Aplikace matematiky

Jan Hurt

On estimation of reliability in the exponential case

Aplikace matematiky, Vol. 21 (1976), No. 4, 263–272

Persistent URL: http://dml.cz/dmlcz/103646

ON ESTIMATION OF RELIABILITY IN THE EXPONENTIAL CASE

JAN HURT

(Received December 17, 1975)

1. INTRODUCTION

Let us consider the family of exponential probability distributions with density functions

$$f(x;\theta) = \theta^{-1}\exp(-\theta^{-1}x)\,, \quad \text{for} \quad x > 0 \tag{1}$$
$$= 0 \qquad \text{otherwise,}$$

where θ is an unknown parameter. Let c be a fixed positive number. If the random variable X is distributed according to the density (1), then

$$P_\theta(X > c) = \exp(-\theta^{-1}c)\,. \tag{2}$$

In terms of reliability theory, X means the time to failure of a system, whereas the quantity (2), called often reliability, means the probability that the system will operate at time c.

In this paper, we will study four different estimates of (2) based on a random sample $X_1, \ldots, X_n$ from the parent population (1). The best unbiased estimator of (2) was derived by Pugh [4]. With the notation $T_n = n^{-1}\sum X_i$, the best unbiased estimator may be written as

$$R_1 = \left(1 - \frac{c}{nT_n}\right)^{n-1} \quad \text{if} \quad T_n > c/n\,, \tag{3}$$
$$= 0 \qquad \text{otherwise.}$$

Using the well-known fact that T_n is the maximum likelihood estimator of θ, we get the frequently used maximum likelihood estimator of (2),

$$R_2 = \exp(-cT_n^{-1})\,; \tag{4}$$

this estimate is not unbiased, in general.

Suppose now that the parameter $\lambda = \theta^{-1}$ is a random variable with the a priori distribution of the gamma-type with parameters α, p, i.e., with the density function

$$g(\lambda) = \alpha^p \lambda^{p-1} e^{-\alpha\lambda} / \Gamma(p) \quad \text{if} \quad \lambda > 0\,,$$
$$= 0 \qquad \text{otherwise}\,.$$

Then the a posteriori distribution is also of the gamma-type, with parameters $\alpha + \sum x_i$, $p + n$, i.e., with the density function

$$g(\lambda \mid x_1, \ldots, x_n) = (\alpha + \sum x_i)^{p+n} \lambda^{p+n-1} \times$$
$$\times \exp\left[-\lambda(\alpha + \sum x_i)\right] / \Gamma(p + n) \quad \text{if} \quad \lambda > 0\,,$$
$$= 0 \qquad \text{otherwise.}$$

The Bayes estimate of (2) can then be obtained as the expectation of $e^{-\lambda c}$ with respect to the a posteriori distribution,

$$R_3 = \int_0^\infty e^{-\lambda c} g(\lambda \mid x_1, \ldots, x_n)\, \mathrm{d}\lambda\,.$$

After a short calculation we get

$$R_3 = \left[1 + \frac{c}{n}\left(T_n + \frac{\alpha}{n}\right)^{-1}\right]^{-(n+p)}. \tag{5}$$

Finally, we can consider a naive estimator of the probability (2), given by the frequency of the event $\{X_i > c\}$. Defining $Z_1, \ldots, Z_n$ by

$$Z_i = 1 \quad \text{if} \quad X_i > c\,,$$
$$= 0 \quad \text{otherwise,}$$

the naive estimator of (2) can be written as

$$R_4 = n^{-1} \sum Z_i\,.$$

In the following we shall investigate asymptotic properties of all the estimators, defined above.

2. ASYMPTOTIC EXPANSIONS OF MOMENTS

We begin with a theorem which will enable us to find asymptotic expansions for expected values, variances, and mean squared errors of the estimates introduced in Section 1. This theorem is of practical interest by itself.

