $\mathscr{I}^k$ for $k = 1$ is a direct consequence of the choices made in Step (O). Thus, we conclude that the assumption $\mathscr{I}^k$ is valid for every $k \geqq 1$.

The inequalities (6.4) follow from the relations (6.6) and (6.9) because of $\tilde{\theta}_\varepsilon \leqq \theta_1$.

Now we are going to investigate individually the possibilities in Theorem 6.1:

1. If for some k the problem $\tilde{T}_\varepsilon^k$ has no solution, then also $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$ has none. Indeed, if it were not so, then with regard to $\mathscr{I}^k$ the vector $(\tilde{y}_\varepsilon, \tilde{\theta}_\varepsilon)$ would be a feasible solution of the problem $\tilde{T}_\varepsilon^k$, but this has been excluded.

2. Since $\tilde{\lambda}_0(y^k, \theta^k) = \lambda^k > -\infty$, we have on the basis of Lemmas 1.5, 1.4 and 1.3

$$\begin{aligned} u^{(i)}\, q(y^k, \theta^k) &\geqq \lambda^k \quad (i \in G), \\ v^{(j)}\, q(y^k, \theta^k) &\geqq 0 \quad (j \in H), \\ w^{(j)}\, q(y^k, \theta^k) &= 0 \quad (j \in K). \end{aligned} \tag{6.11}$$

Using (6.2), we derive from (6.11) the relations

$$dy^k + u^{(i)}\, q(y^k, \theta^k) \geqq dy^k + \lambda^k \geqq \varphi(y_0, \theta^k) + \varepsilon \quad (i \in G).$$

Finally, because of

$$\theta^k \in \langle \theta^{k-1}, \theta_1 \rangle \subseteq \langle \theta_0, \theta_1 \rangle,$$

the vector (y^k, θ^k) is a feasible solution of the problem $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$. Thus the problem $\tilde{T}_\varepsilon(\theta_0, \theta_1, y_0)$, being feasible, has an optimal solution $(\tilde{y}_\varepsilon, \tilde{\theta}_\varepsilon)$. Then $\tilde{\theta}_\varepsilon \leqq \theta^k$, which yields with regard to (6.10) the equality $\theta^k = \tilde{\theta}_\varepsilon$.

Now it remains to prove the finiteness of the iteration process. Since it is $U_0 \neq \emptyset$ as a consequence of Assumption 5.1, one of the following possibilities occurs for each $k \geqq 1$ unless the process goes to (E^k):

a) $\lambda^k = -\infty$, $v^{(j)}\, q(y^k, \theta^k) < 0$. Then necessarily $j \notin H^k$, so that according to (6.1) we have

$$G^{k+1} = G^k, \quad H^{k+1} \supset H^k. \tag{6.12}$$ [5]

b) $dy^k + \lambda^k = dy^k + u^{(i)}\, q(y^k, \theta^k) < \varphi(y_0, \theta^k) + \varepsilon$. Then $i \notin G^k$, so that according to (6.3) we have

$$G^{k+1} \supset G^k, \quad H^{k+1} = H^k. \tag{6.13}$$

[5]) The symbol $\supset$ stands for the sharp inclusion.

From (6.12), (6.13), (6.7), and from the finiteness of the sets G and H it follows that eventually

$$G^{\bar{k}} = G\,, \quad H^{\bar{k}} = H \tag{6.14}$$

for some $k = \bar{k}$. If the problem $\tilde{T}_\varepsilon^{\bar{k}}$ has a solution, then it is $\lambda^{\bar{k}} > -\infty$, since otherwise it would be $H^{\bar{k}+1} \supset H^{\bar{k}}$ according to (6.12), which contradicts (6.14). In virtue of Lemma 1.5 it is $\lambda^{\bar{k}} = u^{(i_*)}\, q(y^{\bar{k}}, \theta^{\bar{k}})$ for some $i_* \in G$. But according to (6.14) it is also $i_* \in G^{\bar{k}}$ so that the problem $\tilde{T}_\varepsilon^{\bar{k}}$ yields the relations

$$dy^{\bar{k}} + \lambda^{\bar{k}} = dy^{\bar{k}} + u^{(i_*)}\, q(y^{\bar{k}}, \theta^{\bar{k}}) \geqq \varphi(y_0, \theta^{\bar{k}}) + \varepsilon\,.$$

But this means that the iteration process goes to $(\mathrm{E}^{\bar{k}})$ in Step $(\mathrm{D}^{\bar{k}})$. Q.E.D.

Let us point out that each iteration of the algorithm PB defines a value θ^k. In case the problem $\tilde{T}_\varepsilon$ has no solution, it is because we put $\theta^k = \theta_1$ in Step (A^k). Therefore on the basis of Theorem 6.1 and (5.1) it is always

$$\theta^{\bar{k}} = \tilde{\theta}_\varepsilon\,. \tag{6.15}$$

We can easily obtain two consequences of Theorem 6.1:

Corollary 6.1. *If Assumptions* 5.1 *and* 5.2 *hold and if it is* $\theta^k > \theta_0$ *in the k-th iteration of the algorithm* PB, *then* y_0 *is an* ε*-suboptimal vector of the problem* $P(\theta)$ *for* $\theta \in \langle \theta_0, \theta^k)$.

Corollary 6.2. *If Assumptions* 5.1 *and* 5.2 *hold, then it is* $\theta^k > \theta_0$ *for some* $k \geqq 1$ *if and only if* y_0 *is an* ε*-suboptimal vector of the problem* $P(\theta_0)$.

In the iteration process of the algorithm PB two kinds of iterations can be distinguished:

1. stationary – for which $\theta^{k-1} = \theta^k$ $(k \geqq 1)$,
2. transitive – for which $\theta^{k-1} < \theta^k$ $(k \geqq 1)$.

If y_0 is an ε-suboptimal vector of the problem $P(\theta_0)$, then it follows from Corollary 6.2 that the iteration process of the algorithm PB will actually have transitive iterations. On the other hand, if the vector y_0 is not ε-suboptimal, then all the iterations will be stationary with $\theta^k = \theta_0$.

Apparently, with regard to the purpose of the algorithm PB, transitive iterations are more desirable than the others. Therefore the following sufficient condition may be useful.

Theorem 6.2. *Let Assumptions* 5.1 *and* 5.2 *hold and let* $\theta^{k-1} < \theta_1$ *in the* $(k-1)$*-st iteration of the algorithm* PB. *Then the k-th iteration is transitive, i.e.* $\theta^{k-1} < \theta^k$, *if for the problem* $P(\theta^{k-1})$ *the sufficient condition of Theorem* 2.1 *is satisfied with the vector* $\bar{y} = y_0$ *and with the sets* $G^* = G^k$, $H^* = H^k$.

Proof. Let the symbol $M(G^k, H^k, \theta^{k-1})$ stand for the auxiliary problem $M(G^k, H^k)$ corresponding to the problem $P(\theta^{k-1})$. Similarly we denote the dependence on the parameter θ in other symbols of Section 2.

