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SOLUTION OF THE FIRST BIHARMONIC PROBLEM BY THE METHOD OF LEAST SQUARES ON THE BOUNDARY

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Some problems of plane elasticity lead to the solution of biharmonic problem (1.1), (1.2). (See, in more details, in Chap. 2, p. 103.) Many methods have been devellopped to the solution of this problem (the method of finite differences, the finite element method, classical variational methods, methods based on the theory of functions of a complex variable, etc.). In this paper, the method of least squares on the boundary is investigated, having its specific preferances. In the first part (Chap. 1 – 5, p. 101 – 114), the algorithm of this method and a numerical example are given. This part is determined first to "consumers" of mathematics and is written in more details. In Chap. 6, the proof of convergence of the method is given. This part is determined first to mathematicians.

Applied to the solution of the biharmonic problem, the method takes an essential use of the form of equation (1.1). As to its idea itself, it can be applied – in proper modifications – also to the solution of other problems.

1. INTRODUCTION

In problems of the theory of elasticity, we meet frequently the so-called first biharmonic problem

$$\Delta^2 U = 0 \quad \text{in} \quad G\,, \tag{1.1}$$

$$U = g_0(s)\,, \quad \frac{\partial U}{\partial \nu} = g_1(s) \quad \text{on} \quad \Gamma\,, \tag{1.2}$$

where G is a plane bounded simply connected region with the boundary Γ. Here, $g_0(s)$, $g_1(s)$ are given functions, s is the length of arc on the boundary, $\partial U/\partial \nu$ the outward-normal derivative of $U(x, y)$ on Γ. Assumptions concerning the boundary Γ

and the functions $g_0(s)$, $g_1(s)$ will be stated later (p. 117). They will be sufficiently general to include boundaries and loadings (single loads including) we meet most frequently in applications of the theory of two-dimensional elasticity.

Many methods have been elaborated for solution, or approximate solution, of problem (1.1), (1.2), having specific preferances and disadvantages: The method of finite differences is rather simple and universal. However, it yields approximate values of the required function $U(x, y)$ only at discrete points of the net. If second derivatives of $U(x, y)$ (which are of particular interest in problems of theory of elasticity, because they give components of the stress tensor) are then replaced by the corresponding second difference-quotients, the accuracy of these approximations is not satisfactory, in general. A similar property has, in a certain degree, the finite element method. As to classical variational methods, the difficulty lies in finding a function $w(x, y)$ (of a certain class of functions) which fulfills conditions (1.2). The method based on application of the theory of functions of a complex variable is rather complicated.

In this paper, we shall investigate a method which makes particular use of the form of equation (1.1). It can be called the method of least squares on the boundary. It is closely connected with the method given in [2], p. 285 and with variational methods given in [3], Part IV. Its idea is the following:

Let us consider the system of biharmonic polynomials

$$z_1(x, y), \quad z_2(x, y), \ldots \tag{1.3}$$

(see, in details, in Chap. 3, p. 106), choose a positive integer n and assume the approximate solution of problem (1.1), (1.2) in the form

$$U_n(x, y) = \sum_{i=1}^{4n-2} a_{ni}\, z_i(x, y)\,. \tag{1.4}$$

(Why we consider precisely $4n-2$ terms, becomes clear in Chap. 3.) Each of functions (1.3) being biharmonic, the function (1.4) is also biharmonic, and thus satisfies equation (1.1), whatever are the constants a_{ni}. We now choose these constants in such a way that

$$\int_\Gamma (U_n - g_0)^2\, \mathrm{d}s + \int_\Gamma \left(\frac{\partial U_n}{\partial s} - \frac{\mathrm{d}g_0}{\mathrm{d}s}\right)^2 \mathrm{d}s + \int_\Gamma \left(\frac{\partial U_n}{\partial \nu} - g_1\right)^2 \mathrm{d}s = \min. \tag{1.5}$$

among all expressions of the form

$$\int_\Gamma (V_n - g_0)^2\, \mathrm{d}s + \int_\Gamma \left(\frac{\partial V_n}{\partial s} - \frac{\mathrm{d}g_0}{\mathrm{d}s}\right)^2 \mathrm{d}s + \int_\Gamma \left(\frac{\partial V_n}{\partial \nu} - g_1\right)^2 \mathrm{d}s\,, \tag{1.6}$$

where

$$V_n(x, y) = \sum_{i=1}^{4n-2} b_{ni}\, z_i(x, y)\,. \tag{1.7}$$

We shall see [cf. (4.10), p. 109] that condition (1.5) leads to a system of $4n-2$ linear algebraic equations to determine the coefficients a_{ni} ($i = 1, \ldots, 4n-2$) in (1.4). This system will be shown to be uniquely solvable. We show further that under rather natural assumptions on the boundary Γ and functions $g_0(s)$, $g_1(s)$, the expression on the left-hand side of (1.5) can be made arbitrarily small if n is sufficiently large (thus, in this sense, the given boundary conditions can be approximated with an arbitrary accuracy) and that for $n \to \infty$ the sequence $\{U_n(x, y)\}$ converges in the mean to a certain function $U(x, y)$ which is uniquely determined by the given functions $g_0(s)$, $g_1(s)$ and which is a solution (eventually in a generalized sense) of the problem (1.1), (1.2). The proof of these assertions is not easy and, therefore, is left to Chap. 6, because our aim is to make first the reader familiar with the method itself and its application. In the following chapter, we show what kinds of problems of the theory of elasticity lead to the solution of the biharmonic problem (which is typical for application of our method). In Chap. 3 we shall briefly discuss the fundamental system of biharmonic polynomials, in Chap. 4 our method itself in details, in Chap. 5 we present a numerical example and in Chap. 6 we give the proof of convergence.

2. THE BIHARMONIC PROBLEM

Problems of the type (1.1), (1.2) are of particular interest in the theory of the so-called plane or two-dimensional elasticity, especially in the theory of wall-beams. As well-known (see e.g. [2], p. 59), if a plane simply connected body G is in a state of stress, characterized by components σ_x, σ_y, τ_{xy} of the stress-tensor fulfilling equations of static equilibrium and the equation of compatibility, then there exists a biharmonic function $U(x, y)$, the so-called *Airy function*, the derivatives of which these components are,

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x\, \partial y}. \tag{2.1}$$

Conversely, if $U(x, y)$ is an arbitrary biharmonic function in G, then functions (2.1) satisfy the equilibrium equations and the equation of compatibility, and thus characterize a certain state of stress in G.

Let s be the parameter of arc on Γ, $0 \leqq s < l$, where l is the length of Γ and let s be increasing if we run through Γ in the positive sense of its orientation (leaving G to the left-hand side). Let the point of Γ corresponding to $s = 0$ be denoted by A. Let, further, the loading on Γ be given by the stress vector $\mathbf{V}(s)$ with components $X(s)$, $Y(s)$. If we put $\partial U/\partial x = 0$, $\partial U/\partial y = 0$ at the point A, then (see [2], p. 73), on Γ, the derivatives $\partial U/\partial x$, $\partial U/\partial y$ of the Airy function $U(x, y)$ are given by

$$\frac{\partial U}{\partial x}(s) = -\int_0^s Y(t)\, \mathrm{d}t, \quad \frac{\partial U}{\partial y}(s) = \int_0^s X(t)\, \mathrm{d}t. \tag{2.2}$$

The functions $\frac{\partial U}{\partial x}(s)$, $\frac{\partial U}{\partial y}(s)$ being known on Γ, we get in the usual way the functions $\frac{\partial U}{\partial s}(s)$, $\frac{\partial U}{\partial \nu}(s)$,

$$\frac{\partial U}{\partial s} = -\frac{\partial U}{\partial x}\nu_y + \frac{\partial U}{\partial y}\nu_x, \tag{2.3}$$

$$\frac{\partial U}{\partial \nu} = \frac{\partial U}{\partial x}\nu_x + \frac{\partial U}{\partial y}\nu_y, \tag{2.4}$$

where $\nu_x(s)$, $\nu_y(s)$ are components of the outward normal ν. Then, putting $U = 0$ at the point A, we get

$$U(s) = \int_0^s \frac{\partial U}{\partial s}(t)\,\mathrm{d}t\,. \tag{2.5}$$

In this way, we come to the problem (1.1), (1.2). See the following Example 2.1.

It follows from (2.2) that the value of the function $\partial U/\partial y$ or $-\partial U/\partial x$ at the point $B(s)$, is given by the y- or x-component, respectively, of the resulting vector of the loading on Γ, considered between the points A, B with parameters 0 and s. In this sense, it is possible to take also single loads into considerations; at the points, where these single loads are acting, the functions $\partial U/\partial x$ and $\partial U/\partial y$ (or at least one of them) are then discontinuous. In what follows, we shall assume that the point A (with the parameter $s = 0$) is chosen in such a way that no single load is acting there. Then the functions $\partial U/\partial x$, $\partial U/\partial y$ are continuous at the point A [i.e. it holds

$$\lim_{s\to l-}\frac{\partial U}{\partial x}(s) = \frac{\partial U}{\partial x}(0)\,, \quad \lim_{s\to l-}\frac{\partial U}{\partial y}(s) = \frac{\partial U}{\partial y}(0)]$$

if and only if the loading satisfies the condition of static equilibrium in forces. The function $U(s)$ is then continuous if and only if the loading satisfies also the condition of equilibrium of moments.

Example 2.1. Let us consider a rectangular body G loaded as shown in Fig. 1. Thus we have $0 \leqq s < 2(a + b)$, while $s = 0$ at the point $A(a, 0)$. In details,

$$\begin{aligned}
0 &\leqq s < b && \text{on } AB\,,\\
b &\leqq s < a + b && \text{on } BC\,,\\
a + b &\leqq s < a + 2b && \text{on } CD\,,\\
a + 2b &\leqq s < 2(a + b) && \text{on } DA\,.
\end{aligned}$$

From Fig. 1 it first follows

$$X(s) \equiv 0\,, \quad \text{so that} \quad \frac{\partial U}{\partial y}(s) \equiv 0\,.$$

Further, according to (2.2) (we have $Y(s) = -q$ on BC, etc.)

