Karel Segeth

Universal approximation by systems of hill functions

Aplikace matematiky, Vol. 19 (1974), No. 6, 403–436

Persistent URL: http://dml.cz/dmlcz/103558

UNIVERSAL APPROXIMATION BY SYSTEMS OF HILL FUNCTIONS

KAREL SEGETH

(Received April 23, 1974)

1. INTRODUCTION

Several results extending the author's work [7] are given in this paper. In particular, these results contribute to a better numerical employment of the ideas presented in [7]. For the sake of simplicity we will confine ourselves to the one-dimensional case as well as we did in [7].

Let ω be an infinitely smooth rapidly decreasing function and Λ its Fourier transform. In [7], the approximation of the form

$$\sum_{k=-\infty}^{\infty} c_k \omega((x/h - k)\,\eta(h)) \tag{1.1}$$

is studied where $\eta(h)$ is a certain increasing function (so-called Λ-admissible function), $\eta(0) = 0$. This approximation is shown to be universal, i.e., for any approximated function f we obtain the best possible order of approximation limited only by the smoothness of f. Analogously to the hill functions of Babuška [1] the function ω is called the universal hill function.

Babuška [1], and Fix and Strang [4], [9] consider the approximation of the form

$$\sum_{k=-\infty}^{\infty} c_k \omega(x/h - k) \tag{1.2}$$

and show that it is necessary for the Fourier transform of the hill function ω to have zeros at the points $2\pi j$ for all non-zero integers j. The multiplicity of these zeros determines the highest order of approximation attainable.

In the universal approximation of the form (1.1), no zeros of the Fourier transform Λ of the universal hill function ω are required in general. The quality of the approximation is achieved only by the employment of the Λ-admissible function $\eta(h)$. However, practical computations show that if Λ has zeros at some of the points $2\pi j/\eta(h)$ (which correspond here to the points $2\pi j$ considered in the approximation of the form (1.2)) then the error of the result may decrease. We will be concerned with this phenomenon in the paper.

Unfortunately, the dependence of Λ (as well as of ω) on η is more complex in this case. We have to work with a whole system of universal hill functions here and the results of [7] need a generalization, which is the subject of this paper, too.

The purpose of such an investigation is far from purely theoretical. In practical computation, the round-off error prevails in the result relatively soon, i.e., for relatively large parameter h. Thus we cannot obtain arbitrarily accurate results choosing sufficiently small h. We are forced to seek ways for obtaining very accurate results for relatively large values of h, which are acceptable in the numerical process involved. •

Basic concepts of the theory of generalized functions are introduced in Sec. 2. We refer to [10], [7] for their principal properties. In addition, the main result of [1] in approximation by hill functions is given in this section.

In Sec. 3 we introduce a system $\{\omega_y\}_{y\in(0,U)}$ of infinitely smooth rapidly decreasing functions and establish a theorem concerned with the approximation by this system of universal hill functions. A proper choice of a Λ-admissible function $\eta(h)$ is shown for a class of systems $\{\omega_y\}$ and a possibility of the approximation by a system of functions not having compact supports is studied. These statements are generalizations of the corresponding theorems of [7] Sec. 4.

Two particular systems of universal hill functions the Fourier transform of which has zeros of a certain multiplicity at some of the points $2\pi j/y$ are constructed in Sec. 4. Moreover, a theorem examining the influence of these zeros on the quality of the approximation is proven here.

In conclusion, a simple numerical example illustrating the statements of Sections 3 and 4 is given in Sec. 5.

2. PRELIMINARIES

We will confine ourselves to the one-dimensional case. Apart from basic definitions and notations, the principal result of [1] is given without proof in the conclusion of this section.

Definition 2.1. *Let R be a one-dimensional Euclidean space. Let us denote the set of complex-valued continuous functions defined in R with derivatives of all orders continuous in R by $C^\infty(R)$. Let us denote by $S(R)$ the set of all rapidly decreasing (at ∞) functions (i.e., the functions $\varphi \in C^\infty(R)$ satisfying the condition*

$$\sup_{x\in R} \left|x^k \varphi^{(l)}(x)\right| < \infty$$

for all non-negative integers k, l) with the usual topology (see [10]). Let $S'(R)$ be the space of generalized functions over $S(R)$. We will write simply C^∞, S, S' etc. instead of $C^\infty(R)$, $S(R)$, $S'(R)$ wherever it will not be ambiguous. ●

We also introduce the convolution $\varphi * \psi$ of functions $\varphi, \psi \in S$, and the product ψf and the convolution $f * \psi$ of functions $\psi \in S$, $f \in S'$ in the usual way.

Definition 2.2. *Let $w(x) = ax + b$ be a non-singular linear mapping of R on R with a, b real, let $f \in S'$. Let us denote by $f(ax + b)$ a function from S' satisfying the relation*

$$(f(ax + b), \varphi(x)) = (f(w(x)), \varphi(x)) = |a|^{-1} (f(x), \varphi(w^{-1}(x)))$$

for any $\varphi \in S$. For the sake of brevity we will sometimes use the notation

$$f^{[a]}(x) = f(ax)$$

with $b = 0$.

Definition 2.3. *Let us denote the Fourier transform of a function $\varphi \in S$ by*

$$\mathscr{F}(\varphi)(x) = \tilde{\varphi}(x) = \int_{-\infty}^{\infty} e^{itx} \varphi(t) \, dt .$$

Let us introduce $\mathscr{F}(f) = \tilde{f}$ for $f \in S'$ by the equality

$$(\mathscr{F}(f), \mathscr{F}(\varphi)) = 2\pi(f, \varphi)$$

valid for any $\varphi \in S$.

Remark 2.1. The Fourier transform $\mathscr{F}$ is a linear continuous mapping of S on S and of S' on S'. The inverse Fourier transform $\mathscr{F}^{-1}(\varphi)$ of the function $\varphi \in S$ is given by the formula

$$\mathscr{F}^{-1}(\varphi)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) \, dt .$$

The inverse Fourier transform $\mathscr{F}^{-1}(f)$ of the function $f \in S'$ is defined by the equality

$$(\mathscr{F}^{-1}(f), \varphi) = (2\pi)^{-1} (f, \mathscr{F}(\varphi))$$

valid for any $\varphi \in S$.

Definition 2.4. *Let $f \in S'$. A closed set $G = \operatorname{supp} f$ is said to be the support of the function f if $(f, \varphi) = 0$ for all $\varphi \in S$ such that $\varphi(x) = 0$ in some neighborhood of G. (A support in the sense of this definition need not mean the minimal support.)*

Definition 2.5. *Let us denote by $H^\alpha(R)$, $\alpha \geqq 0$ the set of all functions $f \in S'$ such that*

$$|\mathscr{F}(f)(x)|^2 (1 + |x|^{2\alpha}) \in L_1(R) .$$

Let us put

$$(2.1) \qquad \|f\|^2_{H^\alpha(R)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathscr{F}(f)(x)|^2 (1 + |x|^{2\alpha})\, dx .$$

The normed linear space $H^\alpha(R)$ with the norm (2.1) is said to be a fractional Sobolev space.

Remark 2.2. Apparently $H^\alpha(R) \supset H^\beta(R)$ for $0 \leqq \alpha \leqq \beta$, and $H^0(R) = L_2(R)$. •

The following theorem is a special case of Theorem 4.1, the basic approximation theorem of [1].

Theorem 2.1. *Let $0 \leqq \alpha' \leqq \beta$ be real numbers. Let $\omega_j \in S'$; $j = 1, \ldots, r$ be functions with compact supports. Let χ_j; $j = 1, \ldots, r$ be (complex-valued) trigonometric polynomials such that the function*

$$\Lambda = \sum_{j=1}^{r} \lambda_j \chi_j$$

where $\lambda_j = \mathscr{F}(\omega_j)$ has the following properties:

1.

$$(2.2) \qquad \Lambda(0) \neq 0 .$$

2. *There exists a function $z(k)$ such that*

$$(2.3) \qquad |\Lambda(x - 2\pi k)| \leqq z(k)\, |x|^t$$

for some $t \geqq 0$, all x such that

$$(2.4) \qquad |x| < \pi ,$$

and all integers $k \neq 0$, and

3.

$$(2.5) \qquad \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} z^2(k)\, |k|^{2\alpha'} < \infty .$$

Then there exists an operator A_h,

$$A_h(f)(x) = \sum_{j=1}^{r} \sum_{k=-\infty}^{\infty} c_j(h, f, k)\, \omega_j(x/h - k)$$

mapping H^β into H^α for any $0 \leqq \alpha \leqq \alpha'$. Moreover,

$$(2.6) \qquad \|f - A_h(f)\|_{H^\alpha} \leqq Ch^\mu \|f\|_{H^\beta}$$

where

$$\mu = \min(t - \alpha, \beta - \alpha) \tag{2.7}$$

and $0 < C < \infty$ *is a constant independent of* h.

If the support T *of* f *is compact then there exists a constant* $0 < L < \infty$ *independent of* h *such that* $A_h(f)$ *has a compact support* T' *where* T' *is an* Lh*-neighborhood of* T.

Proof is given in [1].

Remark 2.3. Further analysis in [1] shows that the conditions (2.2) to (2.5) are not only sufficient but also necessary for the estimate (2.6), (2.7). Condition (2.3) is of particular importance. It says that the function Λ has zeros of multiplicity t' at all the points $2\pi k$ where $k \neq 0$ is an integer and t' is the minimal integer not less than t.

3. UNIVERSAL APPROXIMATION BY SYSTEMS

This section is devoted to the formulation of the results of [7] Sec. 4 for systems $\{\omega_y\}$ of functions introduced in Definition 3.1. The principal theorem of the paper is Theorem 3.1, an analog of [7] Theorem 4.1. In the theorem we examine the approximation by a system of hill functions, i.e., the approximation of the form

$$\sum_{k=-\infty}^{\infty} c_k \omega_{\eta(h)}((x/h - k)\,\eta(h)),$$

where $\mathscr{F}(\omega_y) = \Lambda_y$ and η is a Λ-admissible function introduced in Definition 3.2, and show when this approximation is universal. We attach a brief sketch of the proof of this theorem since we will need some relations and the notation of the proof in Sec. 4. For the same reason, we present also Lemma 3.1 (an analog of [7] Lemma 4.2) with proof. On the other hand, we refer to [7] wherever it is suitable.

In conclusion, we show the choice of Λ-admissible function for a class of systems of universal hill functions in Theorem 3.2 and consider a numerically practicable way for the approximation by a system of functions not having compact supports in Theorem 3.3. The proofs of these theorems follow from the proofs of [7] Theorems 4.2 and 4.3 after obvious changes.

C, D, and L mean general constants (independent, in particular, of the parameter h) taking different finite positive values at different places throughout this and the following section.

Definition 3.1. *Let* $U > 0$ *be given. Let us denote by* $\{\varphi_y\}_{y \in (0,U)}$ *a system of complex-valued functions* φ_y *defined in* R *for all* $y \in (0, U)$. *We will write simply* $\{\varphi_y\}$

wherever it will not be ambiguous. Further we will write $\{\varphi_y\} \subset S(R)$ *if* $\varphi_y \in S(R)$ *for any* $y \in (0, U)$.

A single function φ *will be considered as a system* $\{\varphi_y\}$ *if we put* $\varphi_y = \varphi$, $y \in (0, U)$, *with an arbitrary* $U > 0$.

Remark 3.1. Let $\{\varphi_y\} \subset S$. Let us put $\psi_y = \mathscr{F}(\varphi_y)$ for any $y \in (0, U)$. Then apparently $\{\psi_y\} \subset S$ (cf. Remark 2.1). •

We will introduce the concept of the Λ-admissible function $\eta(h)$, which plays an important role in the universal approximation by hill functions as well as by systems of them.