Theorem 1. *Let $g = g(t, n)$ be a function defined on $E_1 \times N$. Assume that for all n, g admits the continuous $(q + 1)$-st derivative, $q \geqq 1$, for $t \in [\theta - \delta, \theta + \delta]$ where $\delta > 0$ is independent of n. Suppose that g is bounded on $E_1 \times N$ and all the derivatives $g', \ldots, g^{(q+1)}$ are bounded on $[\theta - \delta, \theta + \delta]$. Let $\{T_n\}$ be a sequence of statistics with finite moments up to the order $2(q + 1)$ such that $E|T_n - \theta|^{2(q+1)} = O(n^{-(q+1)})$. Then*

$$(6) \qquad E[g(T_n, n) - g(\theta, n)] = \sum_{j=1}^{q} \frac{1}{j!} \left(\frac{\partial^j g(t, n)}{\partial t^j} \right)_{t=\theta} E(T_n - \theta)^j + O(n^{-(q+1)/2}),$$

and

$$(7) \qquad \operatorname{var}[g(T_n, n) - g(\theta, n)] = \sum_{\substack{j=1 \\ j+k \leqq q+1}}^{q} \sum_{k=1}^{q} \frac{1}{j!} \frac{1}{k!} \left(\frac{\partial^j g(t, n)}{\partial t^j} \right)_{t=\theta} \left(\frac{\partial^k g(t, n)}{\partial t^k} \right)_{t=\theta} \times$$
$$\times \operatorname{cov}[(T_n - \theta)^j, (T_n - \theta)^k] + O(n^{-(q+2)/2}).$$

Proof. The proof is given in [3].

Remark 1. The assumptions of the Theorem are not too limiting. They are fulfilled, e.g., for sample moments. This follows, for s even, from [1] p. 346 and for s odd by the use the well-known inequality $\gamma_s^{1/s} = \gamma_{s+1}^{1/(s+1)}$ where γ_s is the s-th absoute moment.

3. ASYMPTOTIC PROPERTIES OF THE ESTIMATES

First we prove the asymptotic normality of the proposed estimators. Throughout the paper we shall denote $\varkappa = \theta^{-1}c$, $\beta = \theta^{-1}\alpha$.

Theorem 2. *For $i = 1, 2, 3$*

$$\sqrt{(n)}\,(R_i - e^{-\varkappa}) \xrightarrow{L} N(0, \varkappa^2 e^{-2\varkappa})$$

holds. For R_4

$$\sqrt{(n)}\,(R_4 - e^{-\varkappa}) \xrightarrow{L} N(0, e^{-\varkappa}(1 - e^{-\varkappa}))\,.$$

Proof. For $t \geqq 0$ define

$$(8) \qquad g_1(t, n) = \left(1 - \frac{c}{tn}\right)^{n-1} \quad \text{if} \quad t > c/n\,,$$
$$= 0 \qquad \text{otherwise,}$$

$$(9) \qquad g_2(t, n) = \exp(-ct^{-1})\,,$$

$$(10) \qquad g_3(t, n) = \left[1 + \frac{c}{n}\left(t + \frac{\alpha}{n}\right)^{-1}\right]^{-(n+p)}.$$

Without loss of generality we may assume that $\theta > c/n$ so that the function $g_1(t, n)$ admits continuous derivatives in some neighbourhood of the point θ. By the symbols $g_i', g_i'', \ldots$ we shall mean the corresponding derivatives with respect to t. After differentiating of (8), (9), (10) we obtain

$$g_1'(t, n) = ct^{-2}(1 - 1/n)(1 - ct^{-1}/n)^{n-2}, \tag{11}$$

$$g_2'(t, n) = ct^{-2} \exp(-ct^{-1}), \tag{12}$$

$$g_3'(t, n) = c\left(t + \frac{\alpha}{n}\right)^{-2} \left(1 + \frac{p}{n}\right)\left[1 + \frac{c}{n}\left(t + \frac{\alpha}{n}\right)^{-1}\right]^{-(n+p+1)}. \tag{13}$$

From these formulas it follows that for fixed t

$$\lim_{n \to \infty} g_i(t, n) = ct^{-2} \exp(-ct^{-1}) \quad \text{for} \quad i = 1, 2, 3. \tag{14}$$

We have $\operatorname{var} T_n = \theta^2/n$. From the central limit theorem we deduce

$$\sqrt{(n)}\,(T_n - \theta) \xrightarrow{L} N(0, \theta^2). \tag{15}$$

Now, we utilize (6a. 2.5) in [5] and obtain

$$\sqrt{(n)}\,[g_2(T_n, n) - g_2(\theta, n)] \xrightarrow{L} N(0, \varkappa^2 e^{-2\varkappa}).$$

It is easy to show

$$(1 - ct^{-1})^{n-2} = \exp(-ct^{-1})[1 + O(n^{-1})],$$

hence

$$\sqrt{(n)}\,[g_1(t, n) - \exp(-ct^{-1})] \to 0$$

for $n \to \infty$. A similar result is true for g_3, and therefore by using (x), (b) in [5], p. 122 we get the assertion of the theorem. The case of R_4 is straightforward. Q.E.D.