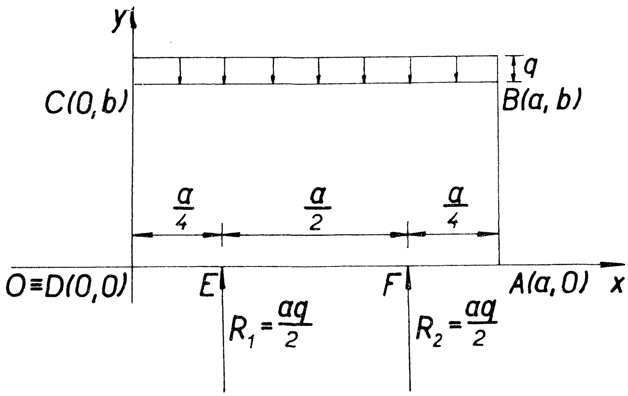

Fig. 1.

$$(2.6) \qquad \begin{aligned} \frac{\partial U}{\partial x}(s) &\equiv 0 && \text{on} \quad AB\,, \\ &= q(s-b) && \text{on} \quad BC\,, \\ &= qa && \text{on} \quad CD\,, \\ &= qa && \text{on} \quad DE\,, \\ &= \frac{qa}{2} && \text{on} \quad EF\,, \\ &\equiv 0 && \text{on} \quad FA\,. \end{aligned}$$

Evidently, we have

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial y} \quad \text{on} \quad AB\,, \quad \frac{\partial U}{\partial s} = -\frac{\partial U}{\partial x} \quad \text{on} \quad BC\,, \quad \frac{\partial U}{\partial s} = -\frac{\partial U}{\partial y} \quad \text{on} \quad CD\,,$$

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x} \quad \text{on} \quad DA\,.$$

Integrating with respect to s and using (2.6), we get

$$(2.7) \qquad \begin{aligned} U(s) &\equiv 0 && \text{on} \quad AB\,, \\ &= -\frac{q(s-b)^2}{2} && \text{on} \quad BC\,, \end{aligned}$$

$$= -\frac{qa^2}{2} \qquad \text{on} \quad CD\,,$$

$$= -\frac{qa^2}{2} + qa(s - a - 2b) \qquad \text{on} \quad DE\,,$$

$$= -\frac{qa^2}{2} + \frac{qa^2}{4} + \frac{qa}{2}\left(s - \frac{5a}{4} - 2b\right) \qquad \text{on} \quad EF\,,$$

$$\equiv 0 \qquad \text{on} \quad FA\,.$$

Further, we have

$$\frac{\partial U}{\partial v} = \frac{\partial U}{\partial x} \quad \text{on} \quad AB\,, \quad \frac{\partial U}{\partial v} = \frac{\partial U}{\partial y} \quad \text{on} \quad BC\,, \quad \frac{\partial U}{\partial v} = -\frac{\partial U}{\partial x} \quad \text{on} \quad CD\,,$$

$$\frac{\partial U}{\partial v} = -\frac{\partial U}{\partial y} \quad \text{on} \quad DA\,.$$

Consequently

$$\frac{\partial U}{\partial v}(s) \equiv 0 \qquad \text{on} \quad AB\,, \tag{2.8}$$

$$\equiv 0 \qquad \text{on} \quad BC\,,$$

$$= -qa \qquad \text{on} \quad CD\,,$$

$$\equiv 0 \qquad \text{on} \quad DA\,.$$

3. BIHARMONIC POLYNOMIALS

Let us consider a system of polynomials (cf. [4])

$$H_1^{(m)}(x, y) = \sum_{i=0}^{k} (-1)^i \binom{m}{2i} x^{m-2i} y^{2i} \qquad \text{for} \quad m = 0, 1, 2, \ldots, \tag{3.1}$$

$$H_2^{(m)}(x, y) = \sum_{i=0}^{k+r-1} (-1)^i \binom{m}{2i+1} x^{m-2i-1} y^{2i+1} \qquad \text{for} \quad m = 1, 2, \ldots, \tag{3.2}$$

$$H_3^{(m)}(x, y) = \sum_{i=1}^{k} (-1)^i\, i \binom{m}{2i} x^{m-2i} y^{2i} \qquad \text{for} \quad m = 2, 3, \ldots, \tag{3.3}$$

$$H_4^{(m)}(x, y) = \sum_{i=1}^{k+r-1} (-1)^i\, i \binom{m}{2i+1} x^{m-2i-1} y^{2i+1} \qquad \text{for} \quad m = 3, 4, \ldots. \tag{3.4}$$

Here, m is the degree of the polynomial,

$$k = \left[\frac{m}{2}\right] = \begin{cases} \dfrac{m}{2} & \text{if } m \text{ is even}, \\ \dfrac{m}{2} - \dfrac{1}{2} & \text{if } m \text{ is odd}, \end{cases} \qquad r = 2 \cdot \left\{\frac{m}{2}\right\} = \begin{cases} 0 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}. \end{cases}$$

For example

$$H_1^{(0)}(x, y) = \sum_{i=0}^{0} (-1)^i \binom{0}{2i} x^{-2i} y^{2i} = (-1)^0 \times 1 x^0 y^0 = 1\,,$$

$$H_2^{(1)}(x, y) = \sum_{i=0}^{0+1-1} (-1)^i \binom{1}{2i+1} x^{1-2i-1} y^{2i+1} = (-1)^0 \times 1 x^0 y^1 = y\,,$$

etc.

The polynomials (3.1)–(3.4) are biharmonic [they satisfy the biharmonic equation (1.1)], the polynomials $H_1^{(m)}(x, y)$, $H_2^{(m)}(x, y)$ are even harmonic.

Let us order the polynomials (3.1)–(3.4) according to their increasing degree, while polynomials of the same degree be ordered according to their increasing lower suffix. In this way, we get a sequence of polynomials

$$(3.5) \quad z_1(x, y) = H_1^{(0)}(x, y)\,, \quad z_2(x, y) = H_1^{(1)}(x, y)\,, \quad z_3(x, y) = H_2^{(1)}(x, y), \ldots.$$

Thus

$$(3.6) \qquad z_1(x, y) \equiv 1\,,$$

$$z_2(x, y) = x\,, \quad z_3(x, y) = y\,,$$

$$z_4(x, y) = x^2 - y^2\,, \quad z_5(x, y) = 2xy\,, \quad z_6(x, y) = -y^2\,,$$

$$z_7(x, y) = x^3 - 3xy^2\,, \; z_8(x, y) = 3x^2y - y^3\,, \; z_9(x, y) = -3xy^2\,, \; z_{10}(x, y) = -y^3\,,$$

etc. It easily follows from the given construction that for every fixed $n \geqq 2$ we shall have precisely $4n - 2$ polynomials of order $\leqq n$.

The following theorem holds (see e.g. [4]):

Theorem 3.1. *In every region G, polynomials (3.5) are linearly independent. Every biharmonic polynomial [thus every polynomial satisfying equation (1.1)] of degree p can be expressed, even in a unique way, as a linear combination of polynomials (3.5) of order $\leqq p$.*

The basic idea of this method has been briefly mentioned at the beginning of this paper. Let us choose a positive integer $n \geqq 2$ and assume an approximate solution of problem (1.1), (1.2) in the form

$$U_n(x, y) = \sum_{i=1}^{4n-2} a_{ni}\, z_i(x, y)\,, \quad n \geqq 2\,, \tag{4.1}$$

where $z_i(x, y)$ are the first $4n-2$ terms of the sequence (3.5) [thus just all polynomials (3.1)–(3.4) of degree $\leqq n$] and a_{ni} are determined (uniquely, as will be shown in Chap. 6) by the condition that

$$F(U_n) = \int_0^l (U_n - g_0)^2\, ds + \int_0^l \left(\frac{\partial U_n}{\partial s} - \frac{dg_0}{ds}\right)^2 ds + \int_0^l \left(\frac{\partial U_n}{\partial v} - g_1\right)^2 ds = \min. \tag{4.2}$$

among all expressions of the form

$$F(V_n) = \int_0^l (V_n - g_0)^2\, ds + \int_0^l \left(\frac{\partial V_n}{\partial s} - \frac{dg_0}{ds}\right)^2 ds + \int_0^l \left(\frac{\partial V_n}{\partial v} - g_1\right)^2 ds\,, \tag{4.3}$$

where

$$V_n(x, y) = \sum_{i=1}^{4n-2} b_{ni}\, z_i(x, y)\,, \tag{4.4}$$

i.e. that the functional F, considered on the set of functions (4.4), be minimal just for the function (4.1).

If we write (4.3) in details, we get

$$\begin{aligned} F(V_n) = \tilde{F}(b_{n1}, \ldots, b_{n,4n-2}) &= \int_0^l (b_{n1} z_1 + \ldots + b_{n,4n-2} z_{4n-2} - g_0)^2\, ds + \\ &+ \int_0^l \left(b_{n1} \frac{\partial z_1}{\partial s} + \ldots + b_{n,4n-2} \frac{\partial z_{4n-2}}{\partial s} - \frac{dg_0}{ds}\right)^2 ds + \\ &+ \int_0^l \left(b_{n1} \frac{\partial z_1}{\partial v} + \ldots + b_{n,4n-2} \frac{\partial z_{4n-2}}{\partial v} - g_1\right)^2 ds\,. \end{aligned} \tag{4.5}$$

Obviously, the value of $F(V_n)$ depends only on $b_{n1}, \ldots, b_{n,4n-2}$ [thus we write $F(V_n) = \tilde{F}(b_{n1}, \ldots, b_{n,4n-2})$], because $g_0(s)$, $g_1(s)$ are given functions and the values of functions $z_i(x, y)$ and of their derivatives with respect to s and v, on the boundary, are also known. It follows from (4.5) that $F(V_n)$ is a quadratic function in b_{ni}. According to (4.2), this function should attain for $b_{n1} = a_{n1}, \ldots, b_{n,4n-2} = a_{n,4n-2}$ its minimal value. A necessary (and in our case obviously also sufficient) condition for this is that the following equations be satisfied:

$$(4.6) \qquad \frac{\partial \tilde{F}}{\partial b_{n1}}(a_{n1}, \ldots, a_{n,4n-2}) = 0, \ldots, \frac{\partial \tilde{F}}{\partial b_{n,4n-2}}(a_{n1}, \ldots, a_{n,4n-2}) = 0\,.$$

The form of the function $\tilde{F}$ obviously permits the differentiation under the sign of integration. For example,

$$\frac{\partial}{\partial b_{n1}} \int_0^l [b_{n1} z_1(s) + \ldots + b_{n,4n-2} z_{4n-2}(s) - g_0(s)]^2 \, ds =$$

$$= 2 \int_0^l [b_{n1}\, z_1(s) + \ldots + b_{n,4n-2} z_{4n-2}(s) - g_0(s)]\, z_1(s) \, ds =$$

$$= 2 \left[b_{n1} \int_0^l z_1(s)\, z_1(s) \, ds + \ldots + b_{n,4n-2} \int_0^l z_1(s)\, z_{4n-2}(s) \, ds - \int_0^l g_0(s)\, z_1(s) \, ds \right],$$

etc. Thus, the condition (4.2) yields the following system of equations:

$$(4.7) \quad \sum_{j=1}^{4n-2} \left[\int_0^l z_i(s)\, z_j(s) \, ds + \int_0^l \frac{\partial z_i}{\partial s}(s) \frac{\partial z_j}{\partial s}(s) \, ds + \int_0^l \frac{\partial z_i}{\partial v}(s) \frac{\partial z_j}{\partial v}(s) \, ds \right] a_{nj} =$$

$$= \int_0^l g_0(s)\, z_i(s) \, ds + \int_0^l \frac{dg_0}{ds}(s) \frac{\partial z_i}{\partial s}(s) \, ds + \int_0^l g_1(s) \frac{\partial z_i}{\partial v}(s) \, ds\,, \quad i = 1, \ldots, 4n-2\,.$$