Definition 3.2. *Let* $\{\Lambda_y\}_{y\in(0,U)} \subset S(R)$ *be given. A bounded continuous increasing real-valued function* $\eta(h)$ *defined on the interval* $\langle 0, 1\rangle$ *is said to be* Λ*-admissible if it satisfies the following conditions*:

1.

$$\eta(0) = 0\,, \quad \eta(1) \leqq U\,. \tag{3.1}$$

2. *There exists a finite positive constant* $C(\eta)$ *such that*

$$h^{\varepsilon}/\eta(h) \leqq C(\eta) \tag{3.2}$$

for $0 < h < 1$ *and any* $\varepsilon > 0$.

3. *For any* $\alpha \geqq 0$ *there exists a function* $z(k) = z(k, \alpha)$ *such that*

$$\left|\Lambda_{\eta(h)}((x - 2\pi k)/\eta(h))\right| \leqq C(\alpha, \gamma)\, h^{\gamma}\, z(k, \alpha) \tag{3.3}$$

holds for all integers k, $k \neq 0$, *any* $\gamma \geqq 0$, $0 < h < 1$ *and* $|x| < \pi$ *with some finite positive constant* $C(\alpha, \gamma)$. *Moreover*

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} z^2(k, \alpha)\, |k|^{2\alpha} < \infty\,, \tag{3.4}$$

i.e., the series converges for any $\alpha \geqq 0$.

Remark 3.2. Let $\Lambda \in S$. We will also speak about the Λ-admissible function $\eta(h)$ in the sense of Definition 3.2 considering the system $\{\Lambda_y\}_{y\in(0,U)}$ given by the equality $\Lambda_y = \Lambda$, $y \in (0, U)$, where $U > 0$ is arbitrary (cf. Definition 3.1 and [7] Definition 4.1).

Theorem 3.1. *Let* $\varepsilon > 0$, $\varepsilon' > 0$ *be given. Let* $\{\omega_y\}_{y\in(0,U)} \subset S(R)$ *and let us denote the Fourier transform of* ω_y *by* $\mathscr{F}(\omega_y) = \Lambda_y$. *Let there exist finite positive constants* Γ_{00}, Γ_0, *and* Γ_s *such that*

$$0 < \Gamma_{00} \leqq |\Lambda_y(0)|\,, \tag{3.5}$$

$$\sup_{x\in R} |\Lambda_y(x)| \leqq \Gamma_0 , \tag{3.6}$$

$$\sup_{x\in R} |\Lambda_y^{(s)}(x)| \leqq \Gamma_s \tag{3.7}$$

for any positive integer s idependently of $y \in (0, U)$. Further, let there exist a Λ-admissible function $\eta(h)$.

Let

$$0 \leqq \alpha \leqq \beta < \infty \tag{3.8}$$

be real numbers and let $f \in H^{\beta}(R)$.

Then there exists an operator $G_{h,\eta}$, $0 < h < 1$,

$$G_{h,\eta}(f)(x) = \eta(h) \sum_{k=-\infty}^{\infty} c(k, h, f)\, \omega_{\eta(h)}((x/h - k)\, \eta(h)) \tag{3.9}$$

such that

$$\|f - G_{h,\eta}(f)\|_{H^{\alpha}(R)} \leqq C(\alpha, \beta, \varepsilon)\, h^{\beta-\alpha-\varepsilon}\, \|f\|_{H^{\beta}(R)} \tag{3.10}$$

for $0 < h < 1$.

Let $\Omega \subset R$ be such a set that

$$\Omega = \operatorname{supp} \omega_y , \quad y \in (0, U) \tag{3.11}$$

(cf. Definition 2.4). If both the support T of f and the set Ω are compact then there exists a constant $L(\alpha, \beta, \varepsilon')$ such that $G_{h,\eta}(f)$ has a compact support T' where T' is an $Lh^{1-\varepsilon'}$-neighborhood of T.

Proof. The course of the proof is based on that of Theorem 4.1 given in [7]. We refer to [7] in the cases when the argument of the proof remains completely unchanged. However, we repeat the substantial steps of the proof here and introduce the notation which will be used in Sec. 4.

The proof consists of four parts. In the first part we approximate the function $f \in H^{\beta}(R)$ by a function $f_h \in C^{\infty}(R)$ and find the bound (3.16) for the error of this approximation. We construct a function $g \in H^{\alpha}(R)$ approximating f_h in the second part and find the bound (3.25) for the norm of their difference in the third part. The fourth part is concerned with the statement on compactness of the support of $G_{h,\eta}(f)$. The proof of an auxiliary statement is removed into Lemma 3.1 that follows the proof of the theorem.

1. We introduce the function $f_h \in C^{\infty}$ exactly in the same way as in [7]. We start from the definition of the functions $\varkappa \in S$, $\nu \in S$:

$$\begin{aligned} \varkappa(x) &= \exp(x^2 - 1)^{-1} , & |x| &\leqq 1 , \\ &= 0 , & |x| &> 1 , \end{aligned}$$

and $v = \mathscr{F}(\varkappa)$. According to [7] Lemma 4.1 there exists a trigonometric polynomial P such that

(3.12) $$|1 - \varphi(xh)| \leqq C|x|^{\beta-\alpha} h^{\beta-\alpha}, \quad |xh| < 1$$

where

(3.13) $$\varphi = vP$$

and $\varphi \in S$. Further let us write $\chi = \mathscr{F}^{-1}(\varphi)$ and

(3.14) $$f_h = h^{-1}(f * \chi^{[h^{-1}]})$$

where $f \in H^\beta(R)$. Then we obtain that $f_h \in C^\infty$, $\mathscr{F}(f_h) \in S'$ (cf. [7] Remark 2.3) and

(3.15) $$\xi_h = \mathscr{F}(f_h) = \varphi^{[h]} \mathscr{F}(f).$$

Moreover, we find that $f_h \in H^\alpha(R)$ and

(3.16) $$\|f_h - f\|_{H^\alpha(R)} \leqq Ch^{\beta-\alpha}\|f\|_{H^\beta(R)}.$$

2. According to Lemma 3.1 there exists a trigonometric polynomial P_h such that

(3.17) $$|\Lambda_{\eta(h)}(xh/\eta(h))\, P_h(xh) - 1| \leqq Ch^{\beta-\alpha}|x|^{\beta-\alpha}, \quad |x| \leqq \pi/h$$

and

(3.18) $$|P_h(x)| \leqq Ch^{-\varepsilon}, \quad x \in R$$

for an arbitrarily chosen $\varepsilon > 0$. Then we may proceed in the same way as in the proof of [7] Theorem 4.1. Let us put

(3.19) $$\zeta_h(x) = P_h(xh) \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi k/h).$$

Then the series in (3.19) converges in $L_2(-\pi/h, \pi/h)$ since we may estimate

(3.20) $$\left\| \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi k/h)\right\|_{L_2(-\pi/h,\pi/h)} \leqq C\|f\|_{L_2(R)}.$$

Therefore $\zeta_h \in L_2(-\pi/h, \pi/h)$. Moreover, the function ζ_h is apparently periodic with period $2\pi/h$. Further let us write

(3.21) $$\zeta_h = P_h^{[h]}(\xi_h + \psi_h)$$

where

(3.22) $$\psi_h(x) = \sum_{k \neq 0} \xi_h(x - 2\pi k/h).$$

This function is denoted by ξ_h^* in [7], i.e.,

$$\xi_h^* = \psi_h \tag{3.23}$$

in our present notation.

Let us construct a Fourier series for the function ζ_h in $(-\pi/h, \pi/h)$,

$$\zeta_h(x) = \sum_{k=-\infty}^{\infty} c_k(h)\, e^{ikhx},$$

where

$$c_k(h) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \zeta_h(x)\, e^{-ikhx}\, \mathrm{d}x .$$

Finally let us write

$$g(x) = h^{-1}\eta(h) \sum_{k=-\infty}^{\infty} c_k(h)\, \omega_{\eta(h)}((x/h - k)\, \eta(h)) .$$

Then we find (cf. [7] Remarks 2.2 and 2.7) that

$$h\eta^{-1}(h)\, \mathscr{F}(g^{[h\eta^{-1}(h)]})\,(x) = \Lambda_{\eta(h)}(x) \sum_{k=-\infty}^{\infty} c_k(h)\, e^{ik\eta(h)x},$$

i.e.,

$$\mathscr{F}(g) = \zeta_h \Lambda_{\eta(h)}^{[h\eta^{-1}(h)]} = P_h^{[h]} \Lambda_{\eta(h)}^{[h\eta^{-1}(h)]} \xi_h + P_h^{[h]} \Lambda_{\eta(h)}^{[h\eta^{-1}(h)]} \psi_h . \tag{3.24}$$

3. Let us now show

$$\|f_h - g\|_{H^\alpha(R)} \leqq C h^{\beta-\alpha-\varepsilon} \|f\|_{H^\beta(R)} . \tag{3.25}$$

Then we obtain also that $g \in H^\alpha(R)$. From (3.15), (3.24) we have

$$\|f_h - g\|^2_{H^\alpha(R)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\xi_h(x) - P_h(xh)\, \Lambda_{\eta(h)}(xh/\eta(h))\, \xi_h(x) - \tag{3.26}$$

$$- P_h(xh)\, \Lambda_{\eta(h)}(xh/\eta(h))\, \psi_h(x)|^2 \,(1 + |x|^{2\alpha})\, \mathrm{d}x \leqq$$

$$\leqq C \bigg(\int_{-\pi/h}^{\pi/h} |1 - P_h(xh)\, \Lambda_{\eta(h)}(xh/\eta(h))|^2\, |\xi_h(x)|^2 \,(1 + |x|^{2\alpha})\, \mathrm{d}x +$$

$$+ \int_{-\pi/h}^{\pi/h} |P_h(xh)\, \Lambda_{\eta(h)}(xh/\eta(h))|^2\, |\psi_h(x)|^2 \,(1 + |x|^{2\alpha})\, \mathrm{d}x +$$

$$+ \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2\, |\Lambda_{\eta(h)}((xh - 2\pi k)/\eta(h))|^2 \,(1 + |x - 2\pi k/h|^{2\alpha})\, \mathrm{d}x +$$

$$+ \int_{\pi/h < |x|} |\xi_h(x)|^2 \,(1 + |x|^{2\alpha})\, \mathrm{d}x \bigg) = C(I_1 + I_2 + I_3 + I_4)$$

since ζ_h is periodic with period $2\pi/h$.

From (3.8), (3.15), (3.17) we obtain

$$(3.27) \qquad I_1 \leqq Ch^{2(\beta-\alpha)} \int_{-\pi/h}^{\pi/h} |\mathscr{F}(f)(x)|^2 |\varphi(xh)|^2 |x|^{2(\beta-\alpha)} (1 + |x|^{2\alpha}) \, dx \leqq$$
$$\leqq Ch^{2(\beta-\alpha)} \|f\|^2_{H^\beta(R)}$$

since the functions φ and

$$(3.28) \qquad (1 + |x|^{2\alpha}) |x|^{2(\beta-\alpha)} (1 + |x|^{2\beta})^{-1}$$

are bounded in R.

We can easily verify that the assumptions of [7] Lemma 4.3 are fulfilled. Putting $\gamma = \alpha$ in the statement of the lemma and using (3.23), we have

$$\int_{-\pi/h}^{\pi/h} |\psi_h(x)|^2 (1 + |x|^{2\alpha}) \, dx \leqq Ch^{2(\beta-\alpha)} \|f\|^2_{H^\beta(R)}$$

which together with (3.6), (3.18) gives

$$(3.29) \qquad I_2 \leqq Ch^{2(\beta-\alpha-\varepsilon)} \|f\|^2_{H^\beta(R)}$$

for an arbitrary $\varepsilon > 0$.

