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BOUNDARY VALUE PROBLEMS FOR THE MILDLY NON-LINEAR ORDINARY DIFFERENTIAL EQUATION OF THE FOURTH ORDER

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1. INTRODUCTION

In this paper, the finite difference method is applied to a boundary value problem for the mildly non-linear ordinary differential equation of the fourth order. The existence of a unique solution of both the differential and the difference problems is proved and an $0(h^2)$ estimate of the discretization error and its first difference quotient is derived. Some numerical examples are given.

The same method has been used in [9] for linear and in [4] for mildly non-linear boundary value problems of the second order. The linear boundary value problem of the fourth order have been considered in [1], [3], [7]. In [7] an approach similar to that used in this paper has been briefly mentioned. [1] have used the estimate of the discrete Dirichlet formula in somewhat different way. [3] deals in addition with discontinuous coefficients, using the discrete Green function for the error estimate.

The present paper is a part of the author's thesis [6].

2. DIFFERENTIAL EQUATION

Let us consider the mildly non-linear boundary value problem of the fourth order

$$(1) \quad L\,y \equiv (p(x)\,y''(x))'' - (q(x)\,y'(x))' + r(x)\,y(x) = f(x, y(x))\,, \quad x \in \langle a, b\rangle\,,$$

$$(2) \qquad y(a) = y'(a) = y(b) = y'(b) = 0$$

under the following assumptions:

$$(3) \qquad p(x) \geqq m > 0\,, \quad q(x) \geqq 0\,, \quad r(x) \geqq 0\,, \quad x \in \langle a, b\rangle\,,$$

(4) $p'''(x)$, $q''(x)$, $r'(x)$ satisfy Lipschitz condition in $\langle a, b\rangle$,

(5) $f(x, y(x))$ has continuous partial derivatives up to the second order inclusive for $x \in \langle a, b\rangle$, $y \in (-\infty, \infty)$,

(6) $f_y(x, y(x)) \leqq \alpha < \lambda_1$ for $x \in \langle a, b\rangle$, $y \in (-\infty, \infty)$ where λ_1 is the smallest eigenvalue of $L\, y(x) = \lambda\, y(x)$ with boundary conditions (2).

Denote

$$\|y\|^2_{L_2} = \int_a^b y^2(x)\, dx ,$$

$$\|y\|^2_2 = \int_a^b [y''^2(x) + y'^2(x) + y^2(x)]\, dx .$$

For any sufficiently smooth function $y(x)$ satisfying (2) it holds

(7) $$\int_a^b y''^2(x)\, dx \geqq K_1^{-1}\, \|y\|^2_2$$

where

$$K_1 = 1 + \tfrac{1}{2}(b - a)^2 \left(1 + \tfrac{1}{2}(b - a)^2\right)$$

and, further,

(8) $$\max_{a \leqq x \leqq b} |y(x)| \leqq \sqrt{(b - a)}\, \|y\|_2 ,$$

(9) $$\max_{a \leqq x \leqq b} |y'(x)| \leqq \sqrt{(b - a)}\, \|y\|_2 .$$

Inequalities (8), (9) are inequalities of Sobolev type. They can be proved in a similar way as their discrete analogues introduced later.

Denote $f_0(x) = f(x, 0)$. From the mean value theorem, it follows

$$f(x, y(x)) = f(x, y) - f_0(x) + f_0(x) = y(x) \int_0^1 f_y(x, \xi y)\, d\xi + f_0(x) .$$

We substitute it into (1), multiply the equation by $y(x)$ and integrate. From (6) and Schwarz inequality we get

$$\int_a^b y(x)\, L\, y(x)\, dx = \int_a^b y^2(x) \int_0^1 f_y(x, \xi y)\, d\xi\, dx + \int_a^b f_0(x)\, y(x)\, dx \leqq$$

$$\leqq \alpha \int_a^b y^2(x)\, dx + \|f_0\|_{L_2}\, \|y\|_{L_2} .$$

If $y(x)$ is a solution of (1), (2) and λ_1 the smallest eigenvalue of $Ly = \lambda y$ and (2), then

$$\lambda_1 \leqq \frac{\int_a^b y(x)\, L\, y(x)\, \mathrm{d}x}{\int_a^b y^2(x)\, \mathrm{d}x}\,,$$

hence

$$\left(1 - \frac{\alpha}{\lambda_1}\right) \int_a^b y(x)\, L\, y(x)\, \mathrm{d}x \leqq \|f_0\|_{L_2} \|y\|_{L_2} \leqq \|f_0\|_{L_2} \|y\|_2 \,.$$

Integrating by parts and using (3) we get on the other hand

$$\int_a^b y(x)\, L\, y(x)\, \mathrm{d}x = \int_a^b (p(x)\, y''^2(x) + q(x)\, y'^2(x) + r(x)\, y^2(x))\, \mathrm{d}x \geqq$$

$$\geqq m \int_a^b y''^2(x) \mathrm{d}x \,.$$

From the last two inequalities and from (7) it follows

$$\left(1 - \frac{\alpha}{\lambda_1}\right) m\, K_1^{-1} \|y\|_2^2 \leqq \left(1 - \frac{\alpha}{\lambda_1}\right) \int_a^b y(x)\, L\, y(x)\, \mathrm{d}x \leqq \|f_0\|_{L_2} \|y\|_2$$

which gives

(10) $$\|y\|_2 \leqq K_1/(m(1 - \alpha/\lambda_1))\, \|f_0\|_{L_2} \,.$$

From (8), (9), (10) we get

(11) $$\max_{a \leqq x \leqq b} |y(x)| \leqq K_2 \|f_0\|_{L_2}\,, \quad \max_{a \leqq x \leqq b} |y'(x)| \leqq K_2 \|f_0\|_{L_2}$$

where

$$K_2 = K_1 \sqrt{(b - a)}/(m(1 - \alpha/\lambda_1))\,.$$

Let us suppose that there exist two solutions $y_1(x)$, $y_2(x)$ of the problem (1), (2). Then their difference $z(x) = y_1(x) - y_2(x)$ is a solution of a similar problem

$$Lz = f(x, y_1(x)) - f(x, y_2(x))\,, \quad z(a) = z'(a) = z(b) = z'(b) = 0\,.$$

By the mean value theorem,

$$f(x, y_1(x)) - f(x, y_2(x)) = f_y(x, \xi y_1 + (1 - \xi)\, y_2)\, (y_1 - y_2) = F(x, z(x))$$

and $F_0(x) = F(x, 0) = 0$. According to (11) it holds

$$\max_{a \leqq x \leqq b} |z(x)| \leqq K_2 \|F_0\|_{L_2} = 0\,,$$

i.e., problem (1), (2) has at most one solution.