If we admitted $\theta^k = \theta^{k-1}$, then according to (2.6) we should have

$$\bar{\psi}_S(G^k, H^k, \theta^k) < \varphi(y_0, \theta^k) + \varepsilon . \tag{6.16}$$

In virtue of $\theta^k < \theta_1$ the vector (y^k, θ^k) is to be considered an optimal solution for the problem $\tilde{T}^k_\varepsilon$, so that it holds

$$\begin{gathered} dy^k + u^{(i)} q(y^k, \theta^k) \geqq \varphi(y_0, \theta^k) + \varepsilon \ (i \in G^k), \\ v^{(j)} q(y^k, \theta^k) \geqq 0 \ (j \in H^k), \quad w^{(j)} q(y^k, \theta^k) = 0 \ (j \in K). \end{gathered} \tag{6.17}$$

Putting $\mu^k = \min_{i \in G^k} u^{(i)} q(y^k, \theta^k)$ we have $(y^k, \mu^k) \in \Omega(G^k, H^k, \theta^k)$ and, according to (6.17) and (6.16),

$$dy^k + \mu^k \geqq \varphi(y_0, \theta^k) + \varepsilon > \bar{\psi}_S(G^k, H^k, \theta^k),$$

which is a contradiction. Thus with respect to the inequalities (6.4) it is $\theta^k > \theta^{k-1}$. Q.E.D.

We can make use of Theorem 6.2 when taking an ε-suboptimal vector y_0 of the problem $P(\theta_0)$ and choosing suitable initial sets G_0, H_0 to guarantee the 1-st iteration of the algorithm PB to be transitive. Using e.g. the algorithm B to obtain the vector y_0, we have $y_0 = y^{\bar{k}}$ and this vector satisfies the sufficient condition of Theorem 2.1 with the sets $G^* = G^{\bar{k}}$, $H^* = H^{\bar{k}}$. So we can take $G_0 = G^{\bar{k}}$, $H_0 = H^{\bar{k}}$.

7. SPECIAL PARAMETRIC PROBLEM ΠT_ε

In this section we shall extend the treatment of the problem $P(\theta)$ to a given finite interval $\langle \theta_0, \theta^*)$ under Assumption 5.2 and

Assumption 7.1. *Let $\varepsilon > 0$. For each $\hat{\theta} \in \langle \theta_0, \theta^*)$ there exists an ε-suboptimal vector $\hat{y}$ of the problem $P(\hat{\theta})$ and a number $\bar{\theta} \in (\hat{\theta}, \theta^*)$ such that $\varphi(\hat{y}, \theta) > -\infty$ for all $\theta \in \langle \hat{\theta}, \bar{\theta} \rangle$.*

Our aim now is, under the assumptions just introduced, to solve approximately in the sense of the ε-suboptimality the problem $P(\theta)$ for $\theta \in \langle \theta_0, \theta^*)$. This problem is called here a *special parametric problem* $\Pi T_\varepsilon(\theta_0, \theta^*)$ or briefly ΠT_ε. The notation hints at a method that we shall propose later for the solution of the problem, namely: The problem will be decomposed into a sequence of elementary parametric problems T_ε which will be solved by means of the problems $\tilde{T}_\varepsilon$ in such a way that the output of one problem is the input for the next one. But this is the subject of the next section; here we first derive an auxiliary assertion:

Lemma 7.1. *Let $U \neq \emptyset$, let Assumption 5.2 hold, and let for some $\hat{\theta}$ a sequence of numbers $\{\theta^\nu\}_{\nu=0}^{\infty}$ and a sequence of vectors $\{y^\nu\}_{\nu=0}^{\infty}$ satisfy*

$$\theta^\nu < \hat{\theta}\,, \quad \theta^\nu \to \hat{\theta}\,, \quad y^\nu \in Y\,, \quad \varphi(y^\nu, \theta^\nu) > -\infty\,.$$

Then there exists a number $\nu_0 \geqq 0$ such that $\varphi(y^\nu, \hat{\theta}) > -\infty$ for all $\nu \geqq \nu_0$.

Proof. Under the assumption $U \neq \emptyset$ we have $\varphi(y, \theta) = dy + \lambda(y, \theta) = dy + \tilde{\lambda}(y, \theta) < +\infty$ for every y and θ. Let us admit that the assertion to be proved does not hold, i.e. for infinitely many ν's it is $\varphi(y^\nu, \hat{\theta}) = -\infty$. We can assume, without loss of generality, that this occurs for all $\nu \geqq 0$. It cannot be $H = K = \emptyset$, since then the polyhedral set U would be bounded, which implies $\tilde{\lambda}(y, \theta) > -\infty$ and thus $\varphi(y, \theta) > -\infty$ for each y, θ.

Let $K = \emptyset$, $H \neq \emptyset$. It follows from Lemma 1.3 that for each $\nu \geqq 0$ there exists $j_\nu \in H$ such that the following inequalities hold:

$$\begin{aligned} v^{(j)}\, q(y^\nu, \theta^\nu) &\geqq 0 \quad (j \in H)\,, \\ v^{(j_\nu)}\, q(y^\nu, \hat{\theta}) &< 0 \quad (j_\nu \in H)\,. \end{aligned} \tag{7.1}$$

The functions $v^{(j)}\, q(y, \theta)$ are linear. Let $\bar{\theta}^\nu$ stand for the root of the equation

$$v^{(j_\nu)}\, q(y^\nu, \theta) = 0\,. \tag{7.2}$$

It is

$$\bar{\theta}^\nu \in \langle \theta^\nu, \hat{\theta})\,, \tag{7.3}$$

so that

$$\bar{\theta}^\nu \to \hat{\theta} \; (\nu \to \infty)\,. \tag{7.4}$$

Since the set H is finite, there exist infinitely many identical members of the sequence $\{j_\nu\}_{\nu=0}^{\infty}$. Let, without loss of generality, be $j_\nu = j_0$ $(\nu = 0, 1, \ldots)$. We reformulate the equation (7.2) as

$$\alpha\theta + \beta y^\nu + \gamma = 0$$

where

$$\alpha = v^{(j_0)} h\,, \quad \beta = -v^{(j_0)} B\,, \quad \gamma = v^{(j_0)} b\,.$$

It is $\alpha \neq 0$ due to (7.1). Then we have for $\nu \geqq 0$

$$\bar{\theta}^{\nu+1} - \bar{\theta}^\nu = -\frac{1}{\alpha}\,\beta(y^{\nu+1} - y^\nu)\,.$$

According to Assumption 5.2 and Lemma 3.1 the vector β is known to have rational components. Thus we can write

$$\bar{\theta}^{\nu+1} - \bar{\theta}^\nu = -\frac{1}{\alpha'}\,\beta'(y^{\nu+1} - y^\nu)$$

where β' has integer components, so that the implication

$$|\bar{\theta}^{\nu+1} - \bar{\theta}^{\nu}| > 0 \Rightarrow |\bar{\theta}^{\nu+1} - \bar{\theta}^{\nu}| \geqq \frac{1}{|\alpha'|} > 0$$

holds which proves, with respect to the convergence (7.4), that the equality $\bar{\theta}^{\nu} = \hat{\theta}$ is valid for all sufficiently large ν. However, this contradicts (7.3).