Further let us write

$$I_3 \leqq \sum_{k \neq 0} \int_{-\pi/h}^{\pi/h} |P_h(xh)|^2 |\xi_h(x)|^2 |\Lambda_{\eta(h)}((xh - 2\pi k)/\eta(h))|^2 (1 + |x - 2\pi k/h|^{2\alpha}) \, dx +$$
$$+ \sum_{k \neq 0} \int_{-\pi/h}^{\pi/h} |P_h(xh)|^2 |\psi_h(x)|^2 |\Lambda_{\eta(h)}((xh - 2\pi k)/\eta(h))|^2 (1 + |x - 2\pi k/h|^{2\alpha}) \, dx =$$
$$= I_{31} + I_{32} .$$

Putting $\gamma = \beta + \varepsilon$ in (3.3) of Definition 3.2, and using (3.18) and the fact that

$$1 + |x - 2\pi k/h|^{2\alpha} \leqq Ch^{-2\alpha} |k|^{2\alpha}$$

we get

$$(3.30) \qquad |P_h(xh)|^2 |\Lambda_{\eta(h)}((xh - 2\pi k)/\eta(h))|^2 (1 + |x - 2\pi k/h|^{2\alpha}) \leqq$$
$$\leqq Ch^{2(\beta-\alpha)} z^2(k) |k|^{2\alpha}$$

for $|x| \leqq \pi/h$, $0 < h < 1$, and any integer k, $k \neq 0$. Then from (3.15), (3.30), and (3.4) of Definition 3.2 we have

$$I_{31} \leqq Ch^{2(\beta-\alpha)} \sum_{k \neq 0} z^2(k) |k|^{2\alpha} \int_{-\pi/h}^{\pi/h} |\mathscr{F}(f)(x)|^2 |\varphi(xh)|^2 \, dx \leqq$$
$$\leqq Ch^{2(\beta-\alpha)} \int_{-\pi/h}^{\pi/h} |\mathscr{F}(f)(x)|^2 (1 + |x|^{2\beta}) \, dx \leqq Ch^{2(\beta-\alpha)} \|f\|^2_{H^\beta(R)}$$

because the function φ is bounded in R. Putting $\gamma = \beta$ in [7] Lemma 4.3 and using (3.23) we obtain

$$\int_{-\pi/h}^{\pi/h} |\psi_h(x)|^2 (1 + |x|^{2\beta}) \, dx \leqq C\|f\|^2_{H^\beta(R)},$$

which together with (3.30) and (3.4) of Definition 3.2 gives

$$I_{32} \leqq Ch^{2(\beta-\alpha)} \sum_{k \neq 0} z^2(k) \, |k|^{2\alpha} \int_{-\pi/h}^{\pi/h} |\psi_h(x)|^2 (1 + |x|^{2\beta}) \, dx \leqq Ch^{2(\beta-\alpha)} \|f\|^2_{H^\beta(R)}.$$

Therefore

$$I_3 \leqq Ch^{2(\beta-\alpha)} \|f\|^2_{H^\beta(R)}. \tag{3.31}$$

Finally we use (3.8), (3.15) and the boundedness of the function φ to show

$$\begin{aligned} I_4 &\leqq C \int_{\pi/h < |x|} |\mathscr{F}(f)(x)|^2 (1 + |x|^{2\alpha}) \, dx \leqq \\ &\leqq C \int_{\pi/h < |x|} |\mathscr{F}(f)(x)|^2 (1 + |x|^{2\beta}) \, |x|^{2(\alpha-\beta)} \, dx \leqq Ch^{2(\beta-\alpha)} \|f\|^2_{H^\beta(R)} \end{aligned} \tag{3.32}$$

since the function (3.28) is bounded in R.

From (3.27), (3.29), (3.31), (3.32) we obtain (3.25), which together with (3.16) completes the proof of (3.10) if we put

$$G_{h,\eta}(f) = g.$$

Thus $G_{h,\eta}(f) \in H^\alpha(R)$.

4. Supposing that the support T of f is compact and repeating the argument of the proof of [7] Theorem 4.1, we establish that the sum in (3.9) is finite, i.e., the summation goes over all integers $k \in K$ where

$$K = [k, kh = v + y, \; v \in T, \; |y| \leqq Lh] \tag{3.33}$$

with some finite positive constant L.

Let the set Ω be also compact, i.e., let there exist a constant D such that

$$\Omega \subset [x, |x| \leqq D].$$

Then according to (3.11)

$$\begin{aligned} \Omega_k &= \operatorname{supp} \omega_{\eta(h)}((x/h - k) \, \eta(h)) \subset [x, |(x - kh) \, \eta(h)/h| \leqq D] = \\ &= [x, kh - Dh/\eta(h) \leqq x \leqq kh + Dh/\eta(h)]. \end{aligned}$$

Finally with respect to (3.33) we find

$$(3.34) \qquad T' = \operatorname{supp} G_{h,\eta}(f) = \bigcup_{k\in K} \Omega_k \subset$$

$$\subset [w, w = v + x + y,\ v \in T,\ |y| \leqq Lh,\ |x| \leqq Dh/\eta(h)] \subset$$

$$\subset [w, w = v + z,\ v \in T,\ |z| \leqq Lh^{1-\varepsilon'}]$$

since the function $h^{\varepsilon'}(L + D/\eta(h))$ is bounded for $0 < h < 1$ and an arbitrary $\varepsilon' > 0$ according to (3.2) of Definition 3.2. Apparently (3.34) is an $Lh^{1-\varepsilon'}$-neighborhood of T. The last statement of the theorem has been proven.

Remark 3.3. Choosing in Theorem 3.1

$$(3.35) \qquad \omega_y = \omega$$

for all $y \in (0, U)$ we obtain immediately [7] Theorem 4.1.

In the sense of Definition 3.1 and Remark 3.2, we say that a function $\omega \in S(R)$ satisfies the assumptions of Theorem 3.1 if the system $\{\omega_y\}$ where ω_y is given by (3.35) (as well as the system $\{\Lambda_y\}$, $\Lambda_y = \mathscr{F}(\omega_y)$) satisfies them. •

The following lemma is used in the proof of Theorem 3.1 and also referred to in Sec. 4.

Lemma 3.1. *Let the assumptions of Theorem* 3.1 *be fulfilled. Then there exists a trigonometric polynomial* P_h *such that*

$$(3.36) \qquad |\Lambda_{\eta(h)}(xh/\eta(h))\, P_h(xh) - 1| \leqq C|x|^{\beta-\alpha}\, h^{\beta-\alpha}$$

for $|x| \leqq \pi/h$ *and*

$$|P_h(x)| \leqq Ch^{-\varepsilon}$$

for $x \in R$ *and an arbitrary* $\varepsilon < 0$.

Proof. The existence of P_h follows from a modification of the proofs of [7] Lemmas 4.1 and 4.2. If $\beta = \alpha$ we may put $P_h(x) \equiv 0$. Thus let $\alpha < \beta$ and let us denote the minimal integer not less than $\beta - \alpha + 1$ by B. Let us choose two integers $N \geqq M$ in such a way that

$$N - M = B - 1 \geqq 0\,.$$

Assuming that P_h is of the form

$$(3.37) \qquad P_h(x) = \sum_{k=M}^{N} b_k(h)\, e^{ikx}$$

we will find its coefficients $b_k(h)$; $k = M, \ldots, N$.

Let us write a Taylor series for the function $(\Lambda_{\eta(h)}^{[\eta^{-1}(h)]}P_h - 1) \in C^\infty(R)$. Putting

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}(\Lambda_{\eta(h)}(x/\eta(h))\, P_h(x) - 1)|_{x=0} = 0\,; \quad j = 0, 1, \ldots, B - 1 \tag{3.38}$$

we get the Taylor series with the first B terms equal to zero, i.e.,

$$\Lambda_{\eta(h)}(x/\eta(h))\, P_h(x) - 1 = x^B(B!)^{-1} \frac{\mathrm{d}^B}{\mathrm{d}x^B}(\Lambda_{\eta(h)}(x/\eta(h))\, P_h(x))|_{x=\hat{x}}$$

for $|x| \leqq \pi$ where $|\hat{x}| < \pi$. Substituting (3.37) for P_h and using (3.6), (3.7), and Definition 3.2, we get

$$|\Lambda_{\eta(h)}(x/\eta(h))\, P_h(x) - 1| \leqq C|x|^B \eta^{-B}(h) \sum_{k=M}^{N} |b_k(h)|\,.$$

We will show that $\max\limits_{k=M,\ldots,N} |b_k(h)| \leqq Ch^{-\varepsilon}$ in the following. Considering this and (3.2) of Definition 3.2, we will have

$$|\Lambda_{\eta(h)}(x/\eta(h))\, P_h(x) - 1| \leqq Ch^{-1}|x|^{\beta-\alpha+1}\,.$$

Substituting finally xh for x, we obtain (3.36).

Let us further examine the conditions (3.38). Differentiating (3.37) and substituting for the derivatives in (3.38), we have the system of B linear algebraic equations for B unknown coefficients $b_k(h)$; $k = M, \ldots, N$, namely

$$\sum_{k=M}^{N} a_{jk}(h)\, b_k(h) = \delta_{0j}\,; \quad j = 0, 1, \ldots, B - 1 \tag{3.39}$$

where

$$a_{jk}(h) = \sum_{l=0}^{j} \binom{j}{l} i^l k^l \eta^{l-j}(h)\, \Lambda_{\eta(h)}^{(j-l)}(0) \tag{3.40}$$

and δ_{mn} is the Kronecker symbol[1]). Making use of the form (3.40) of the elements of the matrix of the system and expressing the determinant $\det(a_{jk}(h))$ of this matrix in terms of the sums in (3.40), we finally obtain

$$\det(a_{jk}(h)) = i^{B(B-1)/2}\, \Lambda_{\eta(h)}^B(0)\, V_B(M, M + 1, \ldots, N)$$

where $V_B(M, M + 1, \ldots, N)$ is the non-zero Vandermonde determinant formed of the B integers $M, M + 1, \ldots, N$, which are different from each other. Considering (3.5), (3.6) we can estimate

$$\Gamma_0^B|V_B(M, M + 1, \ldots, N)| \geqq |\det(a_{jk}(h))| \geqq \Gamma_{00}^B|V_B(M, M + 1, \ldots, N)| > 0 \tag{3.41}$$

[1]) $\delta_{mn} = 0$ unless $m = n$, in which case $\delta_{mn} = 1$.

independently of h. Thus the system (3.39) has a unique solution $b_k(h)$; $k = M, \ldots, N$ for any right-hand part and $0 < h < 1$. The trigonometric polynomial (3.37) satisfying (3.38) has been constructed.