In [8], certain necessary conditions for the existence of a solution of mildly non-linear elliptic boundary value problems are stated. As the problem (1), (2) fulfils these conditions, there exists the unique solution $y(x)$. Assumptions (4), (5) imply that $y^{(5)}(x)$ satisfies Lipschitz condition in $\langle a, b\rangle$.

3. FINITE DIFFERENCE APPROXIMATION

Let N be an integer, $h = (b - a)/N$, $x_i = a + ih$, i integer, $g_i = g(x_i)$, $g_x(x_i) = (g_{i+1} - g_i)/h$, $g_{\bar{x}}(x_i) = (g_i - g_{i-1})/h$, $g_{x\bar{x}}(x_i) = (g_{i+1} - 2g_i + g_{i-1})/h^2$, $g_{\mathring{x}}(x_i) = (g_{i+1} - g_{i-1})/(2h)$ for any function $g(x)$.

Define $\|.\|_{2,h}$ by

$$\|Y\|_{2,h}^2 = h \sum_{i=1}^{N-1} Y_{x\bar{x}}^2(x_i) + h \sum_{i=0}^{N-1} Y_x^2(x_i) + h \sum_{i=0}^{N} Y_i^2 . \tag{12}$$

In the sequel the index h will be omitted, if no misunderstanding can arise.

Lemma. *If* $Y_0 = Y_N = 0$, *then*

$$h \sum_{i=1}^{N-1} Y_{x\bar{x}}(x_i) \geqq K_3\|Y\|_2^2 + K_4 \frac{Y_1^2 + Y_{N-1}^2}{h^2} , \tag{13}$$

$$\max_{0 \leqq i \leqq N} |Y_i| \leqq \sqrt{(b - a)}\, \|Y\|_2 , \tag{14}$$

$$\max_{0 \leqq i \leqq N} |Y_x(x_i)| \leqq \sqrt{(b - a)}\, \|Y\|_2 + h^{-1} Y_1 , \tag{15}$$

where

$$K_3 = \tfrac{1}{2}(1 + 3(b - a)^2 (1 + \tfrac{1}{2}(b - a)^2))^{-1} , \quad K_4 = \tfrac{1}{4}(b - a)^{-1}$$

are positive constants independent of h.

Proof. As $Y_i = h \sum_{j=0}^{i-1} Y_x(x_j) + Y_0$ for $i = 1, 2, \ldots, N$, it follows by Schwarz inequality

$$Y_i^2 = (h \sum_{j=0}^{i-1} Y_x(x_j))^2 \leqq h \sum_{j=0}^{i-1} 1^2 h \sum_{j=0}^{i-1} Y_x^2(x_j) \leqq ih \sum_{j=0}^{N-1} Y_x^2(x_j) .$$

Multiplying by h and adding we obtain

$$h \sum_{i=0}^{N} Y_i^2 \leqq h \sum_{i=1}^{N-1} ih^2 \sum_{j=0}^{N-1} Y_x^2(x_j) = h^2 . \tfrac{1}{2}N(N - 1)\, h \sum_{i=0}^{N-1} Y_x^2(x_j) \leqq \tag{16}$$

$$\leqq \tfrac{1}{2}(b - a)^2\, h \sum_{j=0}^{N-1} Y_x^2(x_i) .$$

Similarly, from

$$Y_x(x_i) = h\sum_{j=1}^{i} Y_{x\bar{x}}(x_j) + Y_x(x_0)$$

and

$$Y_x(x_i) = -h\sum_{j=i+1}^{N-1} Y_{x\bar{x}}(x_j) + Y_x(x_{N-1}),$$

using the obvious inequality $(c \pm d)^2 \leqq 2(c^2 + d^2)$, we get

$$Y_x^2(x_i) \leqq 2(h\sum_{j=1}^{i} Y_{x\bar{x}}(x_j))^2 + 2h^{-2}Y_1^2,$$

$$Y_x^2(x_i) \leqq 2(h\sum_{j=i+1}^{N-1} Y_{x\bar{x}}(x_j))^2 + 2h^{-2}Y_{N-1}^2.$$

Adding and using Schwarz inequality we further get

$$Y_x^2(x_i) \leqq h\sum_{j=1}^{i} Y_{x\bar{x}}^2(x_j)\, h\sum_{j=1}^{i} 1 + h\sum_{j=i+1}^{N-1} Y_{x\bar{x}}^2(x_j)\, h\sum_{j=i+1}^{N-1} 1 + \frac{1}{h^2}(Y_1^2 + Y_{N-1}^2) \leqq$$

$$\leqq (b-a)\,h\sum_{j=1}^{N-1} Y_{x\bar{x}}^2(x_j) + \frac{1}{h^2}(Y_1^2 + Y_{N-1}^2)$$

and from here

$$h\sum_{i=1}^{N-1} Y_x^2(x_i) \leqq (b-a)^2\,h\sum_{i=1}^{N-1} Y_{x\bar{x}}^2(x_i) + (b-a)\frac{1}{h^2}(Y_1^2 + Y_{N-1}^2). \tag{17}$$