Putting

$$(4.8) \quad (z_i, z_j)_\Gamma = \int_0^l z_i(s)\, z_j(s) \, ds + \int_0^l \frac{\partial z_i}{\partial s}(s) \frac{\partial z_j}{\partial s}(s) \, ds + \int_0^l \frac{\partial z_i}{\partial v}(s) \frac{\partial z_j}{\partial v}(s) \, ds\,,$$

$$(4.9) \qquad c_i = \int_0^l g_0(s)\, z_i(s) \, ds + \int_0^l \frac{dg_0}{ds}(s) \frac{\partial z_i}{\partial s}(s) \, ds + \int_0^l g_1(s) \frac{\partial z_i}{\partial v}(s) \, ds$$

$(i, j = 1, \ldots, 4n - 2)$, we can write the system (4.7) in the form

$$(4.10) \qquad \sum_{j=1}^{4n-2} (z_i, z_j)_\Gamma\, a_{nj} = c_i\,, \quad i = 1, \ldots, 4n - 2$$

which represents the system of $4n - 2$ equations for $4n - 2$ unknown constants $a_{n1}, \ldots, a_{n,4n-2}$.

Before giving the proof of existence and uniqueness of the solution of system (4.10) and the proof of convergence if our method, we present, in the following chapter, a numerical example.

Remark 4.1. The reader may be rather surprised by the presence of the middle integral in (4.2), suggesting the question why we do not replace the condition (4.2) by the condition

$$\int_0^l (U_n - g_0)^2 \, ds + \int_0^l \left(\frac{\partial U_n}{\partial v} - g_1 \right)^2 ds = \min. \tag{4.11}$$

The middle term plays an important role in questions of convergence as well as in numerical questions. Examples can be constructed where the method, based on condition (4.11), gives quite unsatisfactory results.

5. A NUMERICAL EXAMPLE

As an example, showing the application of the method of least squares on the boundary, let us consider the biharmonic problem on a square. The reason why the square has been chosen is that it is a sufficiently simple region to make the example very clear, while, on the other side, the boundary of the square contains angular points which often make difficulties in mathematical considerations as well as in applications.

Thus, let the nonhomogeneous boundary value problem (1.1), (1.2), p. 101, be given, where $G = (0, 1) \times (0, 1)$. Let the loading on the boundary be the same as in Example 2.1. (see Fig. 1, p. 105, for $a = 1$, $b = 1$) and let us choose $q = 2$. The boundary conditions are then given by (2.7) and (2.8), p. 105 and 106. For numerical computation, it is convenient to express these conditions in Cartesian coordinates. From (2.7) and (2.8) it then follows

$$\begin{aligned}
g_0 &\equiv 0 && \text{on } AB\,, \\
&= -(1 - x)^2 && \text{on } BC\,, \\
&= -1 && \text{on } CD\,, \\
&= 2x - 1 && \text{on } DE\,, \\
&= x - 0.75 && \text{on } EF\,, \\
&\equiv 0 && \text{on } FA\,, \\[1ex]
g_1 &\equiv 0 && \text{on } AB\,, \\
&\equiv 0 && \text{on } BC\,, \\
&= -2 && \text{on } CD\,, \\
&\equiv 0 && \text{on } DA\,.
\end{aligned} \tag{5.1}$$

An approximate solution of problem (1.1), (1.2) is assumed in the form

$$U_n(x, y) = \sum_{i=1}^{4n-2} a_{ni} \, z_i(x, y)\,. \tag{5.2}$$

For the coefficients a_{ni} we then get the system (4.10).

If n is small (e.g. $n \leqq 3$), the evaluation of coefficients of system (4.10) is very simple. For example, we have [cf. (3.6)]

$$\int_0^1 z_3(s)\, z_4(s)\,\mathrm{d}s = \int_0^1 y(1 - y^2)\,\mathrm{d}y + \int_0^1 1\,.\,(x^2 - 1)\,\mathrm{d}x + \int_0^1 y(0 - y^2)\,\mathrm{d}y + \\ + \int_0^1 0\,.\,(x^2 - 0)\,\mathrm{d}x = -\frac{2}{3}$$

and, similarly,

$$\int_0^1 \frac{\partial z_3}{\partial s}(s)\frac{\partial z_4}{\partial s}(s)\,\mathrm{d}s = \int_0^1 \frac{\partial z_3}{\partial y}(1, y)\frac{\partial z_4}{\partial y}(1, y)\,\mathrm{d}y + \int_0^1\left[-\frac{\partial z_3}{\partial x}(x, 1)\right] \times \\ \times\left[-\frac{\partial z_4}{\partial x}(x, 1)\right]\mathrm{d}x + \int_0^1\left[-\frac{\partial z_3}{\partial y}(0, y)\right]\left[-\frac{\partial z_4}{\partial y}(0, y)\right]\mathrm{d}y + \\ + \int_0^1 \frac{\partial z_3}{\partial x}(x, 0)\frac{\partial z_4}{\partial x}(x, 0)\,\mathrm{d}x = \int_0^1(-2y)\,\mathrm{d}y + \int_0^1(-2y)\,\mathrm{d}y = -2\,,$$

$$\int_0^1 \frac{\partial z_3}{\partial v}(s)\frac{\partial z_4}{\partial v}(s)\,\mathrm{d}s = \int_0^1 \frac{\partial z_3}{\partial x}(1, y)\frac{\partial z_4}{\partial x}(1, y)\,\mathrm{d}y + \int_0^1 \frac{\partial z_3}{\partial y}(x, 1)\frac{\partial z_4}{\partial y}(x, 1)\,\mathrm{d}x + \\ + \int_0^1\left[-\frac{\partial z_3}{\partial x}(0, y)\right]\left[-\frac{\partial z_4}{\partial x}(0, y)\right]\mathrm{d}y + \int_0^1\left[-\frac{\partial z_3}{\partial y}(x, 0)\right] \times \\ \times\left[-\frac{\partial z_4}{\partial y}(x, 0)\right]\mathrm{d}x = -\int_0^1 2\,\mathrm{d}x = -2\,.$$

Thus, we have

$$(z_3, z_4)_r = -\tfrac{14}{3}\,.$$

Other coefficients as well as the right-hand side of the system are received in a similar way.

Table 1

a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	
4·0000	2·0000	2·0000	0·0000	2·0000	−1·6667	−1·6458
2·0000	5·6667	1·0000	4·6667	5·6667	−0·8333	3·8476
2·0000	1·0000	5·6667	−4·6667	5·6667	−5·5000	−0·8333
0·0000	4·6667	−4·6667	14·8000	0·0000	7·4000	1·8983
2·0000	5·6667	5·6667	0·0000	16·0000	−5·5000	3·8333
−1·6667	−0·8333	−5·5000	7·4000	−5·5000	8·0667	0·6667

In Table 1, the scheme of system (4.10) is presented for $n = 2$. Solving this system, we get the corresponding approximate solution (for "negligible" coefficients we give only the order of the first cipher and then write the result symbolically)

$$U_2(x, y) = \sum_{i=1}^{6} a_{2i} z_i(x, y) = -0{\cdot}997\,02 + 2{\cdot}185\,45x + 4 \times 10^{-3}y -$$

$$- 1{\cdot}017\,03(x^2 - y^2) - 2 \times 10^{-3} \times 2xy - 1{\cdot}019\,66(-y^2) \cong$$

$$\cong -0{\cdot}997 + 2{\cdot}185x - 1{\cdot}017x^2 + k \times 10^{-3}(y + 2xy - y^2)\,.$$

As it was to be expected, this approximation is very close to the function

$$u(x, y) = -1 + 2x - x^2$$

which corresponds to the case that single loads on DA are replaced by a uniform load with the same resulting vector.

For the case $n = 3$ we get similarly

$$U_3(x, y) = -0{\cdot}996\,66 + 2{\cdot}054\,89x + 1 \times 10^{-2}y -$$

$$- 1{\cdot}056\,89(x^2 - y^2) - 3 \times 10^{-2} \times 2xy + 1{\cdot}086\,37(-y^2) +$$

$$+ 3 \times 10^{-3}(x^3 - 3xy^2) + 2 \times 10^{-2}(3x^2y - y^3) -$$

$$- 3 \times 10^{-3}(-3xy^2) - 4 \times 10^{-2}(-y^3) \cong$$

$$\cong -0{\cdot}997 + 2{\cdot}055x - 1{\cdot}057x^2 +$$

$$+ k \times 10^{-2}(y + 2xy - y^2 + x^3 - 3xy^2 + 3x^2y - y^3)\,.$$

For comparison, we present also the corresponding result if we use the "method (4.11)":

$$U_3(x, y) = -0{\cdot}990\,18 + 2{\cdot}014\,50x + 5 \times 10^{-3}y -$$

$$- 1{\cdot}015\,55(x^2 - y^2) - 1 \times 10^{-2} \times 2xy + 1{\cdot}059\,71(-y^2) +$$

$$+ 3 \times 10^{-3}(x^3 - 3xy^2) + 4 \times 10^{-3}(3x^2y - y^3) -$$

$$- 5 \times 10^{-3}(-3xy^2) - 3 \times 10^{-3}(-y^3) \cong$$

$$\cong -0{\cdot}990 + 2{\cdot}015x - 1{\cdot}016x^2 +$$

$$+ k \times 10^{-2}(y + 2xy - y^2 + x^3 - 3xy^2 + 3x^2y - y^3)\,.$$

More remarkable differences between results produced by methods (4.2) and (4.11) are to be expected first in higher approximations.

If the accuracy of "lower" approximations is not satisfactory and "higher" approximations should be taken into account, some suitable properties of biharmonic polynomials can be used to prepare the numerical process for a computer. To this aim, let us come back to the original notation used in Chap. 3. The approximate solution can then be written in the form

$$U_n(x, y) = \sum_{j=0}^{n} H_1^{(j)}(x, y)\, a_j^n + \sum_{j=1}^{n} H_2^{(j)}(x, y)\, a_{n+j}^n + \tag{5.3}$$
$$+ \sum_{j=2}^{n} H_3^{(j)}(x, y)\, a_{2n+j-1}^n + \sum_{j=3}^{n} H_4^{(j)}(x, y)\, a_{3n+j-3}^n ,$$

denoting now the unknown constants a_k^n, $k = 0, 1, \ldots, 4n - 3$.