Let us solve the system (3.39) using Cramer's rule. Denoting the matrix, obtained by replacing the rth column in the matrix $(a_{jk}(h))$ by the column of the right-hand parts, by $({}^r a_{jk}(h))$ and treating this matrix in the same way as above, we find

$$\det({}^r a_{jk}(h)) = (-1)^r \sum_{l_1=0}^{1} \ldots \sum_{l_{B-1}=0}^{B-1} i^{l_1} \ldots i^{l_{B-1}} \times \tag{3.42}$$

$$\times \binom{1}{l_1} \ldots \binom{B-1}{l_{B-1}} \eta^{l_1-1}(h) \ldots \eta^{l_{B-1}-B+1}(h)\, \Lambda_{\eta(h)}^{(1-l_1)}(0) \ldots \Lambda_{\eta(h)}^{(B-1-l_{B-1})}(0) \times$$

$$\times \begin{vmatrix} M^{l_1} & \ldots & (M+r-2)^{l_1} & (M+r)^{l_1} & \ldots & N^{l_1} \\ M^{l_2} & \ldots & (M+r-2)^{l_2} & (M+r)^{l_2} & \ldots & N^{l_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ M^{l_{B-1}} & \ldots & (M+r-2)^{l_{B-1}} & (M+r)^{l_{B-1}} & \ldots & N^{l_{B-1}} \end{vmatrix}.$$

According to (3.6), (3.7) we have

$$\left|\Lambda_{\eta(h)}^{(1-l_1)}(0) \ldots \Lambda_{\eta(h)}^{(B-1-l_{B-1})}(0)\right| \leqq C$$

independently of h and for any choice of $l_1, \ldots, l_{B-1}$. Further, the determinant on the right-hand part of (3.42) vanishes apparently whenever $l_p = l_s$ for a pair of indices from the set $l_1, \ldots, l_{B-1}$. Thus the minimum power of $\eta(h)$ appears in the non-zero term with $l_s = s - 1$. Finally we may estimate

$$\left|\det({}^r a_{jk}(h))\right| \leqq C\eta^{1-B}(h) + o(\eta^{1-B}(h)), \quad h \to 0.$$

Since the determinant $\det(a_{jk}(h))$ of the system (3.39) is bounded from below independently of h according to (3.41), we obtain

$$|b_k(h)| \leqq Ch^{-\varepsilon}, \quad 0 < h < 1$$

for any $\varepsilon > 0$ with respect to (3.2) of Definition 3.2. This completes the proof because then (3.36) holds and, moreover,

$$|P_h(x)| \leqq \sum_{k=M}^{N} |b_k(h)| \leqq Ch^{-\varepsilon}, \quad x \in R. \bullet$$

A Λ-admissible function η fulfilling the conditions of Definition 3.2 may be readily found for the class of systems of functions from S the Fourier transform of which decreases (at ∞) as rapidly as $e^{-D|x|}$.

Theorem 3.2. *Let* $\{\Lambda_y\} \subset S$ *satisfy the condition*

$$|\Lambda_y(x)| \leqq Le^{-D|x|}, \quad x \in R, \quad y \in (0, U) \tag{3.43}$$

with some finite positive constants L, D. Then the function

$$\eta(h) = \eta_0(\eta_1 + \log^{1+\varepsilon_0} h^{-1})^{-1} \tag{3.44}$$

where η_0, η_1, ε_0 *are arbitrary positive numbers and* $\eta_0/\eta_1 \leqq U$, *is* Λ*-admissible independently of L, D.*

Proof. Proceeding in the same way as in the proof of [7] Theorem 4.2, we verify the conditions (3.1) to (3.4) of Definition 3.2 since (3.43) is fulfilled independently of y.●

Approximating a function f the support of which is compact we expect the support of $G_{h,\eta}(f)$ to be also compact. However, this is possible only in the case when the approximating function ω_y has also a compact support (cf. Theorem 3.1).

In practice, the functions of the system $\{\omega_y\}$ need not have a compact support but may decrease (at ∞) so rapidly that (from a numerical point of view) their values are negligible for $|x|$ greater than some $Y > 0$. The approximation by a class of systems of such functions is considered in Theorem 3.3.

Theorem 3.3. *Let the assumptions of Theorem 3.1 be fulfilled. Further let K be a non-negative integer such that*

$$|\omega_y^{(j)}(x)| \leqq L_j e^{-D_j|x|} ; \quad x \in R ; \quad y \in (0, U) ; \quad j = 0, 1, \ldots, K \tag{3.45}$$

with some finite positive constants D_j, L_j; $j = 0, 1, \ldots, K$.

Let us introduce a function $\omega_{y,Y}$ *and its derivatives up to the order K by the formula*

$$\omega_{y,Y}^{(j)}(x) = \begin{cases} \omega_y^{(j)}(x), & |x| < Y, \\ 0, & |x| \geqq Y \end{cases} \tag{3.46}$$

for $y \in (0, U)$ *where*

$$Y = Y(h) = Y_0 + Y_1 \log^{1+\varepsilon_1} h^{-1} \tag{3.47}$$

with $0 < h < 1$ *and arbitrary positive constants* Y_0, Y_1, *and* ε_1. *Writing*

$$G_{h,\eta}^{(j)}(f)(x) = \eta^{j+1}(h)\, h^{-j} \sum_{k=-\infty}^{\infty} c(k, h, f)\, \omega_{\eta(h)}^{(j)}((x/h - k)\, \eta(h)),$$

$$G_{h,\eta,Y}^{(j)}(f)(x) = \eta^{j+1}(h)\, h^{-j} \sum_{k=-\infty}^{\infty} c(k, h, f)\, \omega_{\eta(h),Y}^{(j)}((x/h - k)\, \eta(h)) \tag{3.48}$$

for $0 < h < 1$ *with* $c(k, h, f)$ *given in* (3.9), *and*

$$\varrho_{h,\eta,Y}^{(j)} = G_{h,\eta}^{(j)} - G_{h,\eta,Y}^{(j)} ; \quad j = 0, 1, \ldots, K$$

we have

$$\sup_{x \in R} |\varrho_{h,\eta,Y}^{(j)}(f)(x)| \leqq C(j, s)\, h^s \|f\|_{L_2(R)}$$

for any $s \geqq 0$ *and* $j = 0, 1, \ldots, K$.

Proof. Considering that (3.45) holds independently of y, we obtain the statement of the theorem by the same argument as in the proof of [7] Theorem 4.3.

Remark 3.4. Let us put

$$\hat{\omega}_{y,Y}^{(j)}(x) = \omega_y^{(j)}(x)\,, \quad |x| \leqq Y\,,$$
$$= 0\,, \qquad |x| > Y$$

for $y \in (0, U)$. If we substitute $\hat{\omega}_{y,Y}^{(j)}$ for $\omega_{y,Y}^{(j)}$ in Theorem 3.3 the theorem remains true. Its proof needs only minor changes.

4. ON THE SUITABLE CHOICE OF SYSTEMS

Let us consider a function $\omega \in S$ satisfying the assumptions of Theorem 3.1 (i.e., the corresponding system $\{\omega_y\}$, $\omega_y = \omega$ for all $y \in (0, U)$, cf. Definition 3.1 and Remarks 3.2 and 3.3), $\mathscr{F}(\omega) = \Lambda$. Further let $\{\omega_y^*\} \subset S$ be such a system that $\mathscr{F}(\omega_y^*) = \Lambda_y^*$, $\Lambda_y^* = q_y \Lambda$ and the functions q_y as well as Λ_y^* have zeros at some of the points $2\pi j/y$, $j \neq 0$. In Definition 4.1 such a system is called the system associated with the function ω.

Let us study the approximation by the associated system $\{\omega_y^*\}$ of the form (3.9) as compared with the approximation by the single function ω. We will return to the proof of Theorem 3.1 to this end. In (3.26), the principal part of the error of the approximation is bounded by the four integrals, I_1, I_2, I_3, I_4. Let us confine ourselves to $h \in \langle H, 1)$ with some positive number H. Then we may readily see that I_1 and I_2 depend only on the behavior of Λ and Λ_y^* in a certain vicinity of the origin. This behavior of Λ and Λ_y^* is rather similar if we assume $q_y(0) \neq 0$. Further, I_4 does not depend on Λ and Λ_y^* at all.

Since the dependence of the error of the approximation on the behavior of Λ is very complex, we will confine ourselves to the study of the integral I_3 in dependence on Λ. This comparison is the subject of Theorem 4.3, the basic result of this section.

In the introductory part of the section we are concerned with some properties of the associated systems in Lemmas 4.1 and 4.2, and study two particular associated systems that allow a relatively simple direct computation of ω_y^* from ω (without the explicit knowledge of Λ) in Theorems 4.1 and 4.2.

The assumptions of Theorem 4.3 are chosen in order to show the role of the zeros of Λ_y^* and their multiplicity in the error bound. Lemmas 4.3, 4.4, and 4.5 that follow the theorem should explain the nature of these rather restrictive assumptions.

C, D, L, and P mean general constants (independent, in particular, of the parameter h) taking different finite positive values at different places throughout this (as well as the preceding) section.

Without repeating it explicitly, we use the notation

$$\mathscr{F}(\omega) = \Lambda\,, \quad \mathscr{F}(\omega_y^*) = \Lambda_y^*$$

wherever we are concerned with a function $\omega \in S$ or a system $\{\omega_y^*\} \subset S$ in this section.

Definition 4.1. *Let a real number $U > 0$ and a function $\omega \in S(R)$ be given. Let $J > 0$, $M > 0$ be integers, let $q_y \in C^\infty(R)$ be a function having zeros at the points $2\pi j/y$, i.e.,*

$$q_y(2\pi j/y) = 0$$

where $|j| = 1, 2, \ldots, J$ and $y \in (0, U)$. Further let

$$|q_y^{(k)}(x)| \leqq C_k|xy|^{\tau_k} + D_k \tag{4.1}$$

for $x \in R$, $y \in (0, U)$ and arbitrary non-negative integer k with some finite constants $C_k > 0$, $D_k > 0$, and $\tau_k \geqq 0$, and

$$|q_y(0)| \geqq Q > 0\,, \quad y \in (0, U)\,. \tag{4.2}$$

The system $\{\omega_y^*\}_{y \in (0,U)} \subset S$ is said to be J, M-associated with the function ω if

$$\Lambda_y^* = q_y^M \Lambda\,. \bullet \tag{4.3}$$

The dependence of q_y on J and that of ω_y^*, Λ_y^* on J and M is not explicitly expressed in our notation but the corresponding values of J, M will be always apparent from context. In such cases, we will also say that a system is associated instead of J, M-associated.

Remark 4.1. From (4.1) we obtain immediately that $\Lambda_y^* \in S$. Moreover, we may write

$$\omega_y^* = \mathscr{F}^{-1}(\Lambda_y^*) = \mathscr{F}^{-1}(q_y^M \Lambda) =$$
$$= \mathscr{F}^{-1}(q_y^M) * \omega = \mathscr{F}^{-1}(q_y) * \ldots * \mathscr{F}^{-1}(q_y) * \omega$$

where the term $\mathscr{F}^{-1}(q_y)$ appears in the convolution M times since it follows from (4.1) that q_y (as well as q_y^M) is a multiplier and $\mathscr{F}^{-1}(q_y)$ (as well as $\mathscr{F}^{-1}(q_y^M)$) is a convolutor (cf. [7] Definition 2.7 and Remarks 2.3 and 2.4). •

The following two lemmas describe some properties of the associated systems $\{\omega_y^*\}$.

Lemma 4.1. *Let a function $\omega \in S(R)$ be given. Let the system $\{\omega_y^*\}_{y \in (0,U)}$ be J, M-associated with the function ω. Let us have a Λ-admissible function $\eta(h)$ (cf. Remark 3.2). Then the function $\eta(h)$ is Λ^*-admissible.*

Proof. Obviously it is sufficient to verify condition 3 of Definition 3.2 for the system $\{\Lambda_y^*\}$. According to our assumption that $\eta(h)$ is Λ-admissible, there exist $z(k, \alpha)$ and $C(\alpha, \gamma)$ such that

$$|\Lambda((x - 2\pi k)/\eta(h))| \leqq C(\alpha, \gamma)\, h^\gamma\, z(k, \alpha)$$

and

$$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} z^2(k, \alpha)\, |k|^{2\alpha} < \infty$$

for arbitrary $\alpha \geqq 0$, $\gamma \geqq 0$, $0 < h < 1$, and $|x| < \pi$.

