Graham Smith

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ON THE MORÁVEK AND VLACH CONDITIONS FOR THE EXISTENCE OF A SOLUTION TO THE MULTI-INDEX PROBLEM

GRAHAM SMITH

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For the multi-index problem:

maximize

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} c_{ijk} x_{ijk} ,$$

subject to:

$$\sum_{k=1}^{n} x_{ijk} = A_{ij} \quad (i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, m) ,$$

$$\sum_{j=1}^{m} x_{ijk} = B_{ik} \quad (i = 1, 2, \ldots, l;\ k = 1, 2, \ldots, n) ,$$

$$\sum_{i=1}^{l} x_{ijk} = C_{jk} \quad (j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n) ,$$

where:

$$x_{ijk} \geqq 0 \quad (i = 1, 2, \ldots, l;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n) ,$$

$$\sum_{i=1}^{l} A_{ij} = \sum_{k=1}^{n} C_{jk} \quad (j = 1, 2, \ldots, m) ,$$

$$\sum_{j=1}^{m} C_{jk} = \sum_{i=1}^{l} B_{ik} \quad (k = 1, 2, \ldots, n) ,$$

$$\sum_{k=1}^{n} B_{ik} = \sum_{j=1}^{m} A_{ij} \quad (i = 1, 2, \ldots, l)$$

necessary conditions for the existence of a solution have been given by Morávek and Vlach [2] which were restated by Haley [1] in the form:

(1) $$\sum_{i \in I} \sum_{k \in K} B_{ik} + \sum_{j \in J} \sum_{k \in K} C_{jk} - \sum_{i \in I} \sum_{j \in J} A_{ij} \geqq 0$$

where

$$\begin{aligned}
I &\subseteq \{1, 2, \ldots, l\}\,, \\
J &\subseteq \{1, 2, \ldots, m\}\,, \\
K &\subseteq \{1, 2, \ldots, n\}\,, \\
\bar{K} &\subseteq \{1, 2, \ldots, n\}\,, \\
K \cap \bar{K} &= \emptyset\,, \\
K \cup \bar{K} &= \{1, 2, \ldots, n\}.
\end{aligned}$$

It is known [1] that the conditions (1) are sufficient for the existence of a solution if at least one of l, m or n is less than or equal to 2.

The sufficiency of these conditions for other classes of problems has been questioned by Morávek and Vlach [3]. This note will demostrate that the conditions (1) are not sufficient for the existence of a solution for the smallest possible problem with $l, m, n > 2$, that is: $l = m = n = 3$.

A computationally intractable procedure for determining necessary and sufficient conditions for the existence of a solution to the multiindex problem has been given by Smith [5]. In the incomplete application of this procedure to a problem with $l = m = n = 3$, all of the conditions of the set (1) were generated, together with some conditions not belonging to (1). One such condition not belonging to (1) was:

$$(2) \qquad -A_{21} - A_{31} - A_{32} + B_{23} + B_{32} + B_{33} + C_{11} + C_{12} + C_{21} \geqq 0\,.$$

To test whether or not the condition (2) was implied by the conditions (1) the linear programming problem:

Minimize:

$$-A_{21} - A_{31} - A_{32} + B_{23} + B_{32} + B_{33} + C_{11} + C_{12} + C_{21}\,.$$

Subject to

$$\begin{aligned}
&\sum_{i=1}^{l} A_{ij} - \sum_{k=1}^{n} C_{jk} \quad 0 \quad (j = 1, 2, \ldots, m)\,, \\
&\sum_{j=1}^{m} C_{jk} - \sum_{i=1}^{l} B_{ik} = 0 \quad (k = 1, 2, \ldots, n)\,, \\
&\sum_{k=1}^{n} B_{ik} - \sum_{j=1}^{m} A_{ij} = 0 \quad (i = 1, 2, \ldots, l)\,, \\
&\sum_{i\in I} \sum_{k\in K} B_{ik} + \sum_{j\in J} \sum_{k\in \bar{K}} C_{jk} - \sum_{i\in I} \sum_{j\in J} A_{ij} \geqq 0 \quad \text{all} \quad I, J, K\,, \\
&0 \leqq A_{ij} \leqq 1 \quad (i = 1, 2, \ldots, l;\ j = 1, 2. \ldots, m)\,, \\
&0 \leqq B_{ik} \leqq 1 \quad (i = 1, 2, \ldots, l; \quad k = 1, 2, \ldots, n)\,, \\
&0 \leqq C_{jk} \leqq 1 \quad (j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)
\end{aligned}$$

was solved. The bounds on $A_{ij}B_{ik}$ and C_{jk} are required in order that the objective function be bounded.