In case $K \neq \emptyset$, $H = \emptyset$ we proceed analogously starting from the relations

$$w^{(j)}\, q(y^{\nu}, \theta^{\nu}) = 0 \quad (j \in K),$$
$$w^{(j_{\nu})}\, q(y^{\nu}, \hat{\theta}) \neq 0 \quad (j_{\nu} \in K)$$

which give a solvable linear equation

$$w^{(j_{\nu})}\, q(y^{\nu}, \theta) = 0 \tag{7.5}$$

with the root $\bar{\theta}^{\nu} = \theta^{\nu}$. Similarly as above we arrive at a contradiction.

If it is $K \neq \emptyset$, $H \neq \emptyset$, then we have for infinitely many ν's either the equation (7.2) or (7.5), which in either case leads again to contradiction. Q.E.D.

The following assertion can be proved quite similarly:

Lemma 7.2. *Let $U \neq \emptyset$, let Assumption 5.2 hold, and let for some $\hat{\theta}$ sequences of numbers $\{\theta_1^{\nu}\}_{\nu=0}^{\infty}$, $\{\theta_2^{\nu}\}_{\nu=0}^{\infty}$ and a sequence of vectors $\{y^{\nu}\}_{\nu=0}^{\infty}$ satisfy*

$$\theta_1^{\nu} < \theta_2^{\nu} < \hat{\theta}\,, \quad \theta_1^{\nu} \to \hat{\theta}\,, \quad y^{\nu} \in Y\,, \quad \varphi(y^{\nu}, \theta_2^{\nu}) > -\infty\,.$$

Then there exists a number $\nu_0 \geqq 0$ such that for all $\nu \geqq \nu_0$ it holds $\varphi(y^{\nu}, \theta_1^{\nu}) > -\infty$.

If the problem $P(\theta)$ is regular in the interval $\langle\theta_0, \theta^*)$, then it can be guaranteed that Assumption 7.1 is satisfied for an arbitrary $\varepsilon > 0$ by introducing the following assumption regarding the structure of the function $\bar{\varphi}$:

Assumption 7.2. *The function $\bar{\varphi}(\theta)$ is continuous from the right in the interval $\langle\theta_0, \theta^*)$.*

Theorem 7.1. *Let Assumption 5.2 hold. Then Assumption 7.1 holds for an arbitrary $\varepsilon > 0$ if and only if Assumption 7.2 holds.*

Let us remark that the assumption of continuity from the right is a restricting one for the function $\bar{\varphi}$. In general, this function is not continuous from the right (see [25]). Obviously, in a given case it is difficult to make sure of the continuity, so that the theorem has rather a theoretical value. Nevertheless, it gives an idea about the character of the problems $P(\theta)$ considered in this paper.

In this section we propose an algorithm for solving the problem $\Pi T_\varepsilon(\theta_0, \theta^*)$ defined in Section 7. The algorithm constructs a sequence of problems $P(\theta)$ and T_ε which can be solved by the algorithms B and PB, respectively (hence the symbol ΠPB).

When necessary, we shall express the dependence of the problem $P(\theta)$ upon the set Y by the symbol $P_Y(\theta)$. Thus we have $P_Y(\theta) \equiv P(\theta)$. To guarantee the finiteness of the algorithm, it is necessary to add

Assumption 8.1. *For every $\hat{\theta} \in \langle \theta_0, \theta^*)$, among all the ε-suboptimal vectors y of the problem $P(\hat{\theta})$ there exists at most a finite number of those for which $\varphi(y, \theta) = -\infty$ for $\theta > \hat{\theta}$.*

Now let us proceed to the formulation of the algorithm.

Step (O). Put $p = 0$, $S_p = \emptyset$. Go to (A_p).

Step (A_p). Solve the problem $P_{Y-S_p}(\theta_p)$. Let $\bar{y}_p$ be one of its ε-suboptimal vectors. Put $y_p = \bar{y}_p$ and go to (B_p).

Step (B_p). A point $\theta'_p = \min \{\theta^1_p, \theta^*\}$ is required where θ^1_p is the minimal critical point of the parametric problem $\tilde{L}(y_p, \theta)$ in the interval $(\theta_p, +\infty)$. If no such point θ'_p exists, then put

$$\theta_{p+1} = \theta_p\,, \quad S_{p+1} = S_p \cup \{y_p\}$$

and go to (A_{p+1}). If it exists, then put $S_{p+1} = \emptyset$ and go to (C_p).

Step (C_p). Solve the problem $\tilde{T}_\varepsilon(\theta_p, \theta'_p, y_p)$. If it has no solution, then put

$$\theta_{p+1} = \theta'_p\,, \quad y_{p+1} = y_p$$

and: 1. If $\theta'_p = \theta^*$, go to (E_p). 2. If $\theta'_p < \theta^*$, go to (B_{p+1}).

If the problem $\tilde{T}_\varepsilon(\theta_p, \theta'_p, y_p)$ has an optimal solution $(\tilde{y}_p, \tilde{\theta}_p)$, then put

$$\theta_{p+1} = \tilde{\theta}_p\,, \quad y_{p+1} = \tilde{y}_p$$

and go to (B_{p+1}).

Step (E_p). Put $r = p + 1$ and stop.

To interpret the results of the algorithm we define indices:

$$p_0 = 0\,,$$
$$p_{i+1} = \min_{\varkappa} \{p_i + \varkappa \mid \theta_{p_i} < \theta_{p_i+\varkappa},\ \varkappa \geqq 1\} \quad \text{for} \quad i \geqq 0 \quad \text{if} \quad p_i < r\,,$$
$$p_s = r\,.$$

Theorem 8.1. *Let $\varepsilon > 0$ and let Assumptions 5.2, 7.1 and 8.1 hold. Then the realization of the algorithm ΠPB yields a finite increasing sequence of points $\{\theta_{p_i}\}_{i=0}^s$ and a sequence of vectors $\{y_{p_i}\}_{i=0}^s$ with the following properties:*

1. $\theta_{p_0} = \theta_0$, $\theta_{p_s} = \theta^*$;

2. *y_{p_i} is an ε-suboptimal vector of the problem $P(\theta)$ for all $\theta \in \langle\theta_{p_{i-1}}, \theta_{p_i})$ $(i = 1, \ldots, s)$;*
3. *the function $\varphi(y_{p_i}, \theta)$ is linear in the interval $\langle\theta_{p_{i-1}}, \theta_{p_i})$ $(i = 1, \ldots, s)$.*

Proof. From the existence of ε-suboptimal vectors guaranteed by Assumption 7.1 it follows $U \neq \emptyset$. Let us have $y_p \in Y$, $\theta_p \in \langle\theta_0, \theta^*)$ for $p \geqq 0$. Note that Step (B_p) applies for each p – it succeeds either (A_p) or (C_{p-1}). Therefore we may begin our analysis from Step (B_p). Since the nonequivalence $S_{p+1} \neq S_p$ can take place only in Step (B_p), let us suppose $S_p = \emptyset$. There are two possibilities:

(a) It is possible to find the point θ'_p. Then Assumption 5.1 is fulfilled with regard to the vector y_p and the points θ_p, θ'_p instead of y_0, θ_0, θ_1 introduced in the original formulation of this assumption. (See Definition 4.2 and Lemma 4.2.) This along with Assumption 5.2 guarantees regularity of the problem $\tilde{T}_\varepsilon(\theta_p, \theta'_p, y_p)$ according to Lemma 5.1. The following possibilities may occur in the next Step (C_p):

(aa) The vector y_p is an ε-suboptimal vector of the problem $P(\theta_p)$. For the purpose of this proof we shall extend the definition of the symbol $\tilde{\theta}_p$ prescribing $\tilde{\theta}_p = \theta'_p$ when the problem $\tilde{T}_\varepsilon(\theta_p, \theta'_p, y_p)$ has no solution. Then $\theta_p < \tilde{\theta}_p$ according to Theorem 5.2. This means according to Theorem 5.1 that y_p is an ε-suboptimal vector of the problem $P(\theta)$ for all $\theta \in \langle\theta_p, \tilde{\theta}_p)$. But the assignments in Step (C_p) – with regard to the extended definition of $\tilde{\theta}_p$ given above – imply $\theta_{p+1} = \tilde{\theta}_p$. Thus y_p is an ε-suboptimal vector of the problem $P(\theta)$ for all $\theta \in \langle\theta_p, \theta_{p+1})$.