We use also the identities

$$\frac{Y_1}{h} = h\sum_{i=1}^{N-1}\frac{i-N}{N}Y_{x\bar{x}}(x_i), \quad \frac{Y_{N-1}}{h} = -h\sum_{i=1}^{N-1}\frac{i}{N}Y_{x\bar{x}}(x_i)$$

valid for $Y_0 = Y_N = 0$. From here it follows by Schwarz inequality

$$\frac{Y_1^2}{h^2} = \left(h\sum_{i=1}^{N-1}\frac{i-N}{N}Y_{x\bar{x}}(x_i)\right)^2 \leqq h\sum_{i=1}^{N-1}\left(\frac{i-N}{N}\right)^2 h\sum_{i=1}^{N-1} Y_{x\bar{x}}^2(x_i) \leqq$$

$$\leqq (b-a)\,h\sum_{i=1}^{N-1} Y_{x\bar{x}}^2(x_i),$$

which together with the similar inequality for Y_{N-1}^2/h^2 yields

$$\frac{Y_1^2}{h^2} + \frac{Y_{N-1}^2}{h^2} \leqq 2(b-a)\,h\sum_{i=1}^{N-1} Y_{x\bar{x}}^2(x_i). \tag{18}$$

From (16)–(18) we have

$$\|Y\|_2^2 \leqq \left(1 + 3(b-a)^2\left(\tfrac{1}{2}(b-a)^2 + 1\right)\right) h \sum_{i=1}^{N-1} Y_{x\bar{x}}^2(x_i)$$

and from here and (18) it follows (13).

The inequalities (14), (15) follow by Schwarz inequality from

$$Y_i = h \sum_{j=1}^{i-1} Y_x(x_j) + Y_0 \quad \text{and} \quad Y_x(x_i) = h \sum_{j=1}^{i} Y_{x\bar{x}}(x_j) + Y_x(x_0),$$

respectively.

Let us define the linear difference operator L_h by

$$\begin{aligned} L_h Y_i &= (p_i Y_{x\bar{x}}(x_i))_{x\bar{x}} - \tfrac{1}{2}(q_i Y_x(x_i))_{\bar{x}} - \tfrac{1}{2}(q_i Y_{\bar{x}}(x_i))_x + r_i Y_i = \\ &= h^{-4}(p_{i-1}Y_{i-2} - 2(p_{i-1} + p_i) Y_{i-1} + (p_{i-1} + 4p_i + p_{i+1}) Y_i - \\ &\quad - 2(p_i + p_{i+1}) Y_{i+1} + p_{i+1}Y_{i+2}) + h^{-2}(-\tfrac{1}{2}(q_{i-1} + q_i) Y_{i-1} + \\ &\quad + (\tfrac{1}{2}q_{i-1} + q_i + \tfrac{1}{2}q_{i+1}) Y_i - \tfrac{1}{2}(q_i + q_{i+1}) Y_{i+1}) + r_i Y_i . \end{aligned}$$

We approximate the differential problem (1), (2) by the system of non-linear difference equations

$$(19) \qquad L_h Y_i = f(x_i, Y_i), \quad i = 1, 2, \ldots, N-1,$$

$$(20) \qquad Y_0 = 0, \quad Y_{-1} = 3Y_1 - \tfrac{1}{2}Y_2, \quad Y_N = 0, \quad Y_{N+1} = 3Y_{N-1} - \tfrac{1}{2}Y_{N-2}.$$

We first prove the existence of a unique solution of (19) and (20).

We multiply $L_h Y_i$, $i = 1, 2, \ldots, N-1$ by Y_i and by h and add. By Green's difference formulas

$$\begin{aligned} h \sum_{i=1}^{N-1} U_i V_x(x_i) &= -h \sum_{i=1}^{N-1} U_{\bar{x}}(x_i) V_i - U_0 V_1 + U_{N-1} V_N, \\ h \sum_{i=1}^{N-1} U_i V_{\bar{x}}(x_i) &= -h \sum_{i=1}^{N-1} U_x(x_i) V_i - U_1 V_0 + U_N V_{N-1}, \\ h \sum_{i=1}^{N-1} U_i V_{x\bar{x}}(x_i) &= h \sum_{i=1}^{N-1} U_{x\bar{x}}(x_i) V_i + h^{-1}(U_1 V_0 - U_0 V_1 + U_{N-1} V_N - U_N V_{N-1}) \end{aligned}$$

we get (with respect to (20))

$$\begin{aligned} h \sum_{i=1}^{N-1} Y_i L_h Y_i &= h \sum_{i=1}^{N-1} (p_i Y_{x\bar{x}}^2(x_i) + \tfrac{1}{2} q_i Y_x^2(x_i) + \tfrac{1}{2} q_i Y_{\bar{x}}^2(x_i) + r_i Y_i^2) + \\ &\quad + h^{-3}(p_0 Y_1(4Y_1 - \tfrac{1}{2}Y_2) + p_N Y_{N-1}(4Y_{N-1} - \tfrac{1}{2}Y_{N-2})) + \\ &\quad + (2h)^{-1}(q_0 Y_1^2 + q_N Y_{N-1}^2). \end{aligned}$$

There exists $h_0 > 0$ such that for $h \leqq h_0$ it holds

$$h \,.\, \tfrac{1}{2} p_1 Y_{x\bar{x}}^2(x_1) + h^{-3} p_0 Y_1 (4Y_1 - \tfrac{1}{2} Y_2) = h^{-3}\big(p_0\big(\tfrac{1}{2}(\tfrac{5}{2} Y_1 - Y_2)^2 + \tfrac{15}{8} Y_1^2\big) +$$

$$+ \tfrac{1}{2} h p'(x_0 + \Theta_1 h)(Y_2 - 2Y_1)^2\big) \geqq 0\,,$$

$$h^{-3}\big(p_N\big(\tfrac{1}{2}(\tfrac{5}{2} Y_{N-1} - Y_{N-2})^2 + \tfrac{15}{8} Y_{N-1}^2\big) - \tfrac{1}{2} h p'(x_N - \Theta_2 h)(Y_{N-2} - Y_{N-1})^2\big) \geqq 0$$

where $0 < \Theta_1, \Theta_2 < 1$.