From Definition 4.1 we have

$$\Lambda_y^*(x) = q_y^M(x)\, \Lambda(x)$$

where q_y satisfies (4.1). Then

$$\begin{aligned}|\Lambda_{\eta(h)}^*((x - 2\pi k)/\eta(h))| &= |q_{\eta(h)}^M((x - 2\pi k)/\eta(h))|\, |\Lambda((x - 2\pi k)/\eta(h))| \leqq \\ &\leqq (C_0|x - 2\pi k|^{\tau_0} + D_0)^M\, |\Lambda((x - 2\pi k)/\eta(h))| \leqq \\ &\leqq C|k|^{M\tau_0}\, |\Lambda((x - 2\pi k)/\eta(h))| \leqq C(\alpha + M\tau_0, \gamma)\, |k|^{M\tau_0}\, h^\gamma\, z(k, \alpha + M\tau_0)\end{aligned}$$

for $|x| < \pi$ where we used (3.3) with $\alpha + M\tau_0$ instead of α. Writing

$$\text{(4.4)} \qquad \begin{aligned} z^*(k, \alpha) &= z(k, \alpha + M\tau_0)\, |k|^{M\tau_0}\,, \\ C^*(\alpha, \gamma) &= C(\alpha + M\tau_0, \gamma)\,, \end{aligned}$$

we finally obtain

$$|\Lambda_{\eta(h)}^*((x - 2\pi k)/\eta(h))| \leqq C^*(\alpha, \gamma)\, h^\gamma\, z^*(k, \alpha)$$

for all integers k, $k \neq 0$, any $\alpha \geqq 0$, $\gamma \geqq 0$, $0 < h < 1$, and $|x| < \pi$. Further, with respect to (3.4) of Definition 3.2 and (4.1) we have

$$\sum_{k \neq 0} z^{*2}(k, \alpha)\, |k|^{2\alpha} = \sum_{k \neq 0} z^2(k, \alpha + M\tau_0)\, |k|^{2(\alpha + M\tau_0)} < \infty\,.$$

Therefore $\eta(h)$ is Λ^*-admissible.

Lemma 4.2. *Let a function $\omega \in S(R)$ satisfy the assumptions of Theorem* 3.1 (*cf. Remark* 3.3). *Let the system $\{\omega_y^*\}_{y \in (0,U)}$ be J, M-associated with the function ω. Then the system $\{\omega_y^*\}$ satisfies the assumptions of Theorem* 3.1, *too.*

Proof. With respect to Lemma 4.1 it is obviously sufficient to verify the existence of constants Γ_0, Γ_{00}, Γ_s in (3.5) to (3.7).

Since

$$\Lambda_y^*(0) = q_y^M(0)\, \Lambda(0)$$

according to Definition 4.1, the relation (3.5) follows immediately from (4.2). Further from (4.1), (4.3)

$$|\Lambda_y^*(x)| \leqq |\Lambda(x)| (C_0|xy|^{\tau_0} + D_0)^M \leqq |\Lambda(x)| (C|x|^{\tau_0} + D_0)^M$$

which gives (3.6) since $\Lambda \in S$ and $0 < y < U$. Finally we may write

$$\Lambda_y^{*(s)}(x) = \sum_{r=0}^{s} \binom{s}{r} \Lambda^{(r)}(x) (q_y^M(x))^{(s-r)}$$

and thus we obtain (3.7) for any positive integer s by the same argument as above. •

The following two theorems present two particular associated systems $\{\omega_y^*\}$ of practical importance since the function ω_y^* can be determined from ω directly, without the explicit knowledge of $\mathscr{F}(\omega)$.

Theorem 4.1. *Let a real number* $U > 0$ *and a function* $\omega \in S(R)$ *be given. Let*

(4.5) $$p_J(t) = \prod_{j=1}^{J} (t^2 - 4\pi^2 j^2) = \sum_{l=0}^{J} b_l t^{2l}$$

be a polynomial of degree $2J$ *with zeros at the points* $2\pi j$; $|j| = 1, \ldots, J$. *Putting*

(4.6) $$q_y(x) = p_J(xy)$$

in Definition 4.1 *we obtain* a *J, M-associated system* $\{\omega_y^*\}_{y \in (0,U)}$,

(4.7) $$\omega_y^*(x) = \sum_{l=0}^{JM} (-1)^l \, d_l y^{2l} \omega^{(2l)}(x)\,,$$

where we use the notation

$$p_J^M(t) = \sum_{l=0}^{JM} d_l t^{2l}\,.$$

Proof. From (4.5), (4.6) we have

$$q_y(x) = \sum_{l=0}^{J} b_l x^{2l} y^{2l} = y^{2J} \prod_{j=1}^{J} (x^2 - 4\pi^2 j^2/y^2)\,,$$

i.e., q_y has zeros at the points $2\pi j/y$, $|j| = 1, \ldots, J$. Apparently (4.1) holds with $\tau_k = \max(2J - k, 0)$ and suitable constants C_k, D_k. Further

$$q_y(0) = \prod_{j=1}^{J} (-4\pi j^2) = Q > 0$$

independently of y, which implies (4.2).

Considering Remark 4.1 and the relation

$$\widetilde{\delta^{(k)}}(x) = i^k x^k$$

where δ is the Dirac function we obtain (4.7). The theorem has been proven since the relation $\{\omega_y^*\} \subset S$ is obvious.

Theorem 4.2. *Let a real number $U > 0$ and a function $\omega \in S(R)$ be given. Putting*

$$q_y(x) = (\tfrac{1}{2}xy)^{-1} \sin \tfrac{1}{2}xy$$

in Definition 4.1 we obtain a J, M-associated system $\{\omega_y^*\}_{y\in(0,U)}$ for an arbitrary integer $J > 0$. Let δ be the central difference with step y, i.e.,

$$\delta f(x) = f(x + \tfrac{1}{2}y) - f(x - \tfrac{1}{2}y),$$
$$\delta^m f(x) = \delta(\delta^{m-1} f(x)).$$

Further let there exist a function ${}^M\omega \in S$ such that

$${}^M\omega^{(M)} = \omega .$$

Then

$$\omega_y^* = y^{-M}\, \delta^M\, {}^M\omega, \quad y \in (0, U) \tag{4.8}$$

for any positive integer M.

Proof. Since

$$q_y(0) = \lim_{x\to 0} (\tfrac{1}{2}xy)^{-1} \sin \tfrac{1}{2}xy = 1, \quad y \in (0, U),$$

q_y has apparently zeros at all the points $2\pi j/y$, $j \neq 0$. Further (4.2) holds with $Q = 1$. Differentiating the function q_y as the product of $\sin \tfrac{1}{2}xy$ and $(\tfrac{1}{2}xy)^{-1}$ and considering the relation

$$|\sin \tfrac{1}{2}xy| \leqq C|x|\, y, \quad x \in R, \quad y \in (0, U),$$

we have

$$|q_y^{(k)}(x)| = \Big|((\tfrac{1}{2}xy)^{-1})^{(k)} \sin \tfrac{1}{2}xy + \sum_{j=1}^{k} \binom{k}{j} ((\tfrac{1}{2}xy)^{-1})^{(k-j)} (\sin \tfrac{1}{2}xy)^{(j)}\Big| \leqq \tag{4.9}$$
$$\leqq Ck!|x|^{-k} + \sum_{j=1}^{k} \binom{k}{j} (k-j)!\, |x|^{j-k-1} (\tfrac{1}{2}y)^{j-1} \leqq C'_k$$

for $|x| \geqq 1$ and $y \in (0, U)$. To obtain an analogous bound for $|x| \leqq 1$, we use the power series expansion for $\sin \tfrac{1}{2}xy$. The series as well as all those obtained by differentiating converge for $x \in R$, $y \in (0, U)$. In particular,

$$|q_y^{(k)}(x)| = \Big|\sum_{j \geqq k/2} \frac{(-1)^j}{(2j+1)(2j-k)!} (\tfrac{1}{2}y)^{2j} x^{2j-k}\Big| \leqq C''_k \tag{4.10}$$

for $|x| \leqq 1$ and $y \in (0, U)$. Thus (4.1) follows from (4.9) and (4.10) with $\tau_k = 0$; $k \geqq 0$.

Direct computation gives

$$\mathscr{F}^{-1}(q_y) = y^{-1}\varphi^{[y^{-1}]}$$

where

$$\varphi(x) = 1\,, \quad |x| \leqq \tfrac{1}{2}\,,$$
$$= 0\,, \quad |x| > \tfrac{1}{2}\,.$$

Moreover, we may compute

$$\mathscr{F}^{-1}(q_y) * g' = y^{-1}\,\delta g\,, \quad g \in S\,.$$

Considering Remark 4.1, we obtain finally (4.8). The relation $\{\omega_y^*\} \subset S$ is obvious.

Remark 4.2. For both the J, M-associated systems defined in Theorems 4.1 and 4.2 we have

$$\lim_{y\to 0} \omega_y^* = C\omega\,, \quad x \in R$$

as an immediate consequence of (4.7) or (4.8). Therefore, in the approximation by the system $\{\omega_y\}$, we can expect no advantages for small h as compared with the approximation by the function ω. ●

The following definition introduces the notation necessary for the study of the role of zeros of Λ_y^* and their multiplicity performed in Theorem 4.3.

Definition 4.2. *Let a function $\omega \in S(R)$ satisfy the assumptions of Theorem* 3.1 (*cf. Remark* 3.3). *Let the system $\{\omega_y^*\}_{y\in(0,U)}$ be J, M-associated with the function ω and let there exist a Λ-admissible function $\eta(h)$ (cf. Remark* 3.2).

Using the notation of the proof of Theorem 3.1, *let us construct the trigonometric polynomials P_h, P_h^* (corresponding to the functions Λ, $\Lambda_{\eta(h)}^*$) according to Lemma* 3.1. *Let $f \in H^\beta(R)$. Let f_h be the function* (3.14) *constructed in the first part of the proof of Theorem* 3.1. *Let us introduce the function ζ_h by* (3.15), (3.21), *and* (3.22), *and the function ζ_h^* by the analogous relation*

$$\zeta_h^*(x) = P_h^*(xh) \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi k/h) = P_h^*(xh)\,(\xi_h(x) + \psi_h(x))\,.$$

Finally we put

$$I(h, f) = \int_{\pi/h<|x|} |\zeta_h(x)|^2\, |\Lambda(xh/\eta(h))|^2\,(1 + |x|^{2\alpha})\,\mathrm{d}x\,,$$

$$I^*(h, f) = \int_{\pi/h<|x|} |\zeta_h^*(x)|^2\, |\Lambda_{\eta(h)}^*(xh/\eta(h))|^2\,(1 + |x|^{2\alpha})\,\mathrm{d}x\,.$$

We will simply write I, I^* wherever it is not ambiguous.

Remark 4.3. The integrals $I(h,f)$, $I^*(h, f)$ correspond to the integral I_3 in the proof of Theorem 3.1. Our notation introduced in Definition 4.2 underlines the dependence of this integral on h and on the function f. On the other hand, the integral $I^*(h, f)$ depends also on the choice of the J, M-associated system $\{\omega_y^*\}$, and, in particular, on the integers J, M.

Theorem 4.3. *Let a function $\omega \in S(R)$ satisfy the assumptions of Theorem 3.1. Let*

$$(4.11) \qquad |\Lambda(x)| > 0\,, \quad x \in R\,.$$

Let there exist a Λ-admissible function $\eta(h)$.