Since $\theta_{p+1} \leqq \theta'_p$, then in virtue of Assumption 5.1 the function $\varphi(y_p, \theta)$ is linear in the interval $\langle\theta_p, \theta_{p+1})$.

So we have proved parts 2) and 3) of our assertion for $p = p_{i-1}$ and $p_i = p + 1$ $(i \geqq 1)$.

If $\theta'_p = \theta^*$ and if the problem $\tilde{T}_\varepsilon(\theta_p, \theta'_p, y_p)$ has no solution, then and only then it is $\theta_{p+1} = \theta'_p$ and the iteration process ends in Step (E_p). If it was $i = s$, then $r = p + 1 = p_{s-1} + 1 = p_s$ and $\theta_{p_s} = \theta'_p = \theta^*$. This proves part 1) of the assertion.

(ab) The vector y_p is not an ε-suboptimal vector of the problem $P(\theta_p)$. Then according to Theorem 5.2 the problem $\tilde{T}_\varepsilon(\theta_p, \theta'_p, y_p)$ has an optimal solution with $\tilde{\theta}_p = \theta_p$. Step (C_p) is followed by (B_{p+1}). If the point θ'_{p+1} cannot be obtained in this step, then we have the situation which will be discussed in the next section (b). If the point θ'_{p+1} can be obtained, then the process proceeds to (C_{p+1}) in the same way as in (a). Here a sequence of steps

$$(C_p), (B_{p+1}), \ldots, (C_{p+\varkappa-1}), (B_{p+\varkappa}) \quad (\varkappa \geqq 1) \tag{8.1}$$

can arise with $\theta_p = \theta_{p+1} = \ldots = \theta_{p+\varkappa}$. Since the vectors $(y_{p+\varkappa}, \theta_p)$ $(\varkappa \geqq 1)$ are optimal solutions of the problems $\tilde{T}_\varepsilon(\theta_p, \theta'_{p+\varkappa-1}, y_{p+\varkappa-1})$, we have on the basis of the definition conditions of these problems and Lemmas 1.3, 1.4, 1.5 the inequality

$$\varphi(y_{p+\varkappa}, \theta_p) \geqq \varphi(y_{p+\varkappa-1}, \theta_p) + \varepsilon\,.$$

This implies that the sequence (8.1) is finite, since otherwise it would be $\bar{\varphi}(\theta_p) = +\infty$, which contradicts Assumption 7.1.

(b) It is not possible to find the point θ'_p. From Assumption 7.1 it follows that in this case either y_p is not an ε-suboptimal vector of the problem $P(\theta_p)$ or there exists an alternative ε-suboptimal vector y which yields the required point θ'_p. Generally, a sequence of steps

$$(8.2) \qquad (\mathrm{B}_p), (\mathrm{A}_{p+1}), \ldots, (\mathrm{B}_{p+\varkappa-1}), (\mathrm{A}_{p+\varkappa})$$

is carried out, in which $\theta_p = \theta_{p+1} = \ldots = \theta_{p+\varkappa}$ ($\varkappa \geqq 1$) and $S_{p+\varkappa} = \{y_p, \ldots, y_{p+\varkappa-1}\}$, i.e. ε-suboptimal vectors of the problems $P_{Y-S_{p+\varkappa}}(\theta_p)$ are successively obtained forming a set of mutually different ε-suboptimal vectors of the problem $P_Y(\theta_p)$. Finiteness of this sequence is guaranteed by Assumption 8.1. Thus for some $\varkappa = \varkappa_0$ there exists $\theta'_{p+\varkappa}$. Further we proceed according to the section (a) with (aa) taking place.

By this analysis we have proved our assertion for the intervals $\langle \theta_{p_{i-1}}, \theta_{p_i})$ where $p_{i-1} = p$, $p_i = p + \varkappa_0$ ($i \geqq 1$), and $\varkappa_0 = 1$ in the case (aa), $\varkappa_0 > 1$ otherwise. It remains to prove finiteness of the iteration process.

The sequence $\{\theta_p\}$ is nondecreasing and bounded. Equality can take place at most for a finite number of its members – see (8.1) and (8.2). Thus infiniteness of the sequence implies the convergence

$$(8.3) \qquad \theta_p \to \hat{\theta} \leqq \theta^*, \quad \theta_p < \hat{\theta} \ (p = 0, 1, \ldots).$$

Let us admit that this is the case. According to Lemma 7.1 – considering pairs of points $\theta_p, \hat{\theta}$ and vectors y_p – there exists a number p_0 such that for $p \geqq p_0$ it is $\varphi(y_p, \hat{\theta}) > -\infty$. Moreover, on the basis of Lemma 4.2 it is $\varphi(y_p, \theta) > -\infty$ for all $\theta \in \langle \theta_p, \hat{\theta} \rangle$. Thus for $p \geqq p_0$ the point θ'_p can always be found in Step (B_p). Step (A_{p+1}) does not take place.

It follows from (8.3) that for infinitely many p one of the following possibilities for the problem $\tilde{T}_\varepsilon(\theta_{p-1}, \theta'_{p-1}, y_{p-1})$ occurs in Step (C_{p-1}):

(α) $\tilde{T}_\varepsilon$ has no solution, so that $\theta_p = \theta'_{p-1}$, $y_p = y_{p-1}$,

(β) $\tilde{T}_\varepsilon$ has an optimal solution for which

$$\theta_{p-1} < \tilde{\theta}_{p-1} = \theta_p < \theta'_{p-1}.$$

But (α) cannot be the case for all sufficiently large p, say $p \geqq p_1 \geqq p_0$. Should it be true, then all the points $\theta_p = \theta'_{p-1}$ ($p \geqq p_1$) would be critical for the function $\tilde{\lambda}(y_{p_1}, \theta)$, which contradicts the assertion on a finite number of critical points (see Definition 4.2, Lemma 4.2). Thus for infinitely many p the problem $\tilde{T}_\varepsilon$ must have an optimal solution in Step (C_{p-1}). If $\tilde{\theta}_{p-1} > \theta_{p-1}$, we have the case (β) which will be examined below. Therefore we may assume that only the cases (α) or $\tilde{\theta}_{p-1} = \theta_{p-1}$ occur for all $p \geqq p_2 \geqq p_0$. In the latter case, as was derived in (ab), it has to be

$$\varphi(y_p, \theta_p) \geqq \varphi(y_{p-1}, \theta_p) + \varepsilon.$$

But for sufficiently large $p \geqq p_2$ the strict inequality is impossible. Indeed, if it were the case (suppose $\theta_{p-2} < \theta_{p-1}$), then Lemma 7.2 and the continuity of the function φ would imply the inequality $\varphi(y_p, \theta) \geqq \varphi(y_{p-1}, \theta) + \varepsilon$ for some $\theta < \theta_p$, which contradicts the ε-suboptimality of the vector $y_{p-1} = y_{p-2}$ for $\theta \in \langle\theta_{p-2}, \theta_{p-1})$. So, unless the case (β) takes place for infinitely many p, we have for infinitely many p the equality

$$\varphi(y_p, \theta_p) = \varphi(y_{p-1}, \theta_p) + \varepsilon\,. \tag{8.4}$$