Let us note, further – what is of use in preparing the program – that the following relations hold for the outward-normal derivatives on the boundary of the square $ABCD$:

$$\begin{aligned}
\frac{\partial H_1^{(m)}}{\partial v} &= \frac{\partial H_1^{(m)}}{\partial x} = mH_1^{(m-1)} \quad \text{on} \quad AB\,, \\
&= \frac{\partial H_1^{(m)}}{\partial y} = -\, mH_2^{(m-1)} \quad \text{on} \quad BC\,, \\
&= -\frac{\partial H_1^{(m)}}{\partial x} = -\, mH_1^{(m-1)} \quad \text{on} \quad CD\,, \\
&= -\frac{\partial H_1^{(m)}}{\partial y} = mH_2^{(m-1)} \quad \text{on} \quad DA \quad \text{for} \quad m = 0, 1, \ldots\,, \\
\frac{\partial H_2^{(m)}}{\partial v} &= \frac{\partial H_2^{(m)}}{\partial x} = mH_2^{(m-1)} \quad \text{on} \quad AB\,, \\
&= \frac{\partial H_2^{(m)}}{\partial y} = mH_1^{(m-1)} \quad \text{on} \quad BC\,, \\
&= -\frac{\partial H_2^{(m)}}{\partial x} = -\, mH_2^{(m-1)} \quad \text{on} \quad CD\,, \\
&= -\frac{\partial H_2^{(m)}}{\partial y} = -\, mH_1^{(m-1)} \quad \text{on} \quad DA \quad \text{for} \quad m = 1, 2, \ldots\,, \\
\frac{\partial H_3^{(m)}}{\partial v} &= \frac{\partial H_3^{(m)}}{\partial x} = mH_3^{(m-1)} \quad \text{on} \quad AB\,, \\
&= \frac{\partial H_3^{(m)}}{\partial y} = -\, m[H_2^{(m-1)} + H_4^{(m-1)}] \quad \text{on} \quad BC\,, \\
&= -\frac{\partial H_3^{(m)}}{\partial x} = -\, mH_3^{(m-1)} \quad \text{on} \quad CD\,, \\
&= -\frac{\partial H_3^{(m)}}{\partial y} = m[H_2^{(m-1)} + H_4^{(m-1)}] \quad \text{on} \quad DA \quad \text{for} \quad m = 2, 3, \ldots\,,
\end{aligned}$$

$$\frac{\partial H_4^{(m)}}{\partial v} = \frac{\partial H_4^{(m)}}{\partial x} = mH_4^{(m-1)} \quad \text{on} \quad AB\,,$$

$$= \frac{\partial H_4^{(m)}}{\partial y} = mH_3^{(m-1)} \quad \text{on} \quad BC\,,$$

$$= -\frac{\partial H_4^{(m)}}{\partial x} = -mH_4^{(m-1)} \quad \text{on} \quad CD\,,$$

$$= -\frac{\partial H_4^{(m)}}{\partial y} = -mH_3^{(m-1)} \quad \text{on} \quad DA \quad \text{for} \quad m = 3, 4, \ldots\,.$$

Corresponding formulae for derivatives with respect to s can be derived similarly. These formulae enable to express in a simple way coefficients of the system for unknown coefficients a_k^n, while the approximation $U_n(x, y)$ is assumed in the form (5.3).

If n is large, it is available, with regard to the numerical stability of the process, to use polynomials $z_i(x, y)/m$ instead of $z_i(x, y)$, where m is the degree of the polynomial. Of course, it is also possible to orthonormalize these polynomials with respect to the scalar product $(u, v)_\Gamma$. However, it is a labourious procedure, in general.

As to the method itself, the matrix of system (4.10) remains unchanged for different boundary conditions, i.e. for different loadings of the boundary. If n should be increased, the original scalar products remain preserved and only new terms should be evaluated.

We come now to the proof of convergence of our method.

6. CONVERGENCE OF THE METHOD OF LEAST SQUARES ON THE BOUNDARY

a) Some Basic Concepts and Notation

In this chapter we assume that the reader is familiar with fundamental concepts concerning functional-analytical methods in elliptic boundary value problems explained e.g. in [1], Chap. 1 or in [3], Part IV. Speaking about a region G, we shall assume it to be plane, bounded and simply connected, with the so-called Lipschitz boundary Γ. A definition of a region with a Lipschitz boundary can be found e.g. in [3], Chap. 28. Note that to this kind of regions belong, roughly speaking, regions with a smooth or piecewise smooth boundary, without cuspidal points, for example a circle, a square, etc. The closure of the region G is denoted by $\bar{G}$, i.e. $\bar{G} = G + \Gamma$. We speak briefly about the closed region $\bar{G}$.

As usual, we denote by $L_2(G)$ the Hilbert space, the elements of which are real functions, square integrable in G (in the Lebesgue sense), with the scalar product

(6.1) $$(u, v) = \iint_G u(x, y)\, v(x, y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

By $W_2^{(k)}(G)$ we denote the Hilbert space, the elements of which are those functions of $L_2(G)$ which have square integrable generalized derivatives in G, to the k-th order included. The scalar product in $W_2^{(k)}(G)$ is defined by

$$(6.2) \qquad (u, v)_{W_2^{(k)}(G)} = \sum_{|i| \leq k} (D^i u, D^i v) ,$$

where the sum $\sum_{|i| \leq k} (D^i u, D^i v)$ means the sum of scalar products [in $L_2(G)$] of the functions u, v and their generalized derivatives up to the order k included. In particular,

$$(6.3) \qquad (u, v)_{W_2^{(1)}(G)} = (u, v) + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right),$$

$$(6.4) \qquad (u, v)_{W_2^{(2)}(G)} = (u, v) + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right) +$$

$$+ \left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 y}{\partial x^2}\right) + 2\left(\frac{\partial^2 u}{\partial x\, \partial y}, \frac{\partial^2 v}{\partial x\, \partial y}\right) + \left(\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 v}{\partial y^2}\right).$$

On base of the scalar product (6.1), or (6.2), we define, in a usual way, the norm and the metric in $L_2(G)$, or $W_2^{(k)}(G)$, respectively,

$$(6.5) \qquad \|u\|^2 = (u, u), \quad \varrho(u, v) = \|u - v\| ,$$

$$(6.6) \qquad \|u\|^2_{W_2^{(k)}(G)} = (u, u)_{W_2^{(k)}(G)}, \quad \varrho(u, v)_{W_2^{(k)}(G)} = \|u - v\|_{W_2^{(k)}(G)} .$$

In a similar way, the spaces $L_2(\Gamma)$ and $W_2^{(k)}(\Gamma)$, the elements of which are functions on Γ, are defined. See e.g. [3], Chap. 30. In particular, for functions $g_0(s)$, $g_1(s)$ of Example 2.1 (p. 104) we have

$$g_0 \in W_2^{(1)}(\Gamma), \quad g_1 \in L_2(\Gamma) .$$

By $C^{(\infty)}(\bar{G})$[we meet frequently the symbol $E(\bar{G})$ in literature] we denote the set of functions continuous with their derivatives of all orders in $\bar{G}$. It is known (see e.g. [1]) that for regions with a Lipschitz boundary, the space $W_2^{(k)}(G)$ can be defined as the closure of the set $C^{(\infty)}(\bar{G})$ in which the scalar product (6.2) is introduced [in the metric (6.6) given by this scalar product].

By $C_0^{(\infty)}(G)$ [also $D(G)$, in literature] we denote the set of functions with compact support in G, i.e. the set of those functions $u \in C^{(\infty)}(\bar{G})$ the support of which (denoted by supp u) lies in G,

$$(6.7) \qquad \operatorname{supp} u \subset G .$$

Here, by the support of a function $u(x, y)$ we understand the closure of such points $(x, y) \in G$ in which $u(x, y) \neq 0$. Thus, functions of $C_0^{(\infty)}(G)$ have in $\bar{G}$ continuous derivatives of all orders while [according to (6.7)] they are equal to zero in a certain neighbourhood of the boundary Γ [which is different for different functions of $C_0^{(\infty)}(G)$, in general]. The closure in the metric (6.6), of $C_0^{(\infty)}(G)$ is denoted by $\mathring{W}_2^{(k)}(G)$. It is a linear subspace of the space $W_2^{(k)}(G)$. For every $u \in \mathring{W}_2^{(k)}(G)$ we have in the sense of traces (on the concept of a trace see, e.g., in [3], Chap. 30)

$$u = 0, \quad \frac{\partial u}{\partial \nu} = 0, \ldots, \quad \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0 \quad \text{on} \quad \Gamma,$$

where ν is the outward normal[1]) of Γ.

For us the converse of this assertion for $k = 2$ (see e.g. [1], p. 90) will be of use: Let G be a region with a Lipschitz boundary, let $u \in W_2^{(2)}(G)$ and let

$$u = 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \tag{6.8}$$

in the sense of traces. Then $u \in \mathring{W}_2^{(2)}(G)$.

Let $w(x, y)$ be such a function of $W_2^{(2)}(G)$ that

$$w = g_0(s), \quad \frac{\partial w}{\partial \nu} = g_1(s) \quad \text{on} \quad \Gamma \tag{6.9}$$

in the sense of traces.

By a weak solution of the problem

$$\Delta^2 U = 0 \quad \text{in} \quad G, \tag{6.10}$$

$$U = g_0(s) \quad \text{on} \quad \Gamma, \tag{6.11}$$

$$\frac{\partial U}{\partial \nu} = g_1(s) \quad \text{on} \quad \Gamma \tag{6.12}$$

we understand (see e.g. [3], Chap. 32) such a function $U \in W_2^{(2)}(G)$ which satisfies

$$U - w \in V, \tag{6.13}$$

$$A(U, v) = 0 \quad \text{for every} \quad v \in V, \tag{6.14}$$

where

$$A(U, v) = \iint_G \left(\frac{\partial^2 U}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 U}{\partial x \, \partial y} \frac{\partial^2 v}{\partial x \, \partial y} + \frac{\partial^2 U}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \right) \mathrm{d}x \, \mathrm{d}y \tag{6.15}$$

[1]) Which exists almost everywhere on Γ, if Γ is a Lipschitz boundary, see e.g. in [1].

and

$$V = \mathring{W}_2^{(2)}(G)\,. \tag{6.16}$$

It is well known ([3], Chap. 33) that the bilinear form $A(U, v)$ is a so-called V--elliptic form which implies that there exists precisely one weak solution of the problem (6.10)–(6.12) [provided there exists $w \in W_2^{(2)}(G)$ satisfying (6.9)].