Omitting the non-negative terms containing q_i, r_i and taking account of $p_i \geqq m$ we get the estimate of $h \sum_{i=1}^{N-1} Y_i L_h Y_i$ from below

$$h \sum_{i=1}^{N-1} Y_i L_h Y_i \geqq \tfrac{1}{2} m h \sum_{i=1}^{N-1} Y_{x\bar{x}}^2(x_i) + m h^{-3}(Y_1^2 + Y_{N-1}^2)\,. \tag{21}$$

If the right-hand side of (19) does not depend on Y_1 then (19), (20) is a system of linear equations the matrix of which is, according to (21), positive definite and therefore (19), (20) has a unique solution.

In the non-linear case we can again write

$$f(x_i, Y_i) = Y_i \int_0^1 f_y(x_i, \xi Y_i)\, d\xi + f_0(x_i)\,.$$

Let Λ_1 be the smallest eigenvalue of the matrix eigenvalue problem $L_h U_i = \Lambda U_i$, $i = 1, \ldots, N-1$ and (20). If Y is a solution of (19), (20), then

$$\Lambda_1 \leqq h \sum_{i=1}^{N-1} Y_i L_h Y_i / \big(h \sum_{i=1}^{N-1} Y_i^2\big)\,.$$

It can be proved (see [2], [5], [6]) that for $h > 0$ sufficiently small it holds

$$|\lambda_1 - \Lambda_1| = \Theta(h^2) \tag{22}$$

and so we may again assume $\Lambda_1 < \alpha$.

Let us multiply (19) by h and Y_i and add.

We obtain

$$h \sum_{i=1}^{N-1} Y_i L_h Y_i = h \sum_{i=1}^{N-1} Y_i f(x_i, Y_i) = h \sum_{i=1}^{N-1} Y_i^2 \int_0^1 f_y(x, \xi Y)\, d\xi + h \sum_{i=1}^{N-1} Y_i f_0(x_i) \leqq$$

$$\leqq \alpha h \sum_{i=1}^{N-1} Y_i^2 + h \sum_{i=1}^{N-1} Y_i f_0(x_i) \leqq \alpha / \Lambda_1 h \sum_{i=1}^{N-1} Y_i L_h Y_i +$$

$$+ \max_{1 \leqq i \leqq N-1} |f_0(x_i)| \big(h \sum_{i=1}^{N-1} 1\big)^{1/2} \big(h \sum_{i=1}^{N-1} Y_i^2\big)^{1/2}\,,$$

i.e.,

$$(1 - \alpha/\Lambda_1)\, h \sum_{i=1}^{N-1} Y_i L_h Y_i \leqq \max_{1 \leqq i \leqq N-1} |f_0(x_i)| \sqrt{(b - a)}\, \|Y\|_2 \,,$$

which leads together with (21) and (13) to the estimate

$$\|Y\|_2 \leqq K_5 (b - a)^{-1/2} \max_{1 \leqq i \leqq N-1} |f_0(x_i)| \,.$$

Further, by (14) we have

$$\max_{0 \leqq i \leqq N} |Y_i| \leqq K_5 \max_{1 \leqq i \leqq N-1} |f_0(x_i)| \tag{23}$$

where $K_5 = 2m^{-1} K_3^{-1}(1 - \alpha/\Lambda_1)^{-1} (b - a)$ is a positive constant independent of h.

If (19), (20) has two solutions Y^1, Y^2, their difference $Y^1 - Y^2$ would be a solution of the system of difference equations of the same type. Its right-hand side is

$$\begin{aligned} F(x_i, Y_i^1 - Y_i^2) &= f(x_i, Y_i^1) - f(x_i, Y_i^2) = \\ &= f_y(x_i, \xi_i Y_i^1 + (1 - \xi_i)\, Y_i^2)\,(Y_i^1 - Y_i^2)\,, \\ i &= 1, 2, \ldots, N - 1, \ \ \xi_i \in \langle 0, 1 \rangle, \end{aligned}$$

i.e., $F_0(x) = 0$ and therefore it follows from (23) that $Y^1 \equiv Y^2$, i.e., (19), (20) has at most one solution.

Let Ω be the domain of all mesh functions Y satisfying (20) such that

$$\max_{0 \leqq i \leqq N} |Y_i| = K_6 \,, \quad K_6 = K_5 \max_{1 \leqq i \leqq N-1} |f_0(x_i)| \,.$$

Let T be the mapping defined in Ω by $TY = V$, where V is a solution of the system of linear difference equations

$$L_h V_i = V_i \int_0^1 f_y(x_i, \xi Y_i)\, d\xi + f_0(x_i)\,, \quad i = 1, 2, \ldots, N - 1$$

satisfying the boundary conditions (20). From the estimates of $h \sum_{i=1}^{N-1} V_i L_h V_i$ it can be deduced that the matrix of this system is positive definite, so it has just one solution V and it holds

$$\max_{0 \leqq i \leqq N} |V_i| = \max_{0 \leqq i \leqq N} |(TY)_i| \leqq K_6 \,,$$

i.e., T maps Ω into itself.

For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any two mesh functions Y^1, $Y^2 \in \Omega$ such that $\max_{0 \leqq i \leqq N} |Y_i^1 - Y_i^2| < \delta$ it holds

$$\left| \int_0^1 f_y(x_i, \xi Y_i^1)\, d\xi - \int_0^1 f_y(x_i, \xi Y_i^2)\, d\xi \right| < \varepsilon (K_5 K_6)^{-1} \,.$$

Let $V = V^1 - V^2$ be the solution of the equation

$$L_hV_i = L_hV_i^1 - L_hV_i^2 = V_i^1 \int_0^1 f_y(x_i, \xi Y_i^1)\,\mathrm{d}\xi + f_0(x_i) - V_i^2 \int_0^1 f_y(x_i, \xi Y_i^2)\,\mathrm{d}\xi - f_0(x) =$$

$$= (V_i^1 - V_i^2) \int_0^1 f_y(x_i, \xi Y_i^1)\,\mathrm{d}\xi + V_i^2 \int_0^1 (f_y(x_i, \xi Y_i^1) - f_y(x_i, \xi Y_i^2))\,\mathrm{d}\xi\,,$$

$i = 1, 2, \ldots, N - 1$, satisfying (20).