Let $\{\omega_y^*\}_{y\in(0,U)}$ be the system J, M-associated with the function ω for arbitrary positive integers J, M and let

$$(4.12) \qquad |q_y(x)| \leqq 1\,, \quad x \in R$$

where q_y is the function from Definition 4.1. Further let there exist a constant $L(J) > 0$ such that

$$(4.13) \qquad \sup_{x\in\langle -\pi/h,\pi/h\rangle} |\Lambda_{\eta(h)}^{*(M)}((xh - 2\pi j)/\eta(h))| \leqq L^M(J)$$

holds for all integers j, $0 < |j| \leqq J$ and arbitrary positive J, M.

Let $\theta > 0$, $1 > H > 0$ be given. Let $E \subset H^\beta(R)$ be the set of functions satisfying the following two conditions for $H \leqq h < 1$:

1. *There exists a positive constant C such that*

$$(4.14) \qquad I(h, f) \geqq C\|f\|_{L_2(R)}^2$$

for all $f \in E$.

2. *For any non-zero integer k and any $f \in E$, we have*

$$(4.15) \qquad \int_{-\pi/h}^{\pi/h} |\zeta_h^*(x)|^2 (1 + |x - 2\pi k/h|^{2\alpha})\,dx \leqq$$
$$\leqq C(k) \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 (1 + |x - 2\pi k/h|^{2\alpha})\,dx\,,$$

$$(4.16) \qquad \int_{-\pi/h}^{\pi/h} |\zeta_h^*(x)|^2\, |\Lambda_{\eta(h)}^*((xh - 2\pi k)/\eta(h))|^2 (1 + |x - 2\pi k/h|^{2\alpha})\,dx \leqq$$
$$\leqq D(k) \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2\, |\Lambda_{\eta(h)}^*((xh - 2\pi k)/\eta(h))|^2 (1 + |x - 2\pi k/h|^{2\alpha})\,dx$$

where $C(k)$ is a finite positive constant and

$$(4.17) \qquad 0 < D(k) \leqq D|k|^\tau$$

with some constants D, $\tau \geqq 0$.

Then there exist positive integers J_0, M_0 and such a system $\{\omega_y^*\}_{y\in(0,U)}$ J_0, M_0--associated with the function ω that

$$I^*(h,f) \leqq \theta I(h,f)$$

for any $H \leqq h < 1$ and $f \in E$.

Proof. Let us put

$$I_k = I_k(h,f) = \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 \, |\Lambda((xh - 2\pi k)/\eta(h))|^2 \, (1 + |x - 2\pi k/h|^{2\alpha}) \, \mathrm{d}x \,,$$

$$I_k^* = I_k^*(h,f) = \int_{-\pi/h}^{\pi/h} |\zeta_h^*(x)|^2 \, |\Lambda_{\eta(h)}^*((xh - 2\pi k)/\eta(h))|^2 \, (1 + |x - 2\pi k/h|^{2\alpha}) \, \mathrm{d}x$$

for all integers k, $k \neq 0$. Writing the integrals I, I^* as the sum of the integrals from $-2\pi k/h - \pi/h$ to $-2\pi k/h + \pi/h$ and performing a substitution, we obtain

$$I(h,f) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} I_k(h,f) \,, \quad I^*(h,f) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} I_k^*(h,f)$$

since the functions ζ_h, ζ_h^* are periodic with period $2\pi/h$.

The proof of the theorem is divided into two parts. In the first part we will find an integer $J_0 > 0$ such that

$$\Big(\sum_{k=-\infty}^{-J_0-1} + \sum_{k=J_0+1}^{\infty}\Big) I_k^* \leqq \tfrac{1}{2}\theta I \tag{4.18}$$

for an arbitrary $M > 0$, $H \leqq h < 1$, and the functions f satisfying (4.14), (4.16). To this end we will establish the inequality

$$I_k^* \leqq D(k)\, I_k \tag{4.19}$$

for any integer k, $k \neq 0$ and arbitrary $J > 0$, $M > 0$ where the $D(k)$'s are given in (4.16) and fulfill the inequality (4.17). We will use the notation

$$\sum_{|k|=K}^{N} = \sum_{k=-N}^{-K} + \sum_{k=K}^{N}$$

for the sums analogous to those in (4.18).

In the second part we will show that for any $\theta' > 0$ there exists an integer $M_0 > 0$ such that

$$I_k^* \leqq \tfrac{1}{2}\theta' I_k \tag{4.20}$$

for $0 < |k| \leqq J_0$, $H \leqq h < 1$, and the functions f satisfying (4.15). Finally we obtain

$$\sum_{|k|=1}^{J_0} I_k^* \leqq \tfrac{1}{2}\theta I \tag{4.21}$$

which together with (4.18) gives the statement of the theorem.

1. From (4.12) we have

$$|\Lambda_y^*(x)| = |q_y^M(x)|\,|\Lambda(x)| \leqq |\Lambda(x)|\,, \quad x \in R\,, \quad y \in (0, U) \tag{4.22}$$

according to (4.3) of Definition 4.1. Let us estimate

$$D(k)\,I_k - I_k^* = \int_{-\pi/h}^{\pi/h} (D(k)\,|\zeta_h(x)|^2\,|\Lambda((xh - 2\pi k)/\eta(h))|^2 -$$

$$- |\zeta_h^*(x)|^2\,|\Lambda_{\eta(h)}^*((xh - 2\pi k)/\eta(h))|^2)\,(1 + |x - 2\pi k/h|^{2\alpha})\,\mathrm{d}x =$$

$$= D(k) \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2\,(|\Lambda((xh - 2\pi k)/\eta(h))|^2 - |\Lambda_{\eta(h)}^*((xh - 2\pi k)/\eta(h))|^2) \times$$

$$\times (1 + |x - 2\pi k/h|^{2\alpha})\,\mathrm{d}x +$$

$$+ \int_{-\pi/h}^{\pi/h} (D(k)\,|\zeta_h(x)|^2 - |\zeta_h^*(x)|^2)\,|\Lambda_{\eta(h)}^*((xh - 2\pi k)/\eta(h))|^2\,(1 + |x - 2\pi k/h|^{2\alpha})\,\mathrm{d}x \geqq 0$$

since both the integrals are non-negative with respect to (4.16) and (4.22). Therefore the relation (4.19) holds for any integer k, $k \neq 0$, $H \leqq h < 1$ and arbitrary $J > 0$, $M > 0$ with the constants $D(k)$ given in (4.16).

Let us show that

$$\sum_{|k|=1}^{\infty} D(k)\,I_k < \infty\,. \tag{4.23}$$

According to the proof of Theorem 3.1 we have $\zeta_h \in L_2(-\pi/h, \pi/h)$. Further, the bound

$$\|\zeta_h\|_{L_2(-\pi/h,\pi/h)} \leqq C\,\|f\|_{L_2(R)} \tag{4.24}$$

follows from (3.19), (3.20) since the trigonometric polynomial P_h is bounded. Using Definition 3.2 for $H \leqq h < 1$, we obtain

$$|\Lambda((xh - 2\pi k)/\eta(h))| \leqq C(\varrho,\gamma)\,h^\gamma\,z(k, \varrho) \leqq C(\varrho)\,z(k, \varrho) \tag{4.25}$$

for all integers k, $k \neq 0$, any $\varrho \geqq 0$ and $|x| < \pi/h$. Moreover,

$$\sum_{|k|=1}^{\infty} z^2(k, \varrho)\,|k|^{2\varrho} < \infty \tag{4.26}$$

for any ϱ. Further, considering the inequality

$$1 + |x - 2\pi k/h|^{2\alpha} \leqq Ch^{-2\alpha} |k|^{2\alpha} \leqq C|k|^{2\alpha}$$

valid for $k \neq 0$ and $H \leqq h < 1$, and (4.17), (4.25), we may estimate

$$D(k)I_k = D(k)\int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 |\Lambda((xh - 2\pi k)/\eta(h))|^2 (1 + |x - 2\pi k/h|^{2\alpha}) \,\mathrm{d}x \leqq$$

$$\leqq C(\tau, \alpha)\, z^2(k, \tfrac{1}{2}\tau + \alpha)\, |k|^{2\alpha+\tau} \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 \,\mathrm{d}x =$$

$$= C\, z^2(k, \tfrac{1}{2}\tau + \alpha)\, |k|^{2\alpha+\tau} \|\zeta_h\|^2_{L_2(-\pi/h, \pi/h)} .$$

Finally from this and (4.24) we obtain

$$\text{(4.27)} \qquad \sum_{|k|=J}^{\infty} D(k)I_k \leqq C\|f\|^2_{L_2(R)} \sum_{|k|=J}^{\infty} z^2(k, \tfrac{1}{2}\tau + \alpha)\, |k|^{2\alpha+\tau} \leqq$$

$$\leqq C\|f\|^2_{L_2(R)} < \infty$$

with respect to (4.26) for any positive integer J and $H \leqq h < 1$.

Therefore the sum (4.23) converges and we may estimate

$$\sum_{|k|=J}^{\infty} I_k^* \leqq \sum_{|k|=J}^{\infty} D(k)I_k \leqq C\|f\|^2_{L_2(R)} \sum_{|k|=J}^{\infty} z^2(k, \tfrac{1}{2}\tau + \alpha)\, |k|^{2\alpha+\tau}$$

for arbitrary $J > 0$, $M > 0$, and $H \leqq h < 1$ using (4.19), (4.27). Moreover, for an arbitrary $\theta'' > 0$ there exists a $J_0 > 0$ such that

$$\sum_{|k|=J_0+1}^{\infty} I_k^* \leqq \theta'' C\|f\|^2_{L_2(R)}$$

for $H \leqq h < 1$, $M > 0$. Finally we have

$$\sum_{|k|=J_0+1}^{\infty} I_k^* \leqq \theta'' CI \leqq \tfrac{1}{2}\theta I$$

for the functions f satisfying (4.14), (4.16), suitably chosen θ'', $H \leqq h < 1$, and an arbitrary integer $M > 0$. Thus the inequality (4.18) has been proven.

2. In accord with Definition 4.1, let the function $\Lambda_y^* \in S$ have zeros of multiplicity M at the points $2\pi k/y$, $0 < |k| \leqq J_0$. From the Taylor series for $\Lambda^*_{\eta(h)}$ at the point $-2\pi k/\eta(h)$ we obtain

$$\Lambda^*_{\eta(h)}((xh - 2\pi k)/\eta(h)) = \sum_{r=0}^{M-1} \Lambda^{*(r)}_{\eta(h)}(-2\pi k/\eta(h))\,(xh/\eta(h))^r (r!)^{-1} +$$

$$+ \Lambda^{*(M)}_{\eta(h)}((xh\vartheta - 2\pi k)/\eta(h))\,(xh/\eta(h))^M (M!)^{-1}$$

for $0 < |k| \leqq J_0$ and an arbitrary $M > 0$ where $0 < \vartheta < 1$. Since $-2\pi k/\eta(h)$ is a zero of $\Lambda^*_{\eta(h)}$ of multiplicity M we may estimate

$$(4.28)\quad |\Lambda^*_{\eta(h)}((xh - 2\pi k)/\eta(h))| \leqq (\pi/\eta(h))^M (M!)^{-1} \sup_{x \in \langle -\pi/h, \pi/h \rangle} |\Lambda^{*(M)}_{\eta(h)}((xh - 2\pi k)/\eta(h))|$$

for $|x| \leqq \pi/h$, $0 < |k| \leqq J_0$, and an arbitrary $M > 0$.