Consider now the problem (1.1), (1.2), p. 101 and let G be a bounded simply connected region in E_2 with a Lipschitz boundary Γ, $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$.

b) The Solvability of System (4.10)

As we have shown in Chap. 4, our method leads to the solution of $4n - 2$ equations

$$\sum_{j=1}^{4n-2} (z_i, z_j)_\Gamma\, a_{nj} = c_i\,, \quad i = 1, \ldots, 4n - 2 \tag{6.17}$$

for $4n - 2$ unknowns a_{nj}. Here, $(z_i, z_j)_\Gamma$ and c_i are given by (4.8), (4.9). We first prove that system (6.17) has a unique solution. To this purpose it is sufficient to show that its determinant is different from zero.

Let us consider the functions $z_i(x, y)$, $i = 1, \ldots, 4n - 2$, which appear in (6.17) and denote by M the set of all their linear combinations

$$u(x, y) = \sum_{i=1}^{4n-2} \alpha_i\, z_i(x, y) \tag{6.18}$$

with real coefficients α_i. M is a linear set the zero element of which is the function identically equal to zero in G. Denote for $u, v \in M$ [cf. (4.8)]

$$\begin{aligned}(u, v)_\Gamma &= \int_0^l u(s)\, v(s)\, \mathrm{d}s + \int_0^l \frac{\partial u}{\partial s}(s) \frac{\partial v}{\partial s}(s)\, \mathrm{d}s + \int_0^l \frac{\partial u}{\partial \nu}(s) \frac{\partial v}{\partial \nu}(s)\, \mathrm{d}s = \\ &= (u, v)_{L_2(\Gamma)} + \left(\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s}\right)_{L_2(\Gamma)} + \left(\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu}\right)_{L_2(\Gamma)}.\end{aligned} \tag{6.19}$$

Here

$$\frac{\partial u}{\partial s}(s) = -\frac{\partial u}{\partial x}(s)\, \nu_y(s) + \frac{\partial u}{\partial y}(s)\, \nu_x(s)\,, \tag{6.20}$$

$$\frac{\partial u}{\partial \nu}(s) = \frac{\partial u}{\partial x}(s)\, \nu_x(s) + \frac{\partial u}{\partial y}(s)\, \nu_y(s)\,, \tag{6.21}$$

where $\dfrac{\partial u}{\partial x}(s)$, $\dfrac{\partial u}{\partial y}(s)$ are traces of functions $\dfrac{\partial u}{\partial x}(x, y)$, $\dfrac{\partial u}{\partial y}(x, y)$ on Γ and $\nu_x(s)$, $\nu_y(s)$ are components of the unit outward-normal vector (and similarly for v). The integrals in (6.19) have a sense, because traces of the functions $u(x, y)$, $\dfrac{\partial u}{\partial x}(x, y)$, $\dfrac{\partial u}{\partial y}(x, y)$

belong to $L_2(\Gamma)$ and Γ is a Lipschitz boundary so that $v_x(s)$, $v_y(s)$ are bounded measurable functions on Γ([3], Chap. 28).

We prove first that (6.19) is a scalar product on the linear set M. To this purpose it is sufficient to prove

$$(u, u)_\Gamma = 0 \Rightarrow u(x, y) \equiv 0 \quad \text{in} \quad G\,, \tag{6.22}$$

because the remaining axioms of a scalar product are obviously fulfilled. Thus, let $u \in M$, so that $u(x, y)$ is of the form (6.18) and let $(u, u)_\Gamma = 0$. According to (6.19) we have

$$(u, u)_\Gamma = \|u\|^2_{L_2(\Gamma)} + \left\|\frac{\partial u}{\partial s}\right\|^2_{L_2(\Gamma)} + \left\|\frac{\partial u}{\partial v}\right\|^2_{L_2(\Gamma)}.$$

Consequently, from $(u, u)_\Gamma = 0$ it follows

$$u = 0 \quad \text{in} \quad L_2(\Gamma) \quad \text{and} \quad \frac{\partial u}{\partial v} = 0 \quad \text{in} \quad L_2(\Gamma)\,. \tag{6.23}$$

For the function $u(x, y)$ we then have

$$u = 0, \quad \frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \Gamma$$

in the sense of traces.

Thus the function $u(x, y)$, being a linear combination of biharminic polynomials, is a weak solution of the biharmonic problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{6.24}$$

$$u = 0, \quad \frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \Gamma\,. \tag{6.25}$$

From uniqueness of the weak solution of problem (6.24), (6.25) it follows that $u = 0$ in $W_2^{(2)}(G)$ and, because of smoothness of this function in $\bar{G}$,

$$u(x, y) \equiv 0 \quad \text{in} \quad G$$

what we had to prove.

Thus $(.,.)_\Gamma$ is a scalar product in M. But the functions $z_i(x, y)$ are linearly independent in M (Theorem 3.1, p. 107), so that the determinant of the system (6.17) which is the Gram determinant constructed of scalar products of these functions, is different from zero. Consequently, the system (6.17) is uniquely solvable (for every fixed n) what was to be proved.

c) Convergence of the Method

The proof of convergence of the method of least squares on the boundary is rather difficult. We present first some known results which we shall use further. Let us note once more that G is a bounded, simply connected region in E_2 with a Lipschitz boundary.

It follows from Theorem 30. 1 in [3], applied to the functions $u(x, y)$, $\frac{\partial u}{\partial x}(x, y)$ $\frac{\partial u}{\partial y}(x, y)$ and from formulae (6.20), (6.21):

Lemma 6.1. *The mapping of the space $W_2^{(2)}(G)$ into the space $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ is bounded. In details: There exists such a constant $\alpha > 0$, depending only on G that for every $z \in W_2^{(2)}(G)$ we have*

$$(6.26) \qquad \|z\|_{W_2^{(1)}(\Gamma)} \leqq \alpha \|z\|_{W_2^{(2)}(G)}, \quad \left\|\frac{\partial z}{\partial v}\right\|_{L_2(\Gamma)} \leqq \alpha \|z\|_{W_2^{(2)}(G)}.$$

Here

$$(6.27) \qquad \|z\|^2_{W_2^{(1)}(\Gamma)} = \|z\|^2_{L_2(\Gamma)} + \left\|\frac{\partial z}{\partial s}\right\|^2_{L_2(\Gamma)} \quad (\|z\|_{W_2^{(1)}(\Gamma)} \geqq 0).$$

From this lemma and from the linearity of the spaces considered, it follows immediately, if we put $\gamma = \beta/\alpha$:

Lemma 6.2. *To every $\beta > 0$ there exists such a γ that for every two functions $z_1, z_2 \in W_2^{(2)}(G)$ for which*

$$(6.28) \qquad \|z_1 - z_2\|_{W_2^{(2)}(G)} < \gamma$$

we have

$$(6.29) \qquad \|z_1 - z_2\|_{W_2^{(1)}(\Gamma)} < \beta, \quad \left\|\frac{\partial z_1}{\partial v} - \frac{\partial z_2}{\partial v}\right\|_{L_2(\Gamma)} < \beta.$$

Lemma 6.3. ([1], p. 270.) *The traces of functions of the space $W_2^{(2)}(G)$ are dense in the space $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. In details: To every two functions $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$ and to every $\eta > 0$ there exists such a function $z \in W_2^{(2)}(G)$ that the inequalities*

$$(6.30) \qquad \|z - g_0\|_{W_2^{(1)}(\Gamma)} < \eta, \quad \left\|\frac{\partial z}{\partial v} - g_1\right\|_{L_2(\Gamma)} < \eta$$

hold.

Remark 6.1. From Lemma 6.3 it does not follow that to the given functions $g_0(s)$, $g_1(s)$ of the mentioned properties there exists such a function $w \in W_2^{(2)}(G)$

that we have

$$(6.31) \qquad w(s) = g_0(s)\,, \quad \frac{\partial w}{\partial \nu}(s) = g_1(s) \quad \text{on} \quad \Gamma$$

in the sense of traces. Consequently, the existence of a weak solution of the biharmonic problem (6.10)–(6.12), p. 116, is not ensured, because, in formulation (6.13), (6.14), the existence of such a function is assumed. If such a function exists [i.e. if, for example, the boundary Γ and the functions $g_0(s)$, $g_1(s)$ are sufficiently smooth], then problem (6.10)–(6.12) has a weak solution in the sense defined formerly. In the general case, we can come, on base of lemma 6.3, to the concept of a very weak solution:

Let us construct a decreasing sequence of positive numbers ε_n, $\lim \varepsilon_n = 0$ for $n \to \infty$. According to Lemma 6.3, to given functions $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$ and to each of the numbers ε_n, it is possible to find such a function $v_n \in W_2^{(2)}(G)$ that the following inequalities hold:

$$(6.32) \qquad \|v_n - g_0\|_{W_2^{(1)}(\Gamma)} < \varepsilon_n\,, \quad \left\|\frac{\partial v_n}{\partial \nu} - g_1\right\|_{L_2(\Gamma)} < \varepsilon_n\,.$$

Denote $\tilde{v}_n(x, y)$ the weak solution of the biharmonic problem (6.10)–(6.12), corresponding to the function $v_n(x, y)$, thus satisfying the conditions [cf. (6.13), (6.14)]

$$(6.33) \qquad \tilde{v}_n - v_n \in V\,,$$

$$(6.34) \qquad A(\tilde{v}_n, v) = 0 \quad \text{for every} \quad v \in V\,.$$

Lemma 6.4. ([1], p. 274.) *Let $\{v_n(x, y)\}$ be an arbitrary sequence of the just stated properties. Then the sequence of corresponding functions $\tilde{v}_n(x, y)$ converges in $L_2(G)$ to a certain function $U(x, y)$, uniquely determined by the given functions $g_0 \in W_2^{(2)}(\Gamma)$, $g_1 \in L_2(\Gamma)$* [thus independent of the choice of the sequence $\{\varepsilon_n\}$ and of functions $v_n(x, y)$, satisfying conditions (6.32)].

The function $U(x, y)$ is called a *very weak solution* of the problem (6.10)–(6.12).

Remark 6.2. If the problem (6.10)–(6.12) has a weak solution [i.e. if a function $w \in W_2^{(2)}(G)$ exists satisfying conditions (6.31)], then the just defined very weak solution is the weak solution of the problem considered. Obviously, it is sufficient to put $v_n = w$ for every n.