By (23) we get

$$\max_{0 \leqq i \leqq N} |V_i| \leqq K_5 \max_{0 \leqq i \leqq N} \left| V_i^2 \int_0^1 (f_y(x_i, \xi Y_i^1) - f_y(x_i, \xi Y_i^2))\,\mathrm{d}\xi \right| <$$

$$< K_5 \varepsilon K_5^{-1} K_6^{-1} \max_{0 \leqq i \leqq N} |V_i^2| \leqq \varepsilon K_6^{-1} K_6 = \varepsilon\,.$$

i.e., $\max_{0 \leqq i \leqq N} |(TY^1)_i - (TY^2)_i| < \varepsilon$, i.e., the mapping T is continuous.

By Brouwer fixed point theorem there exists $Y \in \Omega$ such that $TY = Y$. According to the definition of T, Y is a solution of (19), (20).

4. DISCRETIZATION ERROR

Theorem. *Let $y(x)$ be the solution of (1), (2), Y_i, $i = -1, 0, \ldots, N + 1$ the solution of (19), (20) and let the assumptions (3)–(6) be fulfilled. Then for a sufficiently small $h > 0$ it holds*

$$\max_{0 \leqq i \leqq N} |y(x_i) - Y_i| \leqq K_7 h^2\,, \tag{24}$$

$$\max_{0 \leqq i \leqq N} \left| y'(x_i) - \frac{Y_{i+1} - Y_{i-1}}{2h} \right| \leqq K_8 h^2\,, \tag{25}$$

where K_7, K_8 are positive constants independent of h.

Proof. Set $y_{-1} = 3y_1 - \frac{1}{2}y_2$, $y_{N+1} = 3y_{N-1} - \frac{1}{2}y_{N-2}$. Then $L_h y_i - L y_i = R_i$, where $R_i = O(h^2)$ for $i = 2, 3, \ldots, N - 2$, $R_1 = O(1)$, $R_{N-1} = O(1)$ as it can be easily seen by Taylor expansion at the points x_i if $i = 2, \ldots, N - 2$, at x_0 if $i = 1$ and at x_N if $i = N - 1$. The mesh function $E_i = y_i - Y_i$, $i = -1, 0, \ldots, N + 1$ is a solution of the system of non-linear difference equations

$$L_h E_i = R_i + f_y(x_i, \xi_i y_i + (1 - \xi_i) Y_i) E_i\,, \quad i = 1, \ldots, N - 1,\ \xi_i \in \langle 0, 1 \rangle\,, \tag{26}$$

$$E_0 = 0\,, \quad E_{-1} = 3E_1 - \tfrac{1}{2}E_2\,, \quad E_N = 0\,, \quad E_{N+1} = 3E_{N-1} - \tfrac{1}{2}E_{N-2} \tag{27}$$

as it holds

$$L_h E_i = L_h y_i - L_h Y_i = L_h y_i - L y_i + L y_i - L_h Y_i = R_i + f(x_i, y_i) - f(x_i, Y_i)$$

for $i = 1, 2, \ldots, N - 1$.

We now estimate the expression $h \sum_{i=1}^{N-1} E_i L_h E_i$. In the same way as (21) we get

$$h \sum_{i=1}^{N-1} E_i L_h E_i \geqq \tfrac{1}{2} m h \sum_{i=1}^{N-1} E_{x\bar{x}}^2(x_i) + m h^{-3}(E_1^2 + E_{N-1}^2)$$

and further, with respect to (13)

$$h \sum_{i=1}^{N-1} E_i L_h E_i \geqq \frac{1}{2} m K_3 \|E\|_2^2 + \left(\frac{m}{h} + K_4\right) \frac{E_1^2 + E_{N-1}^2}{h^2} . \tag{28}$$

On the other hand, we have

$$h \sum_{i=1}^{N-1} E_i L_h E_i = h \sum_{i=1}^{N-1} E_i R_i + h \sum_{i=1}^{N-1} E_i^2 f_y(x_i, \xi_i y_i + (1 - \xi_i) Y_i) \leqq$$

$$\leqq h \sum_{i=1}^{N-1} E_i R_i + h\alpha \sum_{i=1}^{N-1} E_i^2 \leqq h \sum_{i=1}^{N-1} E_i R_i + \frac{\alpha}{\Lambda_1} h \sum_{i=1}^{N-1} E_i L_h E_i .$$

By the inequality $\varphi\psi \leqq \frac{1}{2}\varepsilon\varphi^2 + \frac{1}{2}\varepsilon^{-1}\psi^2$ valid for any φ, ψ and any $\varepsilon > 0$ we get from here

$$\left(1 - \frac{\alpha}{\Lambda_1}\right) h \sum_{i=1}^{N-1} E_i L_h E_i \leqq \frac{E_1}{h} O(h^2) + \frac{E_{N-1}}{h} O(h^2) + h \sum_{i=2}^{N-2} E_i \, O(h^2) \leqq$$

$$\leqq \frac{\varepsilon}{2h^2} (E_1^2 + E_{N-1}^2) + \frac{1}{2\varepsilon} O(h^4) + \frac{\varepsilon}{2} h \sum_{i=2}^{N-2} E_i^2 + \frac{1}{2\varepsilon} h \sum_{i=2}^{N-2} O(h^4)$$

which together with (28) gives

$$A\|E\|_2^2 + B \frac{E_1^2 + E_{N-1}^2}{h^2} \leqq \frac{1}{2\varepsilon} O(h^4) \tag{29}$$

where

$$A = \tfrac{1}{2}((1 - \alpha/\Lambda_1) m K_3 - \varepsilon) ,$$
$$B = (1 - \alpha/\Lambda_1) m(h^{-1} + K_4) - \tfrac{1}{2}\varepsilon .$$