Now we shall examine the case (β). Let us suppose that it occurs for infinitely many p. On the basis of Lemmas 7.2 and 4.2 there exists a number p_3 such that $\varphi(y_p, \theta) > -\infty$ for all $\theta \in \langle\theta_{p-1}, \theta_p\rangle$ and $p \geqq p_3$. Let us have such p's. Since y_{p-1} is an ε-suboptimal vector of the problem $P(\theta)$ for $\theta \in \langle\theta_{p-1}, \theta_p)$, we have

$$\varphi(y_p, \theta) < \varphi(y_{p-1}, \theta) + \varepsilon \quad \text{for} \quad \theta \in \langle\theta_{p-1}, \theta_p)\,. \tag{8.5}$$

According to Lemmas 4.1, 1.3, 1.4 and 1.5 there exist numbers $\tau'_p > 0$, $\tau''_p > 0$ and $i'_p \in G$, $i''_p \in G$ such that

$$\varphi(y_p, \theta) = dy_p + u^{(i'_p)}\, q(y_p, \theta) \qquad \text{for} \quad \theta \in \langle\theta_p - \tau'_p, \theta_p\rangle\,, \tag{8.6}$$

$$\varphi(y_{p-1}, \theta) = dy_{p-1} + u^{(i''_p)}\, q(y_{p-1}, \theta) \quad \text{for} \quad \theta \in \langle\theta_p - \tau''_p, \theta_p\rangle\,. \tag{8.7}$$

Considering (y_p, θ_p) to be an optimal solution of the problem $\tilde{T}_\varepsilon(\theta_{p-1}, \theta'_{p-1}, y_{p-1})$, we obtain by the relation (8.6) the inequality

$$\varphi(y_p, \theta_p) \geqq \varphi(y_{p-1}, \theta_p) + \varepsilon\,. \tag{8.8}$$

Finally — in virtue of the continuity of the function $\varphi(y_p, \theta)$ on a closed region — it follows from (8.5) and (8.8) again that the equality (8.4) is satisfied for infinitely many p.

Let us return to the expressions (8.6) and (8.7), which obviously apply to the case (α), too. Let us consider all p satisfying the equality (8.4). Since the set G is finite, then among them there are infinitely many such p that it holds, say, $i'_p = i_1$, $i''_p = i_2$. Let it be so for all p_r $(r = 1, 2, \ldots)$. Then the equality (8.4) can be rewritten as

$$\alpha_1\theta_{p_r} + \beta_1 y_{p_r} + \gamma_1 = \alpha_2\theta_{p_r} + \beta_2 y_{p_r-1} + \gamma_2 + \varepsilon \tag{8.9}$$

where

$$\left.\begin{aligned} \alpha_\varkappa &= u^{(i_\varkappa)}h \\ \beta_\varkappa &= d - u^{(i_\varkappa)}B \\ \gamma_\varkappa &= u^{(i_\varkappa)}b \end{aligned}\right\} \; (\varkappa = 1, 2)\,.$$

It is $\alpha_1 \neq \alpha_2$, since otherwise (8.9), (8.6) and (8.7) would imply $\varphi(y_{p_r}, \theta) = \varphi(y_{p_r-1}, \theta) + \varepsilon$ for some $\theta < \theta_{p_r}$, which is in contradiction to the ε-suboptimality of the vector y_{p_r-1} for the problem $P(\theta)$, $\theta \in \langle \theta_{p_r-1}, \theta_{p_r} \rangle$. Thus for $\alpha = \alpha_1 - \alpha_2$ it is $\alpha \neq 0$ and from (8.9)

$$\theta_{p_r} = \frac{1}{\alpha} (\beta_2 y_{p_r-1} - \beta_1 y_{p_r} + \gamma_2 - \gamma_1 + \varepsilon) .$$

For the difference $\Delta_r = \theta_{p_{r+1}} - \theta_{p_r}$ we obtain after an arrangement of the fraction

$$\Delta_r = \frac{1}{\alpha'} [\beta'_2(y_{p_{r+1}-1} - y_{p_r-1}) + \beta'_1(y_{p_r} - y_{p_{r+1}})] \tag{8.10}$$

where β'_2, β'_1 are integer-component vectors. Due to (8.10) we have the implication

$$|\Delta_r| > 0 \Rightarrow |\Delta_r| \geqq \frac{1}{|\alpha'|} > 0 \quad (r \geqq 1) ,$$

which contradicts the convergence (8.3). This contradiction proves the finiteness of the iteration process. Q.E.D.

We add three remarks concerning the algorithm.

Remark 8.1. In Step (C_p) the algorithm ΠPB can be slightly modified as follows: If the problem $\tilde{T}_\varepsilon$ has an optimal solution with $\tilde{\theta}_p = \theta_p$, then put $\theta_{p+1} = \theta_p$ and go to (A_{p+1}). The purpose of this modification is to avoid the possibility of getting a long cycle of steps (8.1) – especially when ε is small. The cost of this – loss of theoretical finiteness – is not so important from the practical point of view.

Remark 8.2. We can strengthen Assumption 7.1 as follows: For each $\hat{\theta} \in \langle \theta_0, \theta^*)$ there exists an *optimal* vector $\hat{y}$ of the problem $P(\hat{\theta})$ and a number $\bar{\theta} \in (\hat{\theta}, \theta^*)$ such that $\varphi(\hat{y}, \theta) > -\infty$ for all $\theta \in \langle \hat{\theta}, \bar{\theta} \rangle$. Then in Step (A_p) of the algorithm ΠPB we can find the optimal vector of the problem $P(\hat{\theta})$ and Assumption 8.1 can be replaced by a requirement that each problem $P(\theta)$ should have a finite number of optimal vectors y for $\theta \in \langle \theta_0, \theta^*)$.

Remark 8.3. If the problems $P(\theta_p)$ are solved by the algorithm B and the problems $\tilde{T}_\varepsilon(\theta_p, \theta'_p, y_p)$ by the algorithm PB – and this may be naturally assumed – then it is of advantage to use currently the sets G^k, H^k resulting from one solution as initial sets G_0, H_0 for the next one. Then: 1. We have nontrivial initial sets G_0, H_0. 2. Always after the algorithm B has been applied the 1-st iteration of the algorithm PB is guaranteed to be transitive in virtue of Theorem 6.2.