Remark 6.3. The aim of this Chapter is to prove that the sequence of functions

$$(6.35) \qquad U_n(x, y) = \sum_{i=1}^{4n-2} a_{ni}\, z_i(x, y)$$

constructed by the method of least squares on the boundary [so that the coefficients a_{ni}, $i = 1, \ldots, 4n - 2$ are given by the system (4.10)], converges, in $L_2(G)$, to the very weak solution $U(x, y)$ of the problem (6.10)–(6.12), given by functions $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$. To this purpose, we prove that not only the traces of functions of the space $W_2^{(2)}(G)$ are dense in $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$, but that the same property have traces of the set of all linear combinations of biharmonic polynomials $z_i(x, y)$, $i = 1, 2, \ldots$. First we present some known results:

Lemma 6.5. *The weak solution of a biharmonic problem has in every closed region $\bar{O}$ which is contained in G, continuous derivatives of all orders and is in $\bar{O}$ a classical solution of the biharmonic equation.*

This lemma is a consequence of Theorem 1.1 or 1.2 in [1], p. 197 and 199 and of Sobolev's Imbedding Theorems.

Further, it follows from Theorem 2.4.1 in [2], p. 63 and from the construction of functions $\varphi(z)$, $\chi(z)$ on p. 63 and 64 of the quoted book:

Lemma 6.6. *If $u(x, y)$ is a biharmonic function in G* (thus a classical solution of the biharmonic equation in G), *then there exist such functions $\varphi(z)$, $\chi(z)$ of complex variable z, holomorphic in G that in G we have*

$$u(x, y) = \operatorname{Re}\left[\bar{z}\, \varphi(z) + \chi(z)\right] . \tag{6.36}$$

Conversely, if $\varphi(z)$, $\chi(z)$ are holomorphic functions in G, then the function (6.36) *is biharmonic in G.*

If, moreover, the function $u(x, y)$ and its derivatives of all orders are continuous in $\bar{G}$ [i.e. if $u \in C^{(\infty)}(\bar{G})$], *the functions $\varphi(z)$, $\chi(z)$ and all their derivatives* (with respect to z) *are continuous in $\bar{G}$.*

Lemma 6.7. (The *Walsh Theorem*, [2], p. 490.) *Let $f(z)$ be holomorphic in G and continuous in $\bar{G}$. Then to every $\varkappa$ a polynomial $P_m(z)$* (of a sufficiently high degree m) *can be found that in $\bar{G}$ we have*

$$|f(z) - P_m(z)| < \varkappa .$$

Remark 6.4. If $\varphi(z)$ and $\varphi'(z)$ are holomorphic functions in G and continuous in $\bar{G}$, then to every $\mu > 0$ it is possible to find such a polynomial $P(z)$ that in $\bar{G}$

$$|\varphi(z) - P(z)| < \mu , \quad |\varphi'(z) - P'(z)| < \mu$$

holds simultaneously. Indeed, it is sufficient to find, according to Lemma 6.7, such a polynomial $P'(z)$ that $|\varphi'(z) - P'(z)|$ be sufficiently small in $\bar{G}$ and then to integrate this polynomial over G, taking a proper constant of integration.

Similarly, if $\varphi(z), \varphi'(z), \ldots, \varphi^{(k)}(z)$ are holomorphic in G and continuous in $\bar{G}$, it is possible to find, to every $v > 0$, such a polynomial $P(z)$ that in $\bar{G}$ we have

$$|\varphi(z) - P(z)| < v , \quad |\varphi'(z) - P'(z)| < v, \ldots, |\varphi^{(k)}(z) - P^{(k)}(z)| < v . \tag{6.37}$$

Let, now, the function $u(x, y)$ be biharmonic in G and continuous, including derivatives of all orders, in $\bar{G}$. According to Lemma 6.6, the functions $\varphi(z)$, $\chi(z)$ and their derivatives of all orders are then continuous in $\bar{G}$. Let k be an arbitrary, but fixed positive integer. Then to every $v > 0$ it is possible to find such a polynomial $P(z)$ that (6.37) holds. Similarly, to the same $v > 0$ we can find such a polynomial $Q(z)$ that for the functions $\chi(z)$ and $Q(z)$ similar relations hold as in (6.37). From (6.36) it follows that when calculating partial derivatives of the function $u(x, y)$ up to the order k including, we use derivatives of the functions $\varphi(z)$, $\chi(z)$ also up to the order k including. Consequently, if we replace in (6.36) the functions $\varphi(z)$, $\chi(z)$ by their "sufficiently close" [in the sense (6.37)] approximations $P(z)$, $Q(z)$, then there will be "sufficiently close" not only the functions

$$u(x, y) = \operatorname{Re}[\bar{z}\,\varphi(z) + \chi(z)] \quad \text{and} \quad p(x, y) = \operatorname{Re}[\bar{z}\,P(z) + Q(z)]\,, \tag{6.38}$$

but also their partial derivatives with respect to x, y up to the order k including. Here, $P(z)$ and $Q(z)$ are polynomials in z, so that $p(x, y)$ is a polynomial in x and y, according to Lemma 6.6 biharmonic (thus satisfying the biharmonic equation).

If we take into account that the region G is bounded and that the expression for the norm of functions from the space $W_2^{(k)}(G)$ contains only these functions and their derivatives to the order k including, we can present the following lemma which is itself of interest:

Lemma 6.8. *Let the function $u(x, y)$ be biharmonic in G and let it be including partial derivatives of all orders continuous in $\bar{G}$* [thus $u \in C^{(\infty)}(\bar{G})$]. *Let k be an arbitrary positive integer. Then to every $\sigma > 0$ there exists such a biharmonic polynomial $p(x, y)$ that*

$$\|u - p\|_{W_2^{(k)}(G)} < \sigma\,. \tag{6.39}$$

Further we shall use this lemma for the special case $k = 2$.

We now prove the fundamental lemma of this Chapter.

Lemma 6.9. *To every function $z \in W_2^{(2)}(G)$ and to every $\tau > 0$ it is possible to find such a function $\tilde{z}(x, y)$, biharmonic in G that*

$$\tilde{z} \in C^{(\infty)}(\bar{G})$$

and

$$\|\tilde{z} - z\|_{W_2^{(1)}(\Gamma)} < \tau\,, \quad \left\|\frac{\partial \tilde{z}}{\partial v} - \frac{\partial z}{\partial v}\right\|_{L_2(\Gamma)} < \tau\,. \tag{6.40}$$

Proof. Let us construct such a sequence of plane bounded simply connected regions G_j with Lipschitz boundaries Γ_j that

$$\bar{G} \subset G_j\,, \quad \bar{G}_{j+1} \subset G_j \quad \text{for every} \quad j = 1, 2, \ldots, \tag{6.41}$$

$$\lim_{j \to \infty} m(G_j - \bar{G}) = 0\,, \tag{6.42}$$

where $m(G_j - \bar{G})$ is the Lebesque measure of the region $G_j - \bar{G}$. Thus, the region G is "approximated from outside" by a sequence of regions G_j with a Lipschitz boundary (Fig. 2). Such a sequence exists – to its construction we can use – if we want – conformal mapping of the complement of G on the complement of a unit circle K and choose the regions G_j in such a way that they correspond to circles K_j with centres at the origin and with radii $r_j = 1 + 1/j$, $j = 1, 2, \ldots$. In this case, the boundaries of G_j will be even very smooth.

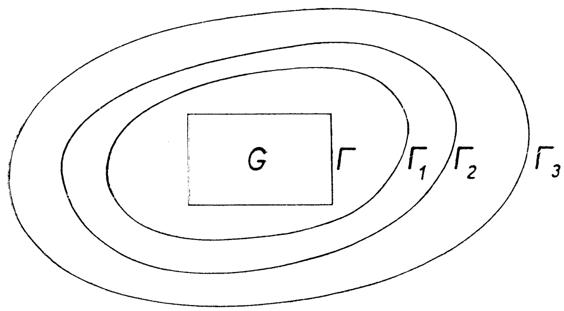

Fig. 2.

To the given function $z \in W_2^{(2)}(G)$ there exists a unique function $u_0 \in W_2^{(2)}(G)$ which is a weak solution of the biharmonic problem given by the condition

$$u_0 - z \in \mathring{W}_2^{(2)}(G)\,. \tag{6.43}$$

However, this function does not belong to $C^{(\infty)}(\bar{G})$, in general.

Let us extend the function $u_0(x, y)$ in a usual way ([1], p. 80) on the whole region G_1 (thus on the "largest" of regions G_j) in order that the so extended function – denote it by $U_0(x, y)$ – belongs to $W_2^{(2)}(G_1)$. Let us denote by $U_{0j}(x, y)$ the restriction of the function $U_0(x,y)$ on G_j, thus the function $U_0(x, y)$ considered only on the region G_j. [Here, $U_{01}(x, y) = U_0(x, y)$, of course.] Evidently, $U_{0j} \in W_2^{(2)}(G_j)$. Denote, further, by $u_j(x, y)$ the weak solution of the biharmonic problem on G_j, given by the condition

$$u_j - U_{0j} \in \mathring{W}_2^{(2)}(G_j) \tag{6.44}$$

and by $U_j(x, y)$ a function, defined on G_1 as follows:

$$U_j(x, y) = \begin{cases} u_j(x, y) & \text{on } G_j \\ U_0(x, y) & \text{on } G_1 - G_j\,. \end{cases} \tag{6.45}$$

Thus the function $U_j(x, y)$ is a weak solution of a biharmonic equation in G_j and is equal to $U_0(x, y)$ outside of G_j. Because the function $U_{0j}(x, y)$ is the restriction of $U_0(x, y)$ on G_j, the traces of $U_{0j}(x, y)$ and $U_0(x, y)$ on Γ_j are the same. From (6.44) it then follows $U_j - U_0 \in W_2^{(2)}(G_1)$. Further $U_j(x, y) = [U_j(x, y) - U_0(x, y)] + U_0(x, y)$ and $U_0 \in W_2^{(2)}(G_1)$, consequently $U_j \in W_2^{(2)}(G_1)$.

Intuitively, we can expect that the restriction $\tilde{u}_j(x, y)$ of the function $u_j(x, y)$ on G will be very close, in the metric of the space $W_2^{(2)}(G)$, to the function $u_0(x, y)$, if j will be sufficiently large. If we prove this assertion, i.e. if we prove that

$$\lim_{j \to \infty} \|\tilde{u}_j - u_0\|_{W_2^{(2)}(G)} = 0\,, \tag{6.46}$$

then we can first affirm, on base of Lemma 6.2 that the traces of functions $u_0(x, y)$, $\tilde{u}_j(x, y)$ [and thus, according to (6.43), also the traces of functions $z(x, y)$ and $\tilde{u}_j(x, y)$] will be sufficiently close in $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ [so that inequalities of the type (6.40) be fulfilled if j will be chosen sufficiently large]. Moreover, according to Lemma 6.5, each of the functions $\tilde{u}_j(x, y)$ will be biharmonic in G and $\tilde{u}_j \in C^{(\infty)}(\bar{G})$, because $\bar{G} \subset G_j$. Consequently, for the function $\tilde{z}(x, y)$ to be found it is then possible to take the restriction $\tilde{u}_j(x, y)$ of a function $u_j(x, y)$ on G, for j sufficiently large. Thus, if we prove (6.46), our lemma is proved.