We can choose $\varepsilon > 0$ independent of h such that A, B are positive and independent of h. Therefore (29) implies

$$\|E\|_2 = O(h^2) , \quad E_1 = O(h^3) , \quad E_{N-1} = O(h^3) \tag{30}$$

and from here by (14) and (15)

$$\max_{0 \leqq i \leqq N} |E_i| = \max_{0 \leqq i \leqq N} |y_i - Y_i| = O(h^2),$$

$$\max_{0 \leqq i \leqq N} |E_x(x_i)| = \max_{0 \leqq i \leqq N} |y_x(x_i) - Y_x(x_i)| = O(h^2).$$

As $y'(x_i) - y_{\hat{x}}(x_i) = O(h^2)$, we have

$$y'(x_i) - Y_{\hat{x}}(x_i) = y'(x_i) - y_{\hat{x}}(x_i) + y_{\hat{x}}(x_i) - Y_{\hat{x}}(x_i) =$$
$$= O(h^2) + \tfrac{1}{2}E_x(x_i) + \tfrac{1}{2}E_{\bar{x}}(x_i) = O(h^2)$$

and therefore also

$$\max_{0 \leqq i \leqq N} |y'(x_i) - Y_{\hat{x}}(x_i)| = O(h^2).$$

Remark 1. Let the boundary conditions be

(2′) $\quad y(a) = y(b) = 0\,, \quad y''(a) = \gamma\, y'(a)\,, \quad y''(b) = -\beta\, y'(b)\,, \quad \gamma \geqq 0, \beta \geqq 0\,.$

Denote by (6′) the assumption (6) in which (2) is replaced by (2′). Let the approximation of (2′) be chosen in the following way:

(20) $$Y_0 = Y_N = 0\,, \quad Y_{-1} = Y_1(-1 + \gamma h - \tfrac{1}{2}\gamma^2 h^2)\,, \quad Y_{N+1} = $$
$$= Y_{N-1}(-1 + \beta h - \tfrac{1}{2}\beta^2 h^2)\,.$$

Then the problem (1), (2′) under the assumptions (3)–(6′) has just one solution $y(x)$, the problem (19), (20′) has just one solution Y and their difference $y(x) - Y$ can be estimated by (24), (25) with possibly different constants, i.e., it again holds

$$\max_{0 \leqq i \leqq N} |y_i - Y_i| = O(h^2)\,, \quad \max_{0 \leqq i \leqq N} |y'(x_i) - Y_{\hat{x}}(x_i)| = O(h^2)\,.$$

The verification of these assertions is almost the same as the above analysis of the problem (1), (2) and its approximation (19), (20).

Remark 2. If we use instead of (20) the approximation

(20″) $$Y_0 = 0\,, \quad Y_N = 0\,, \quad Y_{\hat{x}}(x_0) = 0\,, \quad Y_{\hat{x}}(x_N) = 0\,,$$

then $Ly_1 - L_h y_1 = O(h^{-1})$, $Ly_{N-1} - L_h y_{N-1} = O(h^{-1})$ and therefore we get

$$h \sum_{i=1}^{N-1} E_i L_h E_i \leqq \tfrac{1}{2}\varepsilon h^{-3}(E_1^2 + E_{N-1}^2) + \tfrac{1}{2}\varepsilon \|E\|_2^2 + \varepsilon^{-1}\, O(h^3)$$

and from here

$$\|E\|_2 = O(h^{3/2})\,, \quad E_1 = O(h^{5/2})\,, \quad E_{N-1} = O(h^{5/2})$$

and further

$$\max_{0 \leqq i \leqq N} |y_i - Y_i| = O(h^{3/2})\,, \quad \max_{0 \leqq i \leqq N} |y'(x_i) - Y_{\hat{x}}(x_i)| = O(h^{3/2})\,.$$

These estimates are worse than those corresponding to the approximation (20). The reason may be rather in the method of estimation than in the nature of the problem as the numerical results show.

5. NUMERICAL RESULTS

Several boundary value problems chosen so that $y(x)$, $p(x)$, $q(x)$, $r(x)$, $f(x, y(x))$ are polynomials have been solved. The computations have been performed on the computer D 21 in the Computing Centre of the Technical University in Brno.

The system of non-linear difference equations has been solved by the method of successive approximations

$$L_h Y_i^{n+1} + \nu Y_i^{n+1} = f(x_i, Y_i^n) + \nu Y_i^n, \quad i = 1, \ldots, N-1, \; n = 0, 1, \ldots$$

with the corresponding approximation of boundary conditions, which converges for any Y^0 and for a properly chosen parameter ν (see [6]). It has been sufficient to carry out 2 to 5 iterations.

The system of linear difference equations has been solved by Gauss elimination method fitted to five-diagonal matrices.

The results are given in tables, where

$$ME = \max_{0 \leqq i \leqq N} |y_i - Y_i|, \quad ME' = \max_{1 \leqq i \leqq N-1} |y'(x_i) - Y_{\dot{x}}(x_i)|,$$

$$ME/y = \max_{0 \leqq i \leqq N} |y_i - Y_i| / \max_{a \leqq x \leqq b} |y(x)|,$$

$$ME'/y' = \max_{1 \leqq i \leqq N-1} |y'(x_i) - Y_{\dot{x}}(x_i)| / \max_{a \leqq x \leqq b} |y'(x)|.$$

The last two quantities are given in percents. AB denotes the type of approximation of boundary conditions. If $y(a) = y'(a) = 0$, then $AB = 1$ corresponds to $Y_0 = 0$, $Y_{-1} = 3Y_1 - \frac{1}{2}Y_2$, $AB = 2$ to $Y_0 = 0$, $Y_{\dot{x}}(x_0) = 0$, $AB = 5$ to $Y_0 = Y_x(x_0) = 0$, $AB = 6$ to $Y_1 = 0$, $Y_x(x_0) - \frac{1}{2}h\, Y_{xx}(x_0) = 0$ and $AB = 3$ corresponds to $Y_0 = Y_{x\bar{x}}(x_0) = 0$ for $y(a) = y''(a) = 0$. The situation at $x = b$ is similar.