Generally, it may be quite difficult to verify Assumptions 7.1 and 8.1. But these assumptions were only needed in the proof of Theorem 8.1 to make sure that the problem $P_{Y-S_p}(\theta_p)$ has an ε-suboptimal vector in Step (A_p) or that a point θ'_p can be obtained after a finite number of Steps (B_p). Below we present two modifications of the algorithm ΠPB in which finiteness of the iteration process is ensured by suitably defined — with regard to practical purposes given later in Section 9 — stopping criteria.

Modification ΠPB1. Given an integer number $\omega \geqq 1$. If in Step (A_p) the problem $P_{Y-S_p}(\theta_p)$ has no ε-suboptimal solution, then put $\theta_{p+1} = \theta_p$ and go to (E_p). If after ω executions of Step (B_p) the point θ'_p is not obtained, then go to (E_p).

If $\theta_r > \theta_0$, then for the interval $\langle\theta_0, \theta_r)$ the assertions 2) and 3) of Theorem 8.1 obviously remain valid independently of Assumptions 7.1 and 8.1. If Assumptions 5.2 and 8.1 hold, if ω is a sufficiently large number, and if $\theta_r < \theta^*$, then the function $\bar{\varphi}$ is discontinuous from the right at the point θ_r. This follows from Theorem 7.1.

Modification ΠPB2. If in Step (A_p) the problem $P_{Y-S_p}(\theta_p)$ has no ε-suboptimal solution, then put $\theta_{p+1} = \theta_p$ and go to (E_p). If $\theta_{p+1} = \theta_p$ or $\theta_{p+1} < \theta'_p$ for some $p \geqq 0$, then go to (E_p).

Clearly, $p > 0$ may hold only when Steps (B_p) and (C_p) have been repeatedly executed with the point θ'_p existing in (B_p) and with the problem $\tilde{T}_\varepsilon$ having no solution in (C_p). If $\theta_r > \theta_0$, then for the interval $\langle\theta_0, \theta_r)$ the assertions 2) and 3) of Theorem 8.1 remain valid, in particular: The vector y_0 is an ε-suboptimal vector of the problem $P(\theta)$ for $\theta \in \langle\theta_0, \theta_r)$. The interval $\langle\theta_0, \theta_r)$ is one of maximal length with this property. Let us remark that in this modification of the algorithm ΠPB the problem $P(\theta)$ has to be solved just once — at the point θ_0.

9. APPLICATIONS OF THE ALGORITHM ΠPB

1. Special parametric problem. This problem was defined in Section 7 under Assumption 7.1. Its solution by the algorithm ΠPB further requires that Assumptions 5.2 and 8.1 be fulfilled (see Theorem 8.1). Assumption 5.2 is practically always satisfied. One class of problems $P(\theta)$ for which Assumptions 7.1 and 8.1 hold in the interval $\langle\theta_0, \theta^*)$ for an arbitrary $\varepsilon > 0$ is that with the properties:

(i) $\theta_0 \leqq \theta \leqq \theta' < \theta^* \Rightarrow Z(\theta) \subseteq Z(\theta')$, [6]

(ii) $Z(\theta_0) \neq \emptyset$,

(iii) $\sup \{cx + dy \mid (x, y) \in Z(\theta)\} < +\infty$ for all $\theta \in \langle\theta_0, \theta^*)$.

Indeed, in virtue of the property (ii) in connection with the properties (i) and (iii) there exists an ε-suboptimal vector of the problem $P(\hat{\theta})$ for each $\hat{\theta} \in \langle\theta_0, \theta^*)$. The property (i) further guarantees that the implication

$$\lambda(y, \hat{\theta}) > -\infty \Rightarrow \lambda(y, \theta) > -\infty \quad \text{for all} \quad \theta \in \langle\hat{\theta}, +\infty)$$

[6]) Remember that $Z(\theta)$ is a feasible region of the problem $P(\theta)$.

and thus also

$$\varphi(y, \hat{\theta}) > -\infty \Rightarrow \varphi(y, \theta) > -\infty \quad \text{for all} \quad \theta \in \langle \hat{\theta}, \theta^*)$$

is valid.

The condition (i) is always fulfilled, e.g., for problems of the type

$$\max_{x,y} \{cx + dy \mid A^1x + B^1y \leqq b^1 + \theta h^1,\ A^2x + B^2y = b^2,\ x \geqq 0,\ y \in Y\}$$

where $h^1 \geqq 0$. These problems can be transformed easily into the form of $P(\theta)$. The condition (ii) can be verified directly by solving the problem $P(\theta_0)$. Validity of the condition (iii) when the set Y is bounded is ensured by Theorem 2.5; if Y is not bounded and if the assumption (b) from Theorem 2.3 holds, then the validity of the condition (iii) can be tested by solving the problem $\bar{P}(\theta)$ and making use of Theorem 2.4.

2. Parametric exploration. By a parametric problem we mean in general a calculation of the optimal values $\bar{\varphi}(\theta)$ and optimal solutions of the problem $P(\theta)$ if they exist, for θ from a given interval $\langle \theta_0, \theta^*)$. In this work we have focused our attention exclusively on an approximate solution of this problem in the sense of ε-suboptimality. A special case of this problem – under Assumptions 7.1 and 8.1 – has been solved above.

If Assumptions 7.1 and 8.1 are not necessarily satisfied, then we can repeatedly use the algorithm ΠPB1 to get an approximate solution of the parametric problem in separate subintervals of $\langle \theta_0, \theta^*)$. These subintervals result from applying the algorithm ΠPB1 in the interval $\langle \theta_0, \theta^*)$ successively with the initial points

$$\theta_0^{(1)} = \theta_0,\ \theta_0^{(2)} = \theta_r^{(1)} + \eta, \ldots,\ \theta_0^{(t)} = \theta_r^{(t-1)} + \eta,\ \ldots$$

where $\theta_r^{(t-1)}$ is the terminal point of the preceding application of the algorithm while η is a given positive number. Obviously, for some $t = t^*$ it will be $\theta_r^{(t^*)} \geqq \theta^*$.

We call this way of using the algorithm ΠPB1 a parametric exploration of the problem $P(\theta)$ in the interval $\langle \theta_0, \theta^*)$. The number of applications of the algorithm depends – in usual circumstances – on the number of discontinuities of the function $\bar{\varphi}$ from the right and on the value chosen for η. It may be useful to work heuristically. For instance, if it appears in the course of calculations that the function $\bar{\varphi}$ differs not much from a function continuous from the left, then it will be better to proceed from the right to the left, i.e. to transform

$$\theta' = \theta_0 + \theta^* - \theta$$

and solve the problem $P(\theta_0 + \theta^* - \theta')$ for $\theta' \in \langle \theta_0, \theta^*)$.

3. Sensitivity analysis along the direction h.[7]) The problem to be solved is as follows: An ε-suboptimal solution $(\bar{x}_0, \bar{y}_0)$ of the problem $P(\theta_0)$ is given. Find an interval $\langle\theta_0, \theta_0^+) \subseteq \langle\theta_0, \theta^*)$ of maximal length such that $\bar{y}_0$ is an ε-suboptimal vector of the problem $P(\theta)$ for all $\theta \in \langle\theta_0, \theta_0^+)$. If such an interval does not exist, then put $\theta_0^+ = \theta_0$.