Denote

$$Z_j(x, y) = U_j(x, y) - U_0(x, y) \quad \text{in} \quad G_1\,. \tag{6.47}$$

According to the definition of functions $U_0(x, y)$ and $U_j(x, y)$ we have

$$Z_j(x, y) = u_j(x, y) - U_{0j}(x, y) \quad \text{in} \quad G_j\,,$$

$$Z_j(x, y) \equiv 0 \quad \text{in} \quad G_1 - G_j\,.$$

Further, in G we have

$$U_j(x, y) = \tilde{u}_j(x, y)\,, \quad U_0(x, y) = u_0(x, y)$$

and consequently,

$$Z_j(x, y) = \tilde{u}_j(x, y) - u_0(x, y) \quad \text{in} \quad G\,. \tag{6.48}$$

Thus, if we prove that

$$\lim_{j \to \infty} \|Z_j\|_{W_2^{(2)}(G_1)} = 0\,,$$

we shall the more have

$$\lim_{j \to \infty} \|Z_j\|_{W_2^{(2)}(G)} = 0\,,$$

and this is just the required result (6.46).

As we shall see further, we prove a rather weaker assertion: We prove that it is possible to find such a subsequence $\{Z_{j_k}(x, y)\}$ of the sequence $\{Z_j(x, y)\}$ that

$$\lim_{k \to \infty} \|Z_{j_k}\|_{W_2^{(2)}(G_1)} = 0\,. \tag{6.49}$$

From here, we get

$$\lim_{k\to\infty} \|\tilde{u}_{j_k} - u_0\|_{W_2^{(2)}(G)} = 0 \tag{6.50}$$

which is, indeed, a weaker result than the result (6.46), but quite sufficient for our aim, because, for the required function $z(x, y)$, it is possible to choose the restriction $\tilde{u}_{j_k}(x, y)$, on G, of a function $u_{j_k}(x, y)$ with a sufficiently large j_k. Consequently, in order to prove Lemma 6.9, it is sufficient to prove the assertion (6.49).

Therefore, let us proceed to the proof of this assertion.

Denote by $A_j(u, v)$ the form (6.15) considered on the region G_j, i.e.

$$A_j(u, v) = \iint_{G_j} \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2}\right) \mathrm{d}x\,\mathrm{d}y\,. \tag{6.51}$$

As shown e.g. in [3], p. 408, for the weak solution $u_j(x, y)$ of the biharmonic problem, given by the condition (6.44), the following estimate holds:

$$\|u_j\|_{W_2^{(2)}(G_j)} \leqq K\|U_{0j}\|_{W_2^{(2)}(G_j)}\,, \tag{6.52}$$

where K is a constant given by the constant of V_j-ellipticity of the form $A_j(u, v)$ in the region G_j. Because $G_j \subset G_1$ for every $j = 2, 3, \ldots$, it is possible to choose the same K in (6.52) for all $j = 1, 2, \ldots$[2]). Now, the function $U_{0j}(x, y)$ is a restriction, on G_j, of the function $U_0(x, y)$. From (6.52) we then get

$$\|u_j\|_{W_2^{(2)}(G_j)} \leqq K\|U_0\|_{W_2^{(2)}(G_1)}\,.$$

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[2]) Let (a, b) or (c, d) be projections of the region G_1 into the x- or y-axis, respectively. For every $v \in \mathring{W}_2^{(1)}(G_j)$, $j = 1, 2, \ldots$, the Friedrichs inequality holds,

$$\iint_{G_j} v^2\,\mathrm{d}x\,\mathrm{d}y \leqq c_1 \iint_{G_j} \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y\,,$$

where we can put [see (18.46) in [3], p. 205]

$$c_1 = \frac{1}{\pi^2\left[\dfrac{1}{(b-a)^2} + \dfrac{1}{(d-c)^2}\right]}\,.$$

Now, from (23.35), (23.36), (23.38) in [3], p. 285 and 286, it easily follows

$$A_j(v, v) \geqq \frac{1}{c_1^2 + c_1 + 1}\,\|v\|^2_{\mathring{W}_2^{(2)}(G_j)}\,.$$

Consequently, the forms $A_j(u, v)$ are "uniformly V-elliptic".

From the definition of the functions $U_j(x, y)$ it follows immediately

$$\|U_j\|^2_{W_2^{(2)}(G_1)} \leqq \|u_j\|^2_{W_2^{(2)}(G_j)} + \|U_0\|^2_{W_2^{(2)}(G_1)}$$

so that

$$\|U_j\|_{W_2^{(2)}(G_1)} \leqq \|u_j\|_{W_2^{(2)}(G_j)} + \|U_0\|_{W_2^{(2)}(G_1)}$$

and in consequence of (6.52)

$$\|U_j\|_{W_2^{(2)}(G_1)} \leqq (K + 1) \|U_0\|_{W_2^{(2)}(G_1)} . \tag{6.53}$$

Thus the functions $U_j(x, y)$, $j = 1, 2, \ldots$, and according to (6.47), also the functions $Z_j(x, y)$ are uniformly bounded in $W_2^{(2)}(G_1)$. Consequently, a subsequence $\{Z_{j_k}(x, y)\}$ of the sequence $\{Z_j(x, y)\}$ can be found, converging weakly in $W_2^{(2)}(G_1)$ to a function $Z \in W_2^{(2)}(G_1)$,

$$Z_{j_k} \rightharpoonup Z \quad \text{in} \quad W_2^{(2)}(G_1) . \tag{6.54}$$

Without loss of generality we can assume that corresponding subsequences of (generalized) derivatives of functions $Z_{j_k}(x, y)$ up to the second order including converge weakly in $L_2(G_1)$ to corresponding (generalized) derivatives of the function $Z(x, y)$, so that we have

$$D^i Z_{j_k} \rightharpoonup D^i Z , \quad |i| \leqq 2 \quad \text{in} \quad L_2(G_1) , \tag{6.55}$$

because $Z_j(x, y)$, being uniformly bounded in $W_2^{(2)}(G_1)$, all these derivatives are uniformly bounded in $L_2(G_1)$. For $|i| = 0$ we get, in particular, $Z_{j_k} \rightharpoonup Z$ in $L_2(G_1)$. We prove now that

$$Z = 0 \quad \text{in} \quad W_2^{(2)}(G_1) . \tag{6.56}$$

We prove first that the function $Z(x, y)$ cannot be different from zero on any set of positive measure lying in $G_1 - G$. Let the contrary be true, i.e. let

$$\|Z\|_{L_2(G_1 - G)} > 0 .$$

But $Z_{j_k} \rightharpoonup Z$ in $L_2(G_1)$ and, consequently, also in $L_2(G_1 - G)$, so that

$$\lim_{k \to \infty} (Z, Z_{j_k})_{L_2(G_1 - G)} = (Z, Z)_{L_2(G_1 - G)} > 0 . \tag{6.57}$$

Further, $Z_{j_k}(x, y) \equiv 0$ in $G_1 - G_{j_k}$, thus

$$|(Z, Z_{j_k})_{L_2(G_1 - G)}| = \tag{6.58}$$

$$= |(Z, Z_{j_k})_{L_2(G_{j_k} - G)}| \leqq \|Z\|_{L_2(G_{j_k} - G)} \|Z_{j_k}\|_{L_2(G_{j_k} - G)} .$$

The sequence $\{Z_{j_k}(x, y)\}$ is uniformly bounded in $L_2(G_1)$ and at the same time

$$\lim_{k \to \infty} \|Z\|_{L_2(G_{j_k} - G)} = 0$$

in consequence of (6.42). From here and from (6.58) we get

$$\lim_{k\to\infty} (Z, Z_{j_k})_{L_2(G_1 - G)} = 0$$

in contradiction with (6.57).

Thus $Z = 0$ in $G_1 - G$. Because, moreover, $Z \in W_2^{(2)}(G_1)$, it follows for its traces and for traces of its first derivatives

$$Z = 0\,, \quad \frac{\partial Z}{\partial x} = 0\,, \quad \frac{\partial Z}{\partial y} = 0 \quad \text{on} \quad \Gamma$$

which implies (see p. 116) for the restriction $\tilde{Z}(x, y)$, on G, of this function

$$\tilde{Z} \in \mathring{W}_2^{(2)}(G)\,. \tag{6.59}$$

Let $v(x, y)$ be an arbitrary function of $\mathring{W}_2^{(2)}(G)$. Let us extend it by zero on the whole region G_1 and denote the so extended function by $V(x, y)$. For the restriction $\tilde{V}_{j_k}(x, y)$ of this function on G_{j_k} ($k = 1, 2, \ldots$) we have obviously $\tilde{V}_{j_k} \in \mathring{W}_2^{(2)}(G_{j_k})$.