Remark 3. It follows from Theorem 2 that the estimators R_2, R_3 are weakly asymptotically efficient, i.e. the variances of their asymptotic distributions are the same as the variance of the asymptotic distribution of the best unbiased estimator.

Remark 4. The estimator R_4 is not weakly asymptotically efficient because of the fact

$$\frac{\varkappa^2 e^{-2\varkappa}}{e^{-\varkappa}(1 - e^{-\varkappa})} < 1$$

if $\varkappa > 0$. The last inequality is equivalent to the inequality $e^x - (1 + x^2) > 0, x > 0$, which may be easily verified.

Theorem 3. *For the expected values, variances, and expected squared errors of the estimators R_1, R_2, R_3, we have*

$$ER_1 = e^{-\varkappa},$$

$$ER_2 = e^{-\varkappa}\left[1 + \frac{\varkappa}{2n}(\varkappa - 2)\right] + O(n^{-2}),$$

$$ER_3 = e^{-\varkappa}\left[1 + \frac{\varkappa}{n}(\beta + \varkappa - p - 1)\right] + O(n^{-2}),$$

$$\operatorname{var} R_1 = \frac{\varkappa^2}{n} e^{-2\varkappa}\left[1 + \frac{1}{2n}(\varkappa - 2)^2\right] + O(n^{-5/2}),$$

$$\operatorname{var} R_2 = \frac{\varkappa^2}{n} e^{-2\varkappa}\left\{1 + \frac{1}{2n}[3(\varkappa - 2)^2 - 4]\right\} + O(n^{-5/2}),$$

$$\operatorname{var} R_3 = \frac{\varkappa^2}{n} e^{-2\varkappa}\left\{1 + \frac{1}{2n}[5(\varkappa - 2)^2 + 4(p - 3) -\right.$$

$$\left. -4\varkappa(p - 1) + 4\beta(\varkappa - 2)]\right\} + O(n^{-5/2}),$$

$$E(R_1 - e^{-\varkappa})^2 = \operatorname{var} R_1,$$

$$E(R_2 - e^{-\varkappa})^2 = \frac{\varkappa^2}{n} e^{-2\varkappa}\left\{1 + \frac{1}{2n}\left[\frac{7}{2}(\varkappa - 2)^2 - 4\right]\right\} + O(n^{-5/2}),$$

$$E(R_3 - e^{-\varkappa})^2 = \frac{\varkappa^2}{n} e^{-2\varkappa}\left\{1 + \frac{1}{2n}[5(\varkappa - 2)^2 + 4(p - 3) -\right.$$

$$\left. - 4\varkappa(p - 1) + 4\beta(\varkappa - 2) + 2(\varkappa + \beta - p - 1)^2]\right\} + O(n^{-5/2}).$$

Proof. The asymptotic expansions will be deduced from Theorem 1 if we put $q = 3$. For this purpose we need derivatives of g_i, moments $E(T_n - \theta)^s$, and $\operatorname{cov}[(T_n - \theta)^j, (T_n - \theta)^k]$ which are listed bellow (for the first derivatives see (11), (12), (13)):

$$g_1''(t, n) = ct^{-3}(1 - 1/n)(1 - ct^{-1}/n)^{n-3}(ct^{-1} - 2)$$

$$g_1'''(t, n) = ct^{-4}(1 - 1/n)(1 - ct^{-1}/n)^{n-4}[(ct^{-1} - 2)(ct^{-1} - 3) +$$