As was shown in Section 8, we can solve this problem by the algorithm ΠPB2 when using in Step (A_0) the given ε-suboptimal vector, i.e. $\bar{y} = \bar{y}_0$, and putting $\theta_0^+ = \theta_r$.

10. NUMERICAL REALIZATION OF THE ALGORITHM ΠPB

The main goal that we have followed in this work is how to solve the parametric MILP problem by those means by which the simple optimization is solved. This goal has been attained in two directions: 1. Both the algorithms B and PB are identical in their logical structure; these algorithms are the principle operational components of the algorithm ΠPB. 2. The auxiliary extreme problems of the algorithms B and PB can be solved by the same numerical procedures.

The logical identity of both the algorithms is apparent from their descriptions in Sections 3 and 6. Concerning the second point, it becomes evident for the problems M^k and $\tilde{T}_\varepsilon^k$ from the following elementary transformations (simplified with $K = \emptyset$):

The problem M^k of the algorithm B can be expressed as

$$\psi^k = \max_{y,\mu} \{dy + \mu \mid \alpha_i y + \beta_i \mu \geqq \gamma_i \ (i \in I^k),\ dy + \mu \leqq \psi^{k-1},\ y \in Y,\ \mu \in E^1\}$$

where the introduction of an upper bound of the objective function value is justified by the fact that the feasible region of the problem M^k is included in the feasible region M^{k-1}. Using the transformation $dy + \mu = -\theta$ we get

$$\psi^k = \max_{y,\theta} \{-\theta \mid (\alpha_i - d)\, y - \beta_i \theta \geqq \gamma_i \ (i \in I^k),\ -\psi^{k-1} \leqq \theta,\ y \in Y,\ \theta \in E^1\} =$$
$$= -\min_{y,\theta} \{\theta \mid \alpha_i' y + \beta_i' \theta \geqq \gamma_i' \ (i \in I^k),\ -\psi^{k-1} \leqq \theta,\ y \in Y,\ \theta \in E^1\}\,.$$

The problem $\tilde{T}_\varepsilon^k$ of the algorithm PB can be written analogously as

$$\theta^k = \min_{y,\theta} \{\theta \mid \alpha_i'' y + \beta_i'' \theta \geqq \gamma_i'' \ (i \in J^k),\ \theta^{k-1} \leqq \theta < \theta_1,\ y \in Y,\ \theta \in E^1\}\,.$$

[7]) The term corresponds to that in LP where the sensitivity analysis is done along the coordinate directions.

Here we shall not deal with a numerical solution of auxiliary problems of this type. Methods with a reoptimization facility, like those of Gomory, would be ideal for this purpose. Otherwise, any branch-and-bound method, after a minor modification, can be applied as well.

Concerning the auxiliary problems $\tilde{L}_0(y^k)$ and $\tilde{L}_0(y^k, \theta^k)$, these are identical for both the algorithms B and PB. They can be solved by standard methods of LP. It is of interest from the numerical point of view to mention a problem P with the constraint conditions of the form $Ax + By \leqq b$ (everything will apply to the problem $P(\theta)$ as well). If we convert the inequalities into equalities using a vector of the slack variables $s = b - Ax - By \geqq 0$, then the feasible region of the problem $\tilde{L}(y)$ is

$$U = \{u \mid uA \geqq c,\ u \geqq 0\}\,.$$

If in this case the problem $\tilde{L}(y)$ has a large number of rows – which is typical – then it might be of advantage to solve it by a dual procedure proposed by Benders in [12]. The procedure consists in approximating the feasible region U by a bounded set

$$U_\varrho = \{u \mid uA \geqq c,\ u \geqq 0,\ \sum_{i=1}^{m} u_i \leqq \varrho\}$$

where ϱ is a sufficiently large positive number, and the corresponding problem $\tilde{L}_\varrho(y)$ is then solved dually.

Now let us illustrate the function of the algorithm ПРВ1 on a problem $P(\theta)$ derived from a problem in [14] – page 141:

$$\begin{aligned}
&-2x_1 - 6x_2 - 2y_1 - 3y_2 \to \max \\
&\quad x_1 - 2x_2 - 3y_1 + \ y_2 = -6 - \theta \\
&-x_1 + 3x_2 - 2y_1 - 2y_2 = -4 + 2\theta \\
&\quad x_1 \geqq 0\,,\quad x_2 \geqq 0\,, \\
&\quad Y = \{(y_1, y_2) \mid 0 \leqq y_1 \leqq 2\,,\quad 0 \leqq y_2 \leqq 1 \quad \text{integer}\}\,.
\end{aligned}$$

Our task is to find ε-suboptimal vectors of this problem for $\theta_0 = 0$, $\theta^* = 20$ and $\varepsilon =$ $= 0{,}01$. It is

$$\begin{aligned}
q(y, \theta) &= (-6 - \theta + 3y_1 - y_2,\ -4 + 2\theta + 2y_1 + 2y_2)\,, \\
U &= \{u \mid u_1 - u_2 \geqq -2,\ -2u_1 + 3u_2 \geqq -6\}\,.
\end{aligned}$$

The polyhedral set U has one vertex $u^{(1)} = (-12, -10)$ and two directions of unbounded edges $v^{(1)} = (1,1)$, $v^{(2)} = (3,2)$. It is $U = U_0$. The vector $(2,0)$ is an optimal vector of the problem $P(0)$.

Application of the algorithm ΠPB1 *for* $p = 0$:

$$y_0 = (2{,}0)\,, \quad q(y_0, \theta) = (-\theta, 2\theta)\,,$$

$$\tilde{\lambda}(y_0, \theta) = \min_u \{(-u_1 + 2u_2)\,\theta \mid u \in U\} = -8\theta \quad \text{for all} \quad \theta \geqq 0\,,$$

$$\varphi(y_0, \theta) = -4 - 8\theta\,, \quad \theta_0' = 20\,.$$

Solving the problem $\tilde{T}_\varepsilon(\theta_0, \theta_0', y_0)$ by the algorithm PB:

(O) $G^1 = H^1 = \emptyset,\ \theta^0 = 0$

(A^1) problem $\tilde{T}_\varepsilon^1$: $\min_{y \in Y, \theta} \{\theta \mid 0 \leqq \theta < 20\}$

$y^1 = (0{,}0),\ \theta^1 = 0$

(B^1) problem $\tilde{L}_0(y^1, \theta^1)$: $\min_u \{-6u_1 - 4u_2 \mid u \in U\} = -\infty$ (direction $v^{(1)}$)

(C^1) $G^2 = \emptyset,\ H^2 = \{1\}$

(A^2) $\min_{y \in Y, \theta} \{\theta \mid \theta \geqq 10 - 5y_1 - y_2,\ 0 \leqq \theta < 20\}$

$y^2 = (2{,}0),\ \theta^2 = 0$

(B^2) $\min_u \{u\,.\,0 \mid u \in U\} = u^{(1)}\,.\,0 = 0$

(D^2) $dy^2 + \lambda^2 = -4 < \varphi(y_0, \theta^2) + \varepsilon = -4 + \varepsilon$