Let us denote by $\tilde{Z}_{j_k}(x, y)$ the restriction of the function $Z_{j_k}(x, y)$ on G_{j_k}. It follows from (6.47) [see the text following (6.47)]

$$\tilde{Z}_{j_k}(x, y) = u_{j_k}(x, y) - U_{0j_k}(x, y) \quad \text{in} \quad G_{j_k}\,.$$

In consequence of (6.55) we then have [about A_j see in (6.51)]

$$A(\tilde{Z}, v) = A_1(Z, V) = \lim_{k\to\infty} A_1(Z_{j_k}, V) = \lim_{k\to\infty} A_{j_k}(\tilde{Z}_{j_k}, \tilde{V}_{j_k})\,. \tag{6.60}$$

But for every j_k, $k = 1, 2, \ldots$, we have [consider that $\tilde{V}_{j_k}(x, y) \equiv 0$ outside of G]

$$A_{j_k}(\tilde{Z}_{j_k}, \tilde{V}_{j_k}) = A_{j_k}(u_{j_k}, \tilde{V}_{j_k}) - A_{j_k}(U_{0j_k}, \tilde{V}_{j_k}) =$$
$$A_{j_k}(u_{j_k}, \tilde{V}_{j_k}) - A(u_0, v) = 0\,,$$

because u_{j_k}, or u_0 is a weak solution of the biharmonic equation in G_{j_k}, or G, respectively, so that we have [cf. (6.14), p. 116]

$$A_{j_k}(u_{j_k}, \tilde{V}_{j_k}) = 0$$

and

$$A(u_0, v) = 0\,.$$

Thus, from (6.60) it follows

$$A(\tilde{Z}, v) = 0 \tag{6.61}$$

for every $v \in \mathring{W}_2^{(2)}(G)$ [because $v \in (\mathring{W}_2^{(2)}(G)$ has been chosen arbitrarily]. Consequently, the function $\tilde{Z}(x, y)$ is a weak solution of the biharmonic problem

$$A(u, v) = 0 \quad \text{for every} \quad v \in \mathring{W}_2^{(2)}(G)\,,$$
$$u \in \mathring{W}_2^{(2)}(G)\,.$$

From unicity of the solution of this problem it follows $\tilde{Z} = 0$ in $W_2^{(2)}(G)$. As we have shown in the preceding text (p. 127), $Z(x, y) = 0$ in $G_1 - G$ so that

$$Z = 0 \quad \text{in} \quad W_2^{(2)}(G_1) .$$

From

$$Z_{j_k} \rightharpoonup 0 \quad \text{in} \quad W_2^{(2)}(G_1)$$

and from (6.55) it then follows

$$D^i Z_{j_k} \to 0 \quad \text{in} \quad L_2(G_1) , \quad |i| \leqq 2 . \tag{6.62}$$

We now prove

$$Z_{j_k} \to 0 \quad \text{in} \quad W_2^{(2)}(G_1) \tag{6.63}$$

(strongly, not only weakly). Because the form A_1 is $W_2^{(2)}(G_1)$-elliptic,

$$A_1(u, u) \geqq \alpha \|u\|^2_{W_2^{(2)}(G_1)} \quad \text{for every} \quad u \in \mathring{W}_2^{(2)}(G_1) \tag{6.64}$$

($\alpha > 0$), it is sufficient to show that

$$\lim_{k \to \infty} A_1(Z_{j_k}, Z_{j_k}) = 0 ,$$

which implies [according to (6.64)]

$$\lim_{k \to \infty} \|Z_{j_k}\|_{W_2^{(2)}(G)} = 0 \tag{6.65}$$

what we have to prove. But

$$\begin{aligned} \lim_{k \to \infty} A_1(Z_{j_k}, Z_{j_k}) &= \lim_{k \to \infty} A_1(U_{j_k} - U_0, Z_{j_k}) = \\ &= \lim_{k \to \infty} A_1(U_{j_k}, Z_{j_k}) - \lim_{k \to \infty} A_1(U_0, Z_{j_k}) = \\ &= \lim_{k \to \infty} A_{j_k}(u_{j_k}, \tilde{Z}_{j_k}) - \lim_{k \to \infty} \sum_{|i| = 2} \iint_{G_1} D^i U_0 D^i Z_{j_k} \, dx \, dy = 0 , \end{aligned} \tag{6.66}$$

because, first

$$A_{j_k}(u_{j_k}, \tilde{Z}_{j_k}) = 0 ,$$

u_{j_k} being a weak solution of the biharmonic equation in G_{j_k} and $\tilde{Z}_{j_k} \in \mathring{W}_2^{(2)}(G_{j_k})$, and

$$\lim_{k \to \infty} \iint_{G_1} D^i U_0 \, D^i Z_{j_k} \, dx \, dy = 0$$

for every $|i| \leqq 2$ in consequence of (6.62).

In this way (6.63), i.e. (6.49), is proved. According to the text following (6.50), the proof of Lemma 6.9 is finished.

Now, it follows from Lemma 6.3, from the just proved Lemma 6.9, from Lemma 6.8 in which we put $k = 2$ and from Lemma 6.2:

Lemma 6.10. *The traces of biharmonic polynomials are dense in* $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$. *In details: To every pair of functions* $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$ *and to every* $\varepsilon > 0$ *there exists a polynomial* $p(x, y)$ *which is a solution of the biharmonic equation in* G *and for which the inequalities*

$$\|p - g_0\|_{W_2^{(1)}(\Gamma)} < \varepsilon\,, \quad \left\|\frac{\partial p}{\partial \nu} - g_1\right\|_{L_2(\Gamma)} < \varepsilon \tag{6.68}$$

hold.

Theorem 6.1. *Let* $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$. *Then the sequence of functions* $U_n(x, y)$, *constructed by the method of least squares on the boundary, i.e. the sequence of functions* (4.1) *where the coefficients* a_{ni} *are given by the system* (4.10), *converges in* $L_2(G)$ *to the very weak* (p. 120) *solution of the biharmonic problem*

$$\Delta^2 U = 0 \quad \text{in} \quad G\,, \tag{6.69}$$

$$U = g_0(s), \quad \frac{\partial U}{\partial \nu} = g_1(s) \quad \text{on} \quad \Gamma\,. \tag{6.70}$$

Proof. Let the functions $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$ be given and let $\{\varepsilon_n\}$ be a non-increasing sequence of positive numbers, $\lim \varepsilon_n = 0$. According to Lemma 6.10, to every ε_n such a polynomial $p_{k_n}(x, y)$ of a sufficiently high degree k_n can be found that the inequalities (6.68) hold, where p and ε are replaced by p_{k_n} and ε_n. But $p_{k_n}(x, y)$ being biharmonic, we can write

$$p_{k_n}(x, y) = \sum_{i=1}^{4k_n - 2} b_{k_n i}\, z_i(x, y)\,,$$

where $z_1(x, y)$, $z_2(x, y)$, ... are biharmonic polynomials (3.5) and the coefficients $b_{k_n i}$ are by the polynomial $p_{k_n}(x, y)$ uniquely determined (p. 107). Thus we have

$$\left\|\sum_{i=1}^{4k_n - 2} b_{k_n i} z_i - g_0\right\|_{W_2^{(1)}(\Gamma)} < \varepsilon_n\,, \quad \left\|\sum_{i=1}^{4k_n - 2} b_{k_n i} \frac{\partial z_i}{\partial \nu} - g_1\right\|_{L_2(\Gamma)} < \varepsilon_n\,. \tag{6.71}$$

But for the functions

$$U_{k_n}(x, y) = \sum_{i=1}^{4k_n - 2} a_{k_n i}\, z_i(x, y)\,, \tag{6.72}$$

where the coefficients $a_{k_n i}$ are determined by the method of least squares on the boundary [see (4.1), p. 108], it the more holds

$$\|U_{k_n} - g_0\|_{W_2^{(1)}(\Gamma)} < \varepsilon_n\,, \quad \left\|\frac{\partial U_{k_n}}{\partial \nu} - g_1\right\|_{L_2(\Gamma)} < \varepsilon_n\,. \tag{6.73}$$

Thus, we have

$$(6.74)\qquad \lim_{n\to\infty} U_{k_n} = g_0 \quad \text{in} \quad W_2^{(1)}(\Gamma)\,, \quad \lim_{n\to\infty} \frac{\partial U_{k_n}}{\partial \nu} = g_1 \quad \text{in} \quad L_2(\Gamma)\,.$$

In this way, the convergence [in the sense (6.74)] of the subsequence $\{U_{k_n}(x, y)\}$ of the sequence $\{U_n(x, y)\}$, constructed by the method of least squares on the boundary, is proved. However, we assert that the whole sequence $\{U_n(x, y)\}$ is convergent, in the sense (6.74). In fact, our method being a least squares method, the approximation in the sense (6.74) by polynomials of higher degree can be only better. More precisely: If inequalities (6.73) are valid, then the same inequalities hold for $U_{k_n+1}(x, y)$, $U_{k_n+2}(x, y)$, etc. Thus we really have

$$(6.75)\qquad \lim_{n\to\infty} U_n = g_0 \quad \text{in} \quad W_2^{(1)}(\Gamma)\,, \quad \lim_{n\to\infty} \frac{\partial U_n}{\partial \nu} = g_1 \quad \text{in} \quad L_2(\Gamma)\,.$$

From Lemma 6.4 it then follows

$$(6.76)\qquad \lim_{n\to\infty} U_n(x, y) = U(x, y) \quad \text{in} \quad L_2(G)\,,$$

what completes the proof of Theorem 6.1 and, consequently, the proof of convergence of our method.

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Souhrn

ŘEŠENÍ PRVNÍHO BIHARMONICKÉHO PROBLÉMU METODOU NEJMENŠÍCH ČTVERCŮ NA HRANICI

KAREL REKTORYS a VÁCLAV ZAHRADNÍK

Některé úlohy teorie pružnosti, zejména problémy nosných stěn, vedou k řešení biharmonického problému (1.1), (1.2), str. 101 (podrobněji o něm viz v kapitole 2). K jeho řešení lze aplikovat řadu metod (metodu sítí, metodu konečných prvků, klasické variační metody, metodu založenou na použití funkcí komplexní proměnné, atd.), které mají své specifické přednosti, ale také mnohé nedostatky. Metoda nejmenších čtverců na hranici, vyšetřovaná v této práci, předpokládá přibližné řešení dané úlohy ve tvaru

$$U_n(x, y) = \sum_{i=1}^{4n-2} a_{ni} z_i(x, y), \quad n \geqq 2, \tag{1}$$

kde $z_i(x, y)$, $i = 1, \ldots, 4n - 2$, jsou biharmonické polynomy stupně nejvýše n-tého, popsané v kapitole 3 (str. 106). Koeficienty a_{ni} v (1) jsou určeny podmínkou, aby funkcionál (4.3), str. 108, nabýval právě pro funkci (1) minima na množině všech funkcí tvaru

$$V_n(x, y) = \sum_{i=1}^{4n-2} b_{ni}\, z_i(x, y).$$

Tato podmínka vede na řešení soustavy (4.10), str. 109, $4n - 2$ lineárních rovnic pro hledané koeficienty a_{ni}, $i = 1, \ldots, 4n - 2$. V kapitole 6 (str. 114–130) je dokázáno, že tato soustava je pro každé přirozené n jednoznačně řešitelná a že posloupnost $\{U_n(x, y)\}$ konverguje v $L_2(G)$ k tzv. velmi slabému řešení problému (1.1), (1.2), zavedenému na str. 120. Přitom se předpokládá, že G je rovinná omezená jednoduše souvislá oblast s lipschitzovskou hranicí Γ a že $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$. Pro aplikace v teorii pružnosti jsou tyto předpoklady dostatečně obecné.

První kapitoly práce (str. 101–114) jsou určeny především čtenářům, kteří aplikují matematiku k řešení svých teoretických problémů. Jsou proto psány podrobněji a v kapitole 5 je uveden numerický příklad. Důkaz konvergence je poněkud obtížnější a byl odsunut až do kapitoly 6 (str. 114–130). Tato kapitola je určena především matematikům.

V případě biharmonického problému využívá uvedená metoda podstatně tvaru rovnice (1.1). Lze ji však – vhodně modifikovanou – použít i k řešení problémů jiných.

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