$$+ ct^{-1}(ct^{-1}/n - 1)]$$

$$g_2''(t, n) = ct^{-3}\exp(-ct^{-1})(ct^{-1} - 2)$$

$$g_2'''(t, n) = ct^{-4} \exp(-ct^{-1}) [(ct^{-1} - 2)(ct^{-1} - 3) - ct^{-1}]$$

$$g_3''(t, n) = c(t + \alpha/n)^{-3} (1 + p/n) [1 + cn^{-1}(t + \alpha/n)^{-1}]^{-(n+p+2)} \times$$
$$\times [c(t + \alpha/n)^{-1} (1 + (p - 1)/n) - 2]$$

$$g_3'''(t, n) = c(t + \alpha/n)^{-4} (1 + p/n) [1 + cn^{-1}(t + \alpha/n)^{-1}]^{-(n+p+3)} \times$$
$$\times \{[c(t + \alpha/n)^{-1} (1 + (p - 1)/n) - 2] \times$$
$$\times [c(t + \alpha/n)^{-1} (1 + (p - 1)/n) - 3] - c(t + \alpha/n)^{-1} \times$$
$$\times (1 + (p - 1)/n) [1 + cn^{-1}(t + \alpha/n)^{-1}]\}$$

$$E(T_n - \theta)^2 = \theta^2 n^{-1}$$

$$E(T_n - \theta)^3 = 2\theta^3 n^{-2}$$

$$E(T_n - \theta)^4 = 3\theta^4 n^{-2}(1 + 2/n)$$

$$\operatorname{cov}[(T_n - \theta), (T_n - \theta)] = \theta^2 n^{-1}$$

$$\operatorname{cov}[(T_n - \theta), (T_n - \theta)^2] = 2\theta^3 n^{-2}$$

$$\operatorname{cov}[(T_n - \theta), (T_n - \theta)^3] = 3\theta^4 n^{-2}(1 + 2/n)$$

$$\operatorname{cov}[(T_n - \theta)^2, (T_n - \theta)^2] = 2\theta^4 n^{-2}(1 + 3/n).$$

Now we are in the position to utilize formula (6). For the maximum likelihood estimator R_2 we have

$$ER_2 = e^{-\varkappa} + \tfrac{1}{2}c\theta^{-3} e^{-\varkappa}(\varkappa - 2) \theta^2/n + O(n^{-2}) =$$
$$= e^{-\varkappa} \left[1 + \frac{\varkappa}{2n} (\varkappa - 2)\right] + O(n^{-2}).$$

Further,

$$[1 + cn^{-1}(\theta + \alpha/n)^{-1}]^{-n} = e^{-\varkappa} \left[1 + \frac{\varkappa}{2n} (\varkappa + \beta)\right] + O(n^{-2})$$

so that

$$ER_3 = [1 + cn^{-1}(\theta + \alpha/n)^{-1}]^{-(n+p)} + \tfrac{1}{2}c(\theta + \alpha/n)^{-3} (1 + p/n) \times$$
$$\times [1 + cn^{-1}(\theta + \alpha/n)^{-1}]^{-(n+p+2)} \times$$
$$\times [c(\theta + \alpha/n)^{-1} (1 + (p - 1)/n) - 2] \theta^2/n + O(n^{-2}) =$$
$$= e^{-\varkappa}[1 + \frac{\varkappa}{n}(\beta + \varkappa - p - 1)] + O(n^{-2}).$$

We continue by deriving the expansions for variances. For $\operatorname{var} R_1$ by using $(1 - \varkappa/n)^{2n} = e^{-2\varkappa}[1 - \varkappa^2/n + O(n^{-2})]$ we have

$$\operatorname{var} R_1 = \frac{\varkappa^2}{n} e^{-2\varkappa}(1 - 2/n)(1 - \varkappa^2/n)(1 + 4\varkappa/n) +$$

$$+ \frac{2\varkappa^2}{n^2} e^{-2\varkappa}(\varkappa - 2) + \frac{\varkappa^2}{n^2} e^{-2\varkappa}[(\varkappa - 2)(\varkappa - 3) - \varkappa] +$$

$$+ \frac{\varkappa^2}{2n^2} e^{-2\varkappa}(\varkappa - 2)^2 = \frac{\varkappa^2}{n} e^{-2\varkappa}[1 + \frac{1}{2n}(\varkappa - 2)^2] + O(n^{-5/2}).$$