Let us now show that there exists an integer M_0 such that (4.20) holds. We may use (4.13), (4.28) and write

$$I^*_k \leqq (\pi/\eta(h))^{2M} (M!)^{-2} \sup_{x \in \langle -\pi/h, \pi/h \rangle} |\Lambda^{*(M)}_{\eta(h)}((xh - 2\pi k)/\eta(h))|^2 \times$$
$$\times \int_{-\pi/h}^{\pi/h} |\zeta^*_h(x)|^2 (1 + |x - 2\pi k/h|^{2\alpha})\, dx \leqq$$
$$\leqq C(k) (\pi/\eta(h))^{2M} (M!)^{-2} L^{2M}(J_0) \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 (1 + |x - 2\pi k/h|^{2\alpha})\, dx$$

for the functions f satisfying (4.15) and $0 < |k| \leqq J_0$ with J_0 fixed. Further from (4.11) we obtain

$$|\Lambda((xh - 2\pi k)/\eta(h))| \geqq \hat{C} > 0$$

for $|x| \leqq \pi/h$, $0 < |k| \leqq J_0$, and $H \leqq h < 1$. Therefore

$$\hat{C} \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 (1 + |x - 2\pi k/h|^{2\alpha})\, dx \leqq I_k$$

and, finally, we have

$$(4.29)\qquad I^*_k \leqq C D^{2M} (M!)^{-2} I_k$$

for an arbitrary $M > 0$ where $D = \pi L(J_0)/\eta(H)$ and $C = C(k)/\hat{C}$. With respect to the character of the dependence of the right-hand part of (4.29) on M, there exists an $M_0 > 0$ such that

$$C D^2 M_0 (M_0!)^{-2} \leqq \tfrac{1}{2}\theta'$$

for an arbitrary fixed $\theta' > 0$. Thus (4.20) holds with this M_0 for $H \leqq h < 1$ and $0 < |k| \leqq J_0$. From this the relation (4.21) follows immediately as well as the statement of the theorem according to (4.18).

Remark 4.4. In practical computations it may be advantageous to introduce an associated system $\{\omega^*_y\}$ in a more complex way, namely to consider different multiplicities of the zeros $2\pi j/y$, $|j| = 1, 2, \ldots, J$ of the function Λ^*_y. This may be easily achieved e.g. by constructing the associated system according to Theorem 4.1. On the other hand, the second part of the proof of Theorem 4.3 can apparently be modified in accord with this generalized concept of the associated system.

In our considerations, the nature of which is more or less merely qualitative, we confined ourselves only to the associated systems introduced in Definition 4.1. •

The following two lemmas give the conditions sufficient for fulfilling some of the assumptions of Theorem 4.3.

Lemma 4.3. *Let the assumptions of Theorem* 3.1 *and* (4.11) *be fulfilled for a function* $\omega \in S$. *Let*

$$|P_h(x)| \geqq P > 0\,, \quad x \in R\,, \quad H \leqq h < 1 \tag{4.30}$$

hold for the trigonometric polynomial P_h *constructed in* (3.17) (*cf. Definition* 4.2).

Let $f(z) = f(x + iy)$ *be an entire holomorphic function of complex variable* $z = x + iy$. *For any integer* $N > 0$, *let there exist a finite positive constant* C_N *such that*

$$|f(z)| \leqq C_N(1 + |z|)^{-N} \exp(Q|y|) \tag{4.31}$$

holds in the complex plane with a positive constant Q. *Let the partial function* $f(x) \in H^\beta(R)$.

Then, for a sufficiently small Q, *there exists a positive constant* C *such that*

$$I(h, f) \geqq C\|f\|^2_{L_2(R)} \tag{4.32}$$

for $H \leqq h < 1$.

Proof. We obtain that the support of the function $\mathscr{F}(f)$ is contained in the sphere $|z| \leqq Q$ using the Paley-Wiener theorem (cf. e.g. [10]) and (4.31). Let us suppose $Q < \pi$ and use the notation of the proof of Theorem 3.1. Then we have

$$\xi_h = \mathscr{F}(f_h) = \varphi^{[h]}\mathscr{F}(f)$$

(cf. (3.15)) where $\varphi \in S$ is given in (3.13). Therefore

$$\operatorname{supp} \xi_h = \langle -\pi, \pi \rangle\,. \tag{4.33}$$

Let us estimate the integral $I(h, f)$. For $H \leqq h < 1$, we have

$$\begin{aligned}
I(h, f) &= \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} I_k(h, f) \geqq I_1(h, f) = \\
&= \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 \, |A((xh - 2\pi)/\eta(h))|^2 \, (1 + |x - 2\pi/h|^{2\alpha}) \, \mathrm{d}x \geqq \\
&\geqq C \int_{-\pi/h}^{\pi/h} |P_h(xh)|^2 \left| \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi k/h) \right|^2 \mathrm{d}x \geqq \\
&\geqq C \int_{-\pi/h}^{\pi/h} \left| \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi k/h) \right|^2 \mathrm{d}x
\end{aligned} \tag{4.34}$$

since $I_k(h, f) \geqq 0$ and with respect to (3.19), (4.11), (4.30). Further, considering (4.33) we get

$$\operatorname{supp} \xi_h(x - 2\pi k/h) = \langle 2\pi k/h - \pi/h\,,\quad 2\pi k/h + \pi/h \rangle$$

for any $h < 1$, i.e.,

$$\psi_h(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \xi_h(x - 2\pi k/h) = 0 \tag{4.35}$$

for $|x| \leqq \pi/h$ and $H \leqq h < 1$.

According to (3.12), φ is a continuous function and $\varphi(0) = 1$. Thus for an arbitrary $0 < \Phi < 1$ there exists a positive number X (depending only on α and β) such that

$$|\varphi(x)| > \Phi \quad \text{for} \quad |x| \leqq X\,.$$

Let $Q \leqq X$. Then we have $\operatorname{supp} \xi_h = \langle -X, X \rangle$ instead of (4.33) and, finally, from this and (4.34), (4.35)

$$I(h, f) \geqq C \int_{-\pi/h}^{\pi/h} |\xi_h(x)|^2 \,\mathrm{d}x = C \int_{-X}^{X} |\varphi(xh)|^2 \,|\mathscr{F}(f)(x)|^2 \,\mathrm{d}x \geqq$$

$$\geqq C \int_{-X}^{X} |\mathscr{F}(f)(x)|^2 \,\mathrm{d}x = C \int_{-\infty}^{\infty} |\mathscr{F}(f)(x)|^2 \,\mathrm{d}x = C \|f\|_{L_2(R)}^2$$

for $H \leqq h < 1$. The statement (4.32) of the lemma has been proven.

Lemma 4.4. *Let the assumptions of Theorem* 3.1 *be fulfilled for a function $\omega \in S$, let $\{\omega_y^*\}$ be a J, M-associated system. Let*

$$|P_h(x)| \geqq P > 0\,, \quad x \in R\,, \quad H \leqq h < 1 \tag{4.36}$$

hold for the trigonometric polynomial P_h constructed in (3.17) (*cf. Definition* 4.2).

Then there exist finite positive constants $C(k)$, $D(k)$ satisfying (4.17) *and such that* (4.15), (4.16) *hold independently of J, M for any integer k, $k \neq 0$, an arbitrary $f \in H^\beta(R)$, and $H \leqq h < 1$.*

Proof. Using the notation of the proof of Theorem 3.1, we may write

$$\int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2 \,(1 + |x - 2\pi k/h|^{2\alpha}) \,\mathrm{d}x \geqq \tag{4.37}$$

$$\geqq P^2 \int_{-\pi/h}^{\pi/h} |\xi_h(x) + \psi_h(x)|^2 \,(1 + |x - 2\pi k/h|^{2\alpha}) \,\mathrm{d}x$$

according to (3.21), (4.36). On the other hand, the trigonometric polynomial P_h^* (cf. Definition 4.2) is bounded, therefore

$$\int_{-\pi/h}^{\pi/h} |\zeta_h^*(x)|^2 (1 + |x - 2\pi k/h|^{2\alpha}) \, dx \leqq$$

$$\leqq C \int_{-\pi/h}^{\pi/h} |\xi_h(x) + \psi_h(x)|^2 (1 + |x - 2\pi k/h|^{2\alpha}) \, dx \, .$$

From this and (4.37) we obtain $C(k) = CP^{-2}$ for $k \neq 0$ and $H \leqq h < 1$.

In an analogous way we get $D(k) = CP^{-2}$ for $k \neq 0$ and $H \leqq h < 1$, i.e., (4.17) is fulfilled with $\tau = 0$ and $D = CP^{-2}$. ●

In the preceding two lemmas we supposed that the polynomial P_h has no zeros for $H \leqq h < 1$. We show the existence of such a polynomial in the following lemma. We have to consider the system $\{\Lambda_y\}_{y \in (0,\infty)}$ in this lemma since we choose the value $\eta(H)$ of the Λ-admissible function during the construction of the polynomial P_h. As soon as the value $\eta(H)$ is fixed we find U (as well as the particular system $\{\Lambda_y\}_{y \in (0,U)}$) in accord with (3.1) of Definition 3.2.

Lemma 4.5. *Let $\varepsilon > 0$ and $\beta > \alpha \geqq 0$ be given. Let $\{\Lambda_y\}_{y \in (0,\infty)} \subset S(R)$. Let there exist finite positive constants Γ_{00}, Γ_0, and Γ_s such that* (3.5) *to* (3.7) *hold independently of $y \in (0, \infty)$ for any positive integer s. Let $\eta(h)$ be a Λ-admissible function. Let us choose an $H \in (0, 1)$ and write $\eta(H) = \hat{\eta}$.*

Then there exists a trigonometric polynomial P_h such that (3.17) *holds for $|x| \leqq \leqq \pi/h$ and* (3.18) *for $x \in R$ (cf. Lemma* 3.1). *Moreover, for a sufficiently large $\hat{\eta}$ there exists a constant $P > 0$ such that*

$$|P_h(x)| \geqq P > 0 \tag{4.38}$$

for $x \in R$ and $H \leqq h < 1$.

Proof. We will construct the polynomial P_h of the form (3.37) in the way described in the proof of Lemma 3.1. Using the same notation as there, we put

$$M = 0 \tag{4.39}$$

i.e., $N = B - 1$. Let us write

$$|P_h(x)| = \left| \sum_{k=0}^{B-1} b_k(h) \, e^{ikx} \right| \geqq |b_0(h)| - \sum_{k=1}^{B-1} |b_k(h)| \, . \tag{4.40}$$

We will show that there exists a constant $P > 0$ such that

$$|b_0(h)| - \sum_{k=1}^{B-1} |b_k(h)| \geqq P > 0 \tag{4.41}$$

for $H \leqq h < 1$ and a sufficiently large $\hat{\eta}$. Then we obtain the statement (4.38) of the lemma from (4.40) and (4.41).

Let us follow the proof of Lemma 3.1 and solve the system (3.39) with the coefficients (3.40) by Cramer's rule. Substituting (4.39) into (3.42) and putting $0^0 = 1$, we have

$$\text{(4.42)} \qquad \det({}^r a_{jk}(h)) = (-1)^r \sum_{l_1=0}^{1} \dots \sum_{l_{B-1}=0}^{B-1} i^{l_1} \dots i^{l_{B-1}} \times$$
$$\times \binom{1}{l_1} \dots \binom{B-1}{l_{B-1}} \eta^{l_1-1}(h) \dots \eta^{l_{B-1}-B+1}(h)\, A^{(1-l_1)}_{\eta(h)}(0) \dots A^{(B-1-l_{B-1})}_{\eta(h)}(0) \times$$
$$\times \begin{vmatrix} 0^{l_1} & \dots & (r-2)^{l_1} & r^{l_1} & \dots & (B-1)^{l_1} \\ 0^{l_2} & \dots & (r-2)^{l_2} & r^{l_2} & \dots & (B-1)^{l_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0^{l_{B-1}} & \dots & (r-2)^{l_{B-1}} & r^{l_{B-1}} & \dots & (B-1)^{l_{B-1}} \end{vmatrix};$$

$r = 1, 2, \dots, B$. The determinant on the right-hand part of (4.42) vanishes whenever $l_p = l_s$ for a pair of indices from the set $l_1, \dots, l_{B-1}$. We can find that the term in $\det({}^r a_{jk}(h))$ with the power $\eta^0(h)$ is that with $l_s = s$.