Feasible solutions to this linear programming problem define the constants of multi-index problems satisfying the conditions (1).

There are in fact several optimal solutions, one of which is given in figure 1.

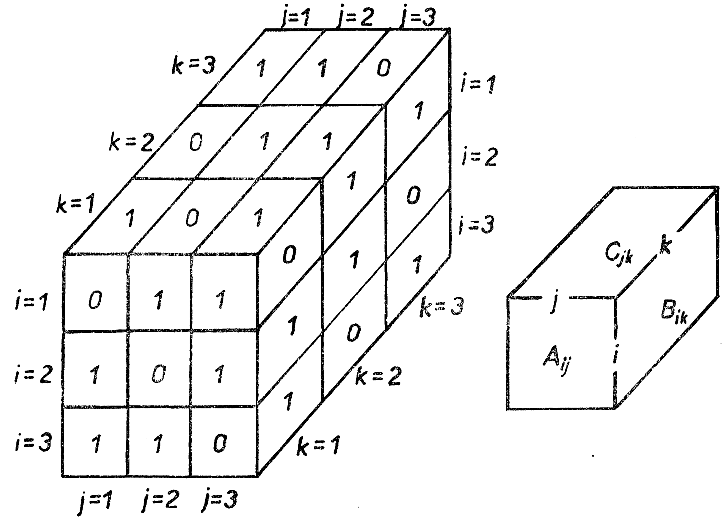

Fig. 1.

For the problem specified by figure 1 condition (2) is not satisfied since:

$$-A_{21} - A_{31} - A_{32} + B_{23} + B_{32} + B_{33} + C_{11} + C_{12} + C_{21} = -1$$

and therefore no solution exists, even though the conditions (1) are satisfied.

If

$$L = \{1, 2, \ldots, l\},$$
$$M = \{1, 2, \ldots, m\},$$
$$N = \{1, 2, \ldots, n\},$$
$$\alpha \subseteq L \times M,$$
$$\beta \subseteq L \times N,$$
$$\gamma \subseteq M \times N$$

such that if $(i, j) \in \alpha$, then for each k, either $(i, k) \in \beta$ or $(j, k) \in \gamma$.

Then for a solution to exist, Smith [4] has shown that it is necessary that

$$(3) \qquad - \sum_{(i,j)\in\alpha} A_{ij} + \sum_{(i,k)\in\beta} B_{ik} + \sum_{(j,k)\in\gamma} C_{jk} \geqq 0 \quad \text{all} \quad \alpha, \beta .$$

The conditions (1) are the subset of the conditions (3) given by: $\alpha = I \times J$, $\beta = I \times K$ and $\gamma = J \times \bar{K}$.

All of the necessary conditions generated for the problem with $l = m = n = 3$ (for example the condition (2)) are included in the conditions (3). It is tempting therefore to conjecture that the conditions (3) are necessary and sufficient for the existence of a solution to the multi-index problem having $l = m = n = 3$.

References

[1] *K. B. Haley:* Note on the Letter by Morávek and Vlach. Opns. Res. *15* (1967), 545—546.

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Souhrn

O MORÁVKOVÝCH A VLACHOVÝCH PODMÍNKÁCH PRO EXISTENCI ŘEŠENÍ VÍCEINDEXOVÉHO PROBLÉMU

GRAHAM SMITH

V článku je dán nejmenší možný příklad trojindexového problému, který vyhovuje podmínkám Morávka a Vlacha z r. 1967 a který nemá přípustné řešení.

Author's address: Professor *Graham Smith*, The University of New South Wales, Sydney, Australia.