From the last expression the assertion of Theorem easily follows. The calculation of $\operatorname{var} R_2$ is straightforward and therefore may be omitted. During the calculation of $\operatorname{var} R_3$ we frequently use the fact

$$[1 + cn^{-1}(t + \alpha/n)^{-1}]^{-(n+k)} = e^{-\varkappa}[1 + \varkappa n^{-1}(\beta + \varkappa/2 - k)] + O(n^{-2}).$$

We have

$$\operatorname{var} R_3 = \frac{\varkappa^2}{n} e^{-2\varkappa}(1 - 4\beta/n)(1 + 2p/n) \times$$

$$\times [1 + \varkappa n^{-1}(2\beta + \varkappa)][1 - 2(p + 1)\varkappa/n] +$$

$$+ \frac{2\varkappa^2}{n^2} e^{-2\varkappa}(\varkappa - 2) + \frac{\varkappa^2}{n^2} e^{-2\varkappa}[(\varkappa - 2)(\varkappa - 3) - \varkappa] +$$

$$+ \frac{\varkappa^2}{2n^2} e^{-2\varkappa}(\varkappa - 2)^2 + O(n^{-5/2}).$$

Now the expression given in Theorem can be directly obtained.

The basic observation required for the calculation of the expected squared errors s the identity

$$E[g(T_n, n) - w(\theta)]^2 = \operatorname{var}[g(T_n, n) - g(\theta, n)] + [Eg(T_n, n) - w(\theta)]^2. \tag{16}$$

Expansions for the expected squared errors of R_i follow immediately after substituting ER_i and $\operatorname{var} R_i$ to (16). Q. E. D.

4. DEFICIENCY

Given the asymptotic expansions for expected squared errors, one can see that all the estimators R_1, R_2, and R_3 are (strongly) asymptotically efficient. For a more detailed comparison of the above estimators we use deficiency (see [2]).

Assume that the expected squared error of the estimator A based on n observations is V_n^A, that of B is V_n^B. Then the deficiency of B with respect to A is the number d_n for which

$$V^B_{n+\mathrm{d}_n} = V^A_n .$$

Roughly speaking, d_n means the number of additional observations which must be performed to obtain the same value of the expected squared error for both estimators. Usually the asymptotic deficiency for $n \to \infty$ is considered. In loc. cit. the authors have shown that if

$$V_n^A = \frac{\gamma}{n^r} + \frac{a}{n^{r+1}} + o(n^{-(r+1)})$$

and

$$V_n^B = \frac{\gamma}{n^r} + \frac{b}{n^{r+1}} + o(n^{-(r+1)})$$

then the asymptotic deficiency of B with respect to A is

$$\mathrm{d}_{BA} = \frac{b-a}{\gamma r} . \tag{17}$$

For our purposes, let us denote the asymptotic deficiency of R_i with respect to R_j by d_{ij}, $i, j = 1, 2, 3$. Given these preliminaries, the following theorem may be proved.

Theorem 4. *The deficiencies satisfy*

$$\mathrm{d}_{21} = \tfrac{5}{4}(\varkappa - 2)^2 - 2$$

$$\mathrm{d}_{31} = 2(\varkappa - 2)^2 + 2(p - 3) - 2\varkappa(p - 1) + 2\beta(\varkappa - 2) + (\varkappa + \beta - p - 1)^2$$

$$\mathrm{d}_{32} = \tfrac{3}{4}(\varkappa - 2)^2 + 2(p - 2) - 2\varkappa(p - 1) + 2\beta(\varkappa - 2) + (\varkappa + \beta - p - 1)^2 .$$

Proof. We derive only d_{21} because the calculation of the remaining deficiencies is quite analogous. We have $r = 1$, $\gamma = \varkappa^2 e^{-2\varkappa}$, $b = \frac{1}{2}\gamma[\frac{7}{2}(\varkappa - 2)^2 - 4]$, $a = \frac{1}{2}\gamma(\varkappa - 2)^2$ so that $\mathrm{d}_{21} = \frac{5}{4}(\varkappa - 2)^2 - 2$. Q. E. D.