The determinant

$$\begin{vmatrix} 1 & 2 & \dots & B-1 \\ 1 & 2^2 & \dots & (B-1)^2 \\ \vdots & \vdots & & \vdots \\ 1 & 2^{B-1} & \dots & (B-1)^{B-1} \end{vmatrix} = V_{B-1}(1, 2, \dots, B-1) \neq 0$$

corresponds to this term T in the expression (4.42) for the $\det({}^1 a_{jk}(h))$. Formulae (3.5), (4.42) imply the existence of a positive constant D_0'' independent of η and such that $|T| \geqq D_0''$.

Let us choose an arbitrary $\theta > 0$. Considering the inequality

$$\eta(h) \geqq \hat{\eta} \quad \text{for} \quad H \leqq h < 1$$

(following from Definition 3.2) and choosing a sufficiently large $\hat{\eta}$, we obtain that $|\det({}^1 a_{jk}(h)) - T| < \theta$ for $H \leqq h < 1$. Fixing θ in such a way that $D_0'' - \theta = D_0' > 0$, we further have

$$\text{(4.43)} \qquad |\det({}^1 a_{jk}(h))| = |T + \det({}^1 a_{jk}(h)) - T| \geqq$$
$$\geqq |T| - |\det({}^1 a_{jk}(h)) - T| \geqq D_0'.$$

It is $b_0(h) = \det({}^1 a_{jk}(h))/\det(a_{jk}(h))$. Considering (3.41) and (4.43) we find that there exists a constant D_0 such that

$$\text{(4.44)} \qquad |b_0(h)| \geqq D_0 > 0$$

for a sufficiently large $\hat{\eta}$ and $H \leqq h < 1$.

By the same argument we can show that the term in $\det({}^r a_{jk}(h))$ with the power $\eta^0(h)$ is equal to zero for $r > 1$ since the corresponding determinant in (4.42) with

$l_s = s$ vanishes. Finally, for an arbitrary $\theta > 0$, a sufficiently large $\hat{\eta}$, and $H \leqq h < 1$ we get

$$|b_r(h)| \leqq \theta\,, \quad r > 1\,. \tag{4.45}$$

The inequality (4.41) follows from (4.44) and (4.45). The lemma has been proven.

5. A NUMERICAL EXAMPLE

The following simple numerical example illustrates the statements of Sections 3 and 4. We solved the same problem as in [3], [7], i.e., the ordinary equation

$$- u''(x) + cu(x) = f(x)\,, \quad x \in (0, \pi)\,, \quad c > 0 \tag{5.1}$$

with the boundary conditions

$$u'(0) = u'(\pi) = 0 \tag{5.2}$$

and the right-hand part

$$f(x) = - \sin\left(d(x - \tfrac{1}{2}\pi)\right), \quad d > 0\,. \tag{5.3}$$

The exact solution of this problem is

$$u(x) = \frac{1}{d^2 + c} \sin\left(d(x - \tfrac{1}{2}\pi)\right) + \frac{d}{(d^2 + c)\sqrt{c}} \frac{\cos\left(\tfrac{1}{2}\pi d\right)}{\operatorname{ch}\left(\tfrac{1}{2}\pi\sqrt{c}\right)} \operatorname{sh}\left((x - \tfrac{1}{2}\pi)\sqrt{c}\right).$$

Let us solve the problem (5.1) to (5.3) by the finite element method using a universal hill function ω and the systems associated with it. Let us denote the approximate solution of the problem sought in the form (3.9) by $u_{h,\eta}$. Since $f \in H^\beta(0, \pi)$ for any $\beta \geqq 0$ we obtain from Theorem 3.1

$$\|u_{h,\eta} - u\|_{L_2(0,\pi)} \leqq C(\beta, \varepsilon)\, h^{\beta + 2 - \varepsilon} \|f\|_{H^\beta(0,\pi)}$$

for arbitrary $\varepsilon > 0$ in the way analogous to [2]. Employing Theorem 3 3 and denoting the approximate solution of the problem in the form (3.48) by $u_{h,\eta,Y}$, we have finally

$$\sup_{x \in \langle 0,\pi \rangle} |u_{h,\eta,Y}(x) - u(x)| \leqq C(\beta, \varepsilon) h^{\beta + 2 - \varepsilon} \|f\|_{H^\beta(0,\pi)}\,.$$

The universal hill function

$$\omega(x) = e^{-x^2}$$

and the systems associated with it were used for approximation. These systems were constructed according to Theorem 4.1, i.e., we put

$$q_y(x) = p_J(xy)$$

where p_J is a polynomial of degree $2J$ with zeros at the points $2\pi j$; $|j| = 1, \ldots, J$, and

$$\tilde{\omega}_y^* = \Lambda_y^* = q_y^M \Lambda .$$

Because

$$\tilde{\omega}(x) = \Lambda(x) = e^{-x^2/4} \sqrt{\pi} ,$$

the Λ-admissible function $\eta(h)$ of the form (3.44) may be chosen according to Theorem 3.2. The functions ω_Y and $\omega_{y,Y}^*$ of the form (3.46) with $Y(h)$ given in (3.47) were used for actual computation according to Theorem 3.3.

The computation has been carr'ed out in single precision on a Minsk 22 computer with J, M-associated systems for several values of J, M and various values of the parameters η_0, η_1, ε_0, Y_0, Y_1, ε_1 of the functions $\eta(h)$, $Y(h)$ and the parameters c, d of the problem. The system of linear algebraic equations obtained was solved by the Gauss elimination.

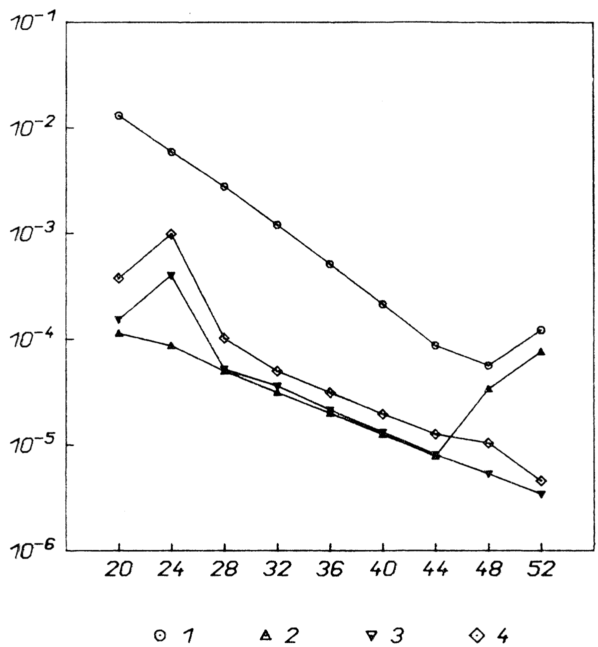

Fig. 5.1.

A typical result is shown in Fig. 5.1 where the scale of the variable $N = \pi/h$ (horizontal) is linear while the scale of the error (vertical) is logarithmic. The actual values of the parameters used in computation of the solution in Fig. 5.1 are: $\eta_0 = 3.8$,

$\eta_1 = 0.5$, $\varepsilon_0 = \varepsilon_1 = 0.0001$, $Y_0 = \frac{11}{6}$, $Y_1 = \frac{2}{3}$, $c = 0.25$, and $d = 3$. The error of the solution is measured by the quantity

$$((N + 1)^{-1} \sum_{n=0}^{N} |u_{h,\eta,Y}(nh) - u(nh)|^2)^{1/2} .$$

Line 1 corresponds to the hill function ω (a "0,0-associated system" in the sense of Definition 4.1), lines 2, 3, and 4 to the 1,1-, 2,1-, and 1,2-associated systems, respectively. It means that line 2 describes the approximation by the function ω_y^* with a simple zero of Λ_y^* at $\pm 2\pi/y$; line 3 that by ω_y^* with simple zeros of Λ_y^* at $\pm\, 2\pi/y$ and $\pm 4\pi/y$, and line 4 the approximation by ω_y^* with a double zero of Λ_y^* at $\pm 2\pi/y$.

The graph shows that the error decreases (as $N \to \infty$) more rapidly than any polynomial of a finite degree in all the cases. For large N, the error increases due to the round-off.

The results confirm that the requirement $\Lambda_y^*(\pm 2\pi/y) = 0$ influences the error very strongly. On the other hand, the results obtained with J, M-associated systems for $J > 1$, $M > 1$ (lines 3 and 4) are very similar to those obtained with the 1,1--associated system (line 2). This means, in accord with our considerations in Sec. 4, that the part of the error influenced by the behavior of Λ_y^* out of the vicinity of the origin is negligible in our example starting from $J = 1$, $M = 1$. The other parts of the error prevail and further zeros of Λ_y^* or their higher multiplicity cannot improve the total error of the approximation.

References

[1] *I. Babuška:* Approximation by hill functions. Comment. Math. Univ. Carolinae 11 (1970), 787—811.

[2] *I. Babuška:* The rate of convergence for the finite element method. SIAM J. Numer. Anal. 8 (1971), 304—315.

[3] *I. Babuška, J. Segethová, K. Segeth:* Numerical experiments with the finite element method I. Tech. Note BN-669, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, August 1970.

[4] *G. Fix, G. Strang:* Fourier analysis of the finite element method in Ritz-Galerkin theory. Studies in Appl. Math. 48 (1969), 265—273.

[5] *J.-L. Lions, E. Magenes:* Problèmes aux limites non homogènes et applications, Vol. 1. Dunod, Paris 1968.

[6] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.

[7] *K. Segeth:* Universal approximation by hill functions. Czechoslovak Math. J. 22 (1972), 612—640.

[8] *K. Segeth:* A remark on a class of universal hill functions. Acta Univ. Carolinae - Math. et Phys. 15 (1974), No. 1—2, to appear.

[9] *G. Strang, G. J. Fix:* An analysis of the finite element method. Prentice-Hall, Englewood Cliffs, N. J. 1973.

[10] *K. Yosida:* Functional analysis. Academic Press, New York—London 1965.

Souhrn

UNIVERZÁLNÍ APROXIMACE SYSTÉMY KOPEČKOVÝCH FUNKCÍ

KAREL SEGETH

Buď $\{\omega_y\}$ systém nekonečně hladkých rychle klesajících funkcí (definice 3.1) a $\eta(h)$ jistá rostoucí funkce, $\eta(0) = 0$ (Λ-přípustná funkce, definice 3.2). Pak je aproximace tvaru

$$\sum_{k=-\infty}^{\infty} c_k \omega_{\eta(h)}((x/h - k)\,\eta(h))$$

univerzální, tj. pro každou aproximovanou funkci f dává systém $\{\omega_y\}$ kopečkových funkcí nejlepší možný řád aproximace omezený pouze hladkostí funkce f (věta 3.1).

Systém $\{\omega_y\}$ lze vybrat tak, aby Fourierova transformace funkce ω_y měla kořeny v bodech $\pm 2\pi j/y$; $j = 1, \ldots, J$, kde J je jisté přirozené číslo (definice 4.1, věty 4.1 a 4.2). V důsledku toho se nepřesnost aproximace zmenší (věta 4.3).

Numerické výsledky potvrzují správnost uvedených tvrzení.

Author's address: RNDr. *Karel Segeth*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.