EXAMPLE 1:

The function $y(x) = x^2(x-1)^2$ is the exact solution of the linear problem

$$\text{(A)} \qquad y^{\text{VI}} = 24,$$
$$y(0) = y'(0) = y(1) = y'(1) = 0.$$

The same $y(x)$ is also the solution of the non-linear problem

(B) $y^{IV} + x^4 y = -y^3 + x^{12} - 6x^{11} + 15x^{10} - 20x^9 + 16x^8 - 8x^7 + 2x^6 + 24$

$y(0) = y'(0) = y(1) = y'(1) = 0\,.$

Another polynomial of the 4th degree $y(x) = (x-1)^2(x+1)^2$ is the solution of the problem

(C) $$y^{IV} + y = x^4 - 2x^2 + 25\,,$$
$$y(-1) = y'(-1) = y(1) = y'(1) = 0\,.$$

EXAMPLE 2:

The function

$$y(x) = (x-1)^2\left(x-\tfrac{1}{2}\right)\left(x-\tfrac{1}{4}\right)\left(x+\tfrac{1}{4}\right)\left(x+\tfrac{1}{2}\right)(x+1)^2$$

is the exact solution of the non-linear equations (D), (E)

(D) $$4y^{IV} - y'' + 8y = -y^3 + P_1(x)\,,$$

(E) $$((x^4 + 0{\cdot}01)\,y'')'' - (x^2 y')' + (x^2 + 8)\,y = -y^3 + P_2(x)$$

where $P_1(x)$, $P_2(x)$ are certain polynomials of 24th degree in x. The above $y(x)$ is also the solution of the linear equations (F), (G), (H)

(F) $4y^{IV} - y'' + 8y = 8x^8 - 74{\cdot}5x^6 + 6802{\cdot}5x^4 - 3352{\cdot}4375x^2 + 158{\cdot}3125$

(G) $$((x^4 + 0{\cdot}01)\,y'')'' - (x^2 y')' + (x^2 + 8)\,y = P_3(x)$$

where $P_3(x)$ is a polynomial of the 10th degree in x,

(H) $$y^{IV} = 1680x^4 - 832{\cdot}5x^2 + 39{\cdot}375\,.$$

The boundary conditions are always

$$y(-1) = y'(-1) = y(1) = y'(1) = 0\,.$$

EXAMPLE 3:

The function

$$y(x) = 2x^7 - 7x^6 + 6x^5 + 3x^4 - 5x^3 + x$$

is the exact solution of the problem

(I) $y^{IV} + 4y = 8x^7 - 28x^6 + 24x^5 + 12x^4 + 1660x^3 - 2520x^2 + 724x + 72$

$y(0) = y''(0) = y(1) = y''(1) = 0\,.$

The first column in the tables denotes the problem considered.

Table 1

	h	ME	ME/y	ME'	ME'/y'	$h\,\Sigma E^2_{x\bar{x}}$	AB
A	·10	$\cdot50\times10^{-3}$	$\cdot80\times10^{0}$	$\cdot14\times10^{-1}$	$\cdot74\times10^{1}$	$\cdot14\times10^{-4}$	1
	·01	$\cdot47\times10^{-6}$	$\cdot72\times10^{-3}$	$\cdot19\times10^{-3}$	$\cdot10\times10^{0}$	$\cdot16\times10^{-10}$	1
	·01	$\cdot50\times10^{-4}$	$\cdot80\times10^{-1}$	$\cdot12\times10^{-6}$	$\cdot63\times10^{-4}$	$\cdot16\times10^{-6}$	2
B	·10	$\cdot50\times10^{-3}$	$\cdot80\times10^{0}$	$\cdot14\times10^{-1}$	$\cdot74\times10^{1}$	$\cdot14\times10^{-4}$	1
	·05	$\cdot62\times10^{-4}$	$\cdot99\times10^{-1}$	$\cdot43\times10^{-2}$	$\cdot23\times10^{1}$	$\cdot24\times10^{-6}$	1
	·025	$\cdot78\times10^{-5}$	$\cdot12\times10^{-1}$	$\cdot11\times10^{-2}$	$\cdot58\times10^{0}$	$\cdot38\times10^{-8}$	1
C	·01	$\cdot44\times10^{-5}$	$\cdot44\times10^{-3}$	$\cdot39\times10^{-3}$	$\cdot26\times10^{-1}$	$\cdot30\times10^{-9}$	1
	·01	$\cdot20\times10^{-3}$	$\cdot20\times10^{-1}$	$\cdot72\times10^{-5}$	$\cdot48\times10^{-3}$	$\cdot32\times10^{-6}$	2