5. CONCLUSIONS

First we examine the bias of R_2, R_3. Theorem 3 implies that both of the estimators R_2, R_3, are biased, in general. For maximum likelihood estimator (MLE) R_2 the bias increases with increasing $\varkappa$. The only case when R_2 offers an "almost" unbiased estimate (up to the order $O(n^{-2})$) is if $\varkappa = 2$. The bias of R_3 depends substantially on the expression $\beta + \varkappa - p - 1$ which may be of any sign and of any magnitude. If the a priori choice of α, p is successful then the bias of R_3 is approximately $\varkappa n^{-1}(\varkappa - 1)$. Thus the bias increases again if $\varkappa$ increases.

From the formulas for deficiencies we can see that MLE R_2 is a little bit better than best unbiased estimator (BUE) R_1 for $\varkappa$ close to 2. This is the case when we estimate $P(X > c)$ for c comparable with the double of the expected time to failure θ, i.e. $\varkappa = c/\theta \approx 2$. For large $\varkappa$, however, BUE is superior. The comparison of the Bayes estimator R_3 and BUE R_1 is not quite simple because d_{31} dependens on three parameters, $\varkappa$, β, and p. After denoting $\Delta = \beta - p$, let us express

$$d_{31} = 3\varkappa^2 + 4\varkappa(\Delta - 2) + \Delta(\Delta - 6) - 2p + 3\,.$$

The difference $\Delta = \alpha(\theta^{-1} - p/\alpha)$ represents the difference between the actual value θ^{-1} and the mean p/α of the a priori distribution. The first insight may be that for large p the deficiency is favourable for the Bayes estimator. This is true only when p/α is chosen sufficiently close to theoretical θ^{-1}. In the opposite case for large p and α of the moderate size, $|\Delta| \approx \alpha \approx p$ so that $d_{31} \approx p^2$. Hence we must be careful if handling the a priori parameters α and p. For illustration, some d_{31}'s for different values $\varkappa$, Δ are given in Table 1. For the comparison of R_3 and R_2, similar conclusions remain true as for the comparison of R_3 and R_1.

TABLE 1

Quantities $d_{31}^* = d_{31} + 2p$

$\varkappa$	Δ						
	−5	−2	−1	0	1	2	5
0·25	51·19	15·19	7·19	1·19	−2·81	−4·81	1·19
0·50	44·75	11·75	4·75	−0·25	−3·25	−4·25	4·75
1·00	33·00	6·00	1·00	−2·00	−3·00	−2·00	13·00
2·00	14·00	−1·00	−2·00	−1·00	2·00	7·00	34·00
3·00	1·00	−2·00	1·00	6·00	13·00	22·00	61·00

From the above discussion, we can conclude that BUE is generally recommendable. Comparing BUE with MLE, the loss of 2 observations in the worst case is not substantial. Comparing BUE with the Bayes estimator, the regions of preferencies of one or another are rather complex sets in the space of parameters $\varkappa$, β, p.

References

[1] *Cramér, H.*: Mathematical methods of statistics. Princeton Univ. Press, Princeton 1946.
[2] *Hodges, jr., J. L.* and *Lehmann, E. L.*: Deficiency, Ann. Math. Statist. 41 (1970), 783—801.
[3] *Hurt, J.*: Asymptotic expansions of functions of statistics, to appear in Apl. Mat.
[4] *Pugh, E. L.*: The best estimate of reliability in the exponential case. Oper. Res. 11/17 (1963), 57—61.
[5] *Rao, C. R.*: Linear statistical inference and its applications, 2nd ed., Wiley, New York 1973.

Souhrn

O ODHADECH SPOLEHLIVOSTI V EXPONENCIÁLNÍM PŘÍPADĚ

JAN HURT

V článku jsou studovány čtyři odhady spolehlivosti v exponenciálním rozdělení, a to nejlepší nestranný, maximálně věrohodný, bayesovský a tzv. naivní. Je dokázána jejich asymptotická normalita a odvozeny asymptotické rozvoje střední hodnoty a střední čtvercové odchylky odhadů. Tři efficientní odhady (nejlepší nestranný, maximálně věrohodný a bayerovský) jsou studovány z hlediska deficience.

Author's address: RNDr. *Jan Hurt*, Matematicko-fysikální fakulta KU, Sokolovská 83, 186 00 Praha 8.