$G^3 = \{1\},\ H^3 = \{1\}$

(A^3) $\min_{y \in Y, \theta} \{\theta \mid \theta \geqq 10 - 5y_1 - y_2,\ 116 \geqq 58y_1 + 11y_2 + \varepsilon,\ 0 \leqq \theta < 20\}$

$y^3 = (1{,}1),\ \theta^3 = 4$

(B^3) $\min_u \{-8u_1 + 8u_2 \mid u \in U\} = -\infty$ (direction $v^{(2)}$)

(C^3) $G^4 = \{1\},\ H^4 = \{1{,}2\}$

(A^4) $\min_{y \in Y, \theta} \{\theta \mid \theta \geqq 10 - 5y_1 - y_2,\ 116 \geqq 58y_1 + 11y_2 + \varepsilon,$

$\theta \geqq 26 - 13y_1 - y_2,\ 4 \leqq \theta < 20\}$

$y^4 = (1{,}1),\ \theta^4 = 12$

(B^4) $\min_u \{-16u_1 + 24u_2 \mid u \in U\} = -48$

(D^4) $dy^4 + \lambda^4 = -53 \geqq \varphi(y_0, \theta^4) + \varepsilon = -100 + \varepsilon \Rightarrow$ (E^4).

Application of the algorithm ΠPB1 *for* $p = 1$:

$$\theta_1 = 12\,, \quad y_1 = (1,1)\,, \quad q(y_1, \theta) = (-4 - \theta, 2\theta)\,,$$
$$\tilde{\lambda}(y_1, \theta) = u^{(1)}\, q(y_1, \theta) = 48 - 8\theta \quad \text{for all} \quad \theta \geqq 12\,,$$
$$\varphi(y_1, \theta) = 43 - 8\theta\,, \quad \theta_1' = 20\,.$$

Solving the problem $\tilde{T}_\varepsilon(\theta_1, \theta_1', y_1)$ by the algorithm PB:

(O) $G^1 = \{1\}$, $H^1 = \{1,2\}$, $\theta^0 = 12$

(A¹) $\min_{y \in Y, \theta} \{\theta \mid \theta \geqq 10 - 5y_1 - y_2,\ 69 \geqq 58y_1 + 11y_2 + \varepsilon,$

$\theta \geqq 26 - 13y_1 - y_2,\ 12 \leqq \theta < 20\}$

$y^1 = (1,0)$, $\theta^1 = 13$

(B¹) $\min_u \{-16u_1 + 24u_2 \mid u \in U\} = -48$

(D¹) $dy^1 + \lambda^1 = -50 \geqq \varphi(y_1, \theta^1) + \varepsilon = -61 + \varepsilon \Rightarrow$ (E¹).

Application of the algorithm ΠPB1 *for* $p = 2$:

$$\theta_2 = 13\,, \quad y_2 = (1,0)\,, \quad q(y_2, \theta) = (-3 - \theta,\ -2 + 2\theta)\,,$$
$$\tilde{\lambda}(y_2, \theta) = u^{(1)}\, q(y_2, \theta) = 56 - 8\theta \quad \text{for all} \quad \theta \geqq 13\,,$$
$$\varphi(y_2, \theta) = 54 - 8\theta\,, \quad \theta_2' = 20\,.$$

Solving the problem $\tilde{T}_\varepsilon(\theta_2, \theta_2', y_2)$ by the algorithm PB:

(O) $G^1 = \{1\}$, $H^1 = \{1,2\}$

(A¹) $\min_{y \in Y, \theta} \{\theta \mid \theta \geqq 10 - 5y_1 - y_2,\ 58 \geqq 58y_1 + 11y_2 + \varepsilon,$

$\theta \geqq 26 - 13y_1 - y_2,\ 13 \leqq \theta < 20\}$.

The problem $\tilde{T}_\varepsilon^1$ has no solution $\Rightarrow$ (E¹), $\theta_3 = \theta_2'$, $y_3 = y_2$.

On the basis of Theorem 8.1 and our explanations to the algorithm ΠPB1 we come to the following result:

vector y	ε-suboptimal in	$\varphi(y, \theta)$
(2,0)	⟨0,12)	$-4 - 8\theta$
(1,1)	⟨12,13)	$43 - 8\theta$
(1,0)	⟨13,20)	$54 - 8\theta$

It can be seen easily that the same result would be obtained for an arbitrary value of $0 < \varepsilon < 0{,}01$. This means, with respect to Lemma 2.2, that the vectors contained in the table are optimal in the respective intervals.

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Souhrn

BENDERSOVA METODA A PARAMETRIZACE PRAVÝCH STRAN V ÚLOZE SMÍŠENÉHO CELOČÍSELNÉHO LINEÁRNÍHO PROGRAMOVÁNÍ

JAROSLAV HROUDA

V článku se zabýváme otázkou jednorozměrné lineární parametrizace pravých stran úlohy smíšeného celočíselného lineárního programování (s.c.l.p.) se zřetelem na reálnou možnost numerického řešení. Považujeme pro to za rozhodující algoritmickou kompaktnost řešící metody v tom smyslu, aby se bodová i parametrická

optimalizace provedla pomocí týchž numerických procedur – asi tak, jako se v parametrickém lineárním programování využívá simplexové metody. Ukázalo se, že tohoto cíle lze dosáhnout na bázi Bendersovy metody, ovšem za jistých omezení a specializací, a to že:

a) Úloha se řeší pouze aproximativně ve smyslu získání suboptimální hodnoty účelové funkce v mezích zadané odchylky od optimální hodnoty. Je zřejmé, že z praktického hlediska je toto omezení nepodstatné.
b) Vyčerpávající algoritmické řešení je podáno jen pro speciální případ úlohy s.c.l.p., který lze zhruba charakterizovat jako parametrizaci pravých stran typu $\leqq b_i + \theta h_i$, kde $h_i > 0$ a θ je parametr.
c) Obecnou úlohu s.c.l.p. lze parametricky řešit pouze do okamžiku změny daného suboptimálního řešení (tzv. analýza sensitivity v daném směru) nebo do okamžiku porušení spojitosti optimální hodnoty účelové funkce zprava.

Postup, který navrhujeme pro řešení výše uvedených úloh, spočívá v opakované aplikaci algoritmu, který nazýváme algoritmus PB a který je v podstatě pouze formální obměnou Bendersova algoritmu – zde nazývaného B. Celou tuto kombinovanou proceduru souhrnně označujeme jako algoritmus ΠPB.

Algoritmy B a PB jsou založeny na duální dekomposici lineární složky úlohy s.c.l.p. V literatuře byl Bendersův algoritmus popsán pouze pro omezenou množinu celočíselných proměnných Y a omezující podmínky typu nerovností. Zde tyto předpoklady zeslabujeme. Je to umožněno využitím jedné dosud málo připomínané, avšak hluboké vlastnosti úloh s.c.l.p., kterou zde nazýváme regularitou. Regulární úlohy s.c.l.p. se vzhledem k řešitelnosti chovají analogicky jako úlohy lineárního programování se známou trichotomií možností (řešitelnost, neomezenost, neřešitelnost). Regularita je, zhruba řečeno, důsledkem omezenosti množiny Y nebo racionálnosti koeficientů úlohy. Tyto předpoklady dovolují současně dokázat finitnost algoritmů B, PB a ΠPB.

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