Table 2

	h	ME	ME/y	ME'	ME'/y'	$h\,\Sigma E^2_{x\bar{x}}$	AB
D	·02	$\cdot23\times10^{-2}$	$\cdot76\times10^{1}$	$\cdot58\times10^{-2}$	$\cdot28\times10^{1}$	$\cdot35\times10^{-3}$	1
	·01	$\cdot61\times10^{-3}$	$\cdot20\times10^{1}$	$\cdot15\times10^{-2}$	$\cdot73\times10^{0}$	$\cdot23\times10^{-4}$	1
	·01	$\cdot12\times10^{-3}$	$\cdot40\times10^{0}$	$\cdot11\times10^{-4}$	$\cdot54\times10^{-2}$	$\cdot31\times10^{-4}$	2
	·004	$\cdot97\times10^{-4}$	$\cdot32\times10^{0}$	$\cdot25\times10^{-3}$	$\cdot12\times10^{0}$	$\cdot59\times10^{-6}$	1
E	·01	$\cdot55\times10^{-3}$	$\cdot18\times10^{1}$	$\cdot18\times10^{-2}$	$\cdot88\times10^{0}$	$\cdot70\times10^{-4}$	1
F	·01	$\cdot36\times10^{-1}$	$\cdot12\times10^{3}$	$\cdot85\times10^{-1}$	$\cdot41\times10^{2}$	$\cdot12\times10^{0}$	1
	·02	$\cdot23\times10^{-2}$	$\cdot76\times10^{1}$	$\cdot58\times10^{-2}$	$\cdot28\times10^{1}$	$\cdot35\times10^{-3}$	1
	·02	$\cdot49\times10^{-3}$	$\cdot16\times10^{1}$	$\cdot44\times10^{-4}$	$\cdot21\times10^{-1}$	$\cdot46\times10^{-3}$	2
	·01	$\cdot61\times10^{-3}$	$\cdot20\times10^{1}$	$\cdot15\times10^{-2}$	$\cdot73\times10^{0}$	$\cdot23\times10^{-4}$	1
	·01	$\cdot12\times10^{-3}$	$\cdot40\times10^{0}$	$\cdot11\times10^{-4}$	$\cdot54\times10^{-2}$	$\cdot31\times10^{-4}$	2
	·01	$\cdot36\times10^{-3}$	$\cdot12\times10^{1}$	$\cdot15\times10^{-2}$	$\cdot73\times10^{0}$	$\cdot29\times10^{-4}$	right 2 left 1
	·01	$\cdot13\times10^{-1}$	$\cdot43\times10^{2}$	$\cdot28\times10^{-1}$	$\cdot14\times10^{2}$	$\cdot14\times10^{-2}$	5
	·01	$\cdot22\times10^{-2}$	$\cdot73\times10^{1}$	$\cdot16\times10^{-1}$	$\cdot78\times10^{1}$	$\cdot11\times10^{0}$	6
	·005	$\cdot15\times10^{-3}$	$\cdot50\times10^{0}$	$\cdot40\times10^{-3}$	$\cdot20\times10^{0}$	$\cdot15\times10^{-5}$	1
	·005	$\cdot31\times10^{-4}$	$\cdot10\times10^{0}$	$\cdot31\times10^{-5}$	$\cdot15\times10^{-2}$	$\cdot19\times10^{-5}$	2
G	·01	$\cdot55\times10^{-3}$	$\cdot18\times10^{1}$	$\cdot18\times10^{-2}$	$\cdot88\times10^{0}$	$\cdot69\times10^{-4}$	1
	·01	$\cdot28\times10^{-3}$	$\cdot93\times10^{0}$	$\cdot10\times10^{-2}$	$\cdot49\times10^{0}$	$\cdot62\times10^{-4}$	2
	·005	$\cdot14\times10^{-3}$	$\cdot46\times10^{0}$	$\cdot45\times10^{-3}$	$\cdot22\times10^{0}$	$\cdot45\times10^{-5}$	1
H	·01	$\cdot67\times10^{-3}$	$\cdot22\times10^{1}$	$\cdot15\times10^{-2}$	$\cdot73\times10^{0}$	$\cdot26\times10^{-4}$	1
	·01	$\cdot12\times10^{-3}$	$\cdot40\times10^{0}$	$\cdot22\times10^{-6}$	$\cdot11\times10^{-3}$	$\cdot31\times10^{-4}$	2

Table 3

	h	ME	ME/y	ME'	ME'/y'	$h\,\Sigma^2_{x\bar{x}}$	AB
I	·10	$\cdot50 \times 10^{-2}$	$\cdot25 \times 10^{1}$	$\cdot83 \times 10^{-2}$	$\cdot83 \times 10^{0}$	$\cdot14 \times 10^{-1}$	3
	·05	$\cdot13 \times 10^{-2}$	$\cdot65 \times 10^{0}$	$\cdot27 \times 10^{-2}$	$\cdot27 \times 10^{0}$	$\cdot91 \times 10^{-3}$	3
	·02	$\cdot20 \times 10^{-3}$	$\cdot10 \times 10^{0}$	$\cdot50 \times 10^{-3}$	$\cdot50 \times 10^{-1}$	$\cdot24 \times 10^{-4}$	3
	·01	$\cdot50 \times 10^{-4}$	$\cdot25 \times 10^{-1}$	$\cdot13 \times 10^{-3}$	$\cdot13 \times 10^{-1}$	$\cdot15 \times 10^{-5}$	3
	·005	$\cdot16 \times 10^{-4}$	$\cdot80 \times 10^{-2}$	$\cdot70 \times 10^{-4}$	$\cdot70 \times 10^{-2}$	$\cdot91 \times 10^{-7}$	3

The smallest error appears in three problems of Example 1. This could be expected because $y(x)$ is a polynomial of the 4th degree and therefore it holds $L_h y = Ly$ and the discretization error is caused only by the approximation of boundary conditions. Better results in Example 3 than in Example 2 are probably a consequence of the fact that the polynomial from Example 3 has inside the interval (a, b) no roots while the polynomial from example 2 has in (a, b) four roots.

If we compare the results for different approximations of the boundary conditions for the same h, we can see that for max $|y_i - Y_i|$ the approximation $AB = 1$ is much better than $AB = 2$ in Example 1, but in Example 2 is $AB = 1$ almost like $AB = 2$. For max $|y'(x_i) - Y_{\bar{x}}(x_i)|$, $AB = 2$ is surprisingly better in all cases. On the other hand, the sum $h \sum_{i=1}^{N-1} E^2_{x\bar{x}}(x_i)$ is always better for $AB = 1$. The "inner" approximations $AB = 5$ and $AB = 6$ were used only once and they gave rather bad results.

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Souhrn

OKRAJOVÉ PROBLÉMY PRO MÍRNĚ NELINEÁRNÍ OBYČEJNÉ DIFERENCIÁLNÍ ROVNICE 4. ŘÁDU

HELENA RŮŽIČKOVÁ

Práce se zabývá metodou sítí pro řešení okrajových problémů pro mírně nelineární obyčejnou diferenciální rovnici 4. řádu. Je dokázána existence a jednoznačnost řešení diferenciálního a diferenčního problému. Pro diskretizační chybu a její první diferenci je dokázán odhad $O(h^2)$. Dále je uvedeno několik numerických příkladů.

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