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ORTHOTROPIC ALMOST CYLINDRICAL BEAMS: BENDING BY A TRANSVERSE LOAD

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We shall consider a beam bounded by two planes $x_3 = 0$, $x_3 = h$ and a surface $\mathscr{F}$ given by

$$f[x_1(1 - kx_3), \quad x_2(1 - kx_3)] = 0\,, \tag{1}$$

where k is a small parameter, the square and higher powers of which can be neglected.

Such beams are called "almost cylindrical beams" [1].

Many results are known in connection with beams of the shape (1).

Homogeneous and composite beams have been taken into account in the isotropic case [1], [2], [3] etc. as well as in the anisotropic case [4], [5], [6], [7] etc.

In this Note we will study the problem of bending by a transverse load when the material of the beam is orthotropic.

The solution of the problem will be given in two steps.

First, the problem will be reduced to Almansi's problem.

Second, the complete solution will be given. A solution of this problem corresponding to the first step was given in the paper [6] but by a complicated method. Here, the first step is performed in a very simple way and, moreover, it may be easily generalized to the case when the surface $\mathscr{F}$ is of the form

$$f[x_1(1 - k\theta), \; x_2(1 - k\theta)] = 0\,, \tag{2}$$

where $\theta = \theta(x_3)$ is a given function of x_3.

1. GENERAL EQUATIONS

We suppose that the surface $\mathscr{F}$ is free from tractions and that there are no body forces. We denote by $\mathscr{V}$ the region occupied by the beam.

Under these assumptions the stress components σ_{ij} must satisfy the equilibrium equations

$$\sigma_{ij,j} = 0 \quad \text{in} \quad \mathscr{V}\text{*)} \tag{3}$$

and the boundary conditions

$$\sigma_{ij} n_j = 0 \quad \text{on} \quad \mathscr{F}, \tag{4}$$

where n_i are the direction cosines of the exterior normal to the surface $\mathscr{F}$.

The tractions applied at the end at $x_3 = h$ are equivalent with a transverse load F_1 acting in the direction of x_1.

We shall suppose that the material of the beam is orthotropic so that Hooke's law may be written in the form

$$\begin{aligned} \sigma_{11} &= A\gamma_{11} + H\gamma_{22} + G\gamma_{33}, \\ \sigma_{22} &= H\gamma_{11} + B\gamma_{22} + F\gamma_{33}, \\ \sigma_{33} &= G\gamma_{11} + F\gamma_{22} + C\gamma_{33}, \end{aligned} \tag{5a}$$

$$\sigma_{12} = D\gamma_{12}, \quad \sigma_{23} = L\gamma_{23}, \quad \sigma_{31} = M\gamma_{31}, \tag{5b}$$

where $A, B, \ldots, L, M$ are moduli of elasticity.

We express the components of strain γ_{ij} in terms of stress components by

$$\begin{aligned} \gamma_{11} &= \frac{1}{E}\left(\nu_{11}\sigma_{11} + \nu_{12}\sigma_{22} - \nu_1\sigma_{33}\right), \\ \gamma_{22} &= \frac{1}{E}\left(\nu_{12}\sigma_{11} + \nu_{22}\sigma_{22} - \nu_2\sigma_{33}\right), \\ \gamma_{33} &= \frac{1}{E}\left(-\nu_1\sigma_{11} - \nu_2\sigma_{22} + \sigma_{33}\right), \end{aligned} \tag{6a}$$

$$\gamma_{12} = \frac{1}{D}\sigma_{12}, \quad \gamma_{23} = \frac{1}{L}\sigma_{23}, \quad \gamma_{31} = \frac{1}{M}\sigma_{31}, \tag{6b}$$

where the coefficients of strain E, ν_{ij}, ν_i can be expressed in terms of moduli of elasticity [8].

The components γ_{ij} are given in terms of displacement components u_i by

$$\gamma_{ii} = u_{i,i} \ \text{(not summed)}, \quad \gamma_{ij} = u_{i,j} + u_{j,i} \ (i \neq j)$$

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*) The index j after comma indicates partial differentiation with respect to x_j. We use also the summation convention over the repeated indices.

and they must satisfy the compatibility conditions of Saint-Venant

$$\gamma_{11,22} + \gamma_{22,11} = \gamma_{12,12}, \dots \tag{7a}$$

$$(-\gamma_{23,1} + \gamma_{31,2} + \gamma_{12,3})_{,1} = \gamma_{11,23}, \dots \tag{7b}$$

By means of the transformation [1]

$$\begin{aligned} \xi &= x_1(1 - kx_3)\,, & \eta &= x_2(1 - kx_3)\,, & \zeta &= x_3\,, \\ x_1 &= \xi(1 + k\zeta)\,, & x_2 &= \eta(1 + k\zeta)\,, & x_3 &= \zeta\,, \end{aligned} \tag{8}$$

the surface (1) becomes

$$f(\xi, \eta) = 0\,, \tag{9}$$

which is a cylindrical surface $\mathscr{F}_1$ in the space ξ, η, ζ.

We denote by S the domain of the cross-section of the cylindrical surface (9) and by Γ the boundary of S.

We can easily prove the following formulae

(10a)

$$\frac{\partial}{\partial x_1} = (1 - k\zeta)\frac{\partial}{\partial \xi}\,, \quad \frac{\partial}{\partial x_2} = (1 - k\zeta)\frac{\partial}{\partial \eta}\,, \quad \frac{\partial}{\partial x_3} = \frac{\partial}{\partial \zeta} - k\left(\xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta}\right)$$

and

$$n_1 = \cos\alpha\,, \quad n_2 = \cos\beta\,, \quad n_3 = -k(\xi\cos\alpha + \eta\cos\beta)\,, \tag{10b}$$

where $\cos\alpha$, $\cos\beta$ are the direction cosines of the exterior normal to the curve Γ.

We shall take the axes ξ, η, ζ such that the axis ζ is the central line of the beam (9) and the axes ξ and η are the principal axes of inertia of the end at $\zeta = 0$. In this case we have

$$\iint_S \xi \,\mathrm{d}\xi\,\mathrm{d}\eta = 0\,, \qquad \iint_S \eta\,\mathrm{d}\xi\,\mathrm{d}\eta = 0\,, \qquad \iint_S \xi\eta\,\mathrm{d}\xi\,\mathrm{d}\eta = 0\,. \tag{11}$$

We shall try to find the solution of the above problem supposing that the displacements u_i are of the form

$$\begin{aligned} u_1 &= -\tau\eta\zeta + a\left[\tfrac{1}{2}(h - \zeta)(\nu_1\xi^2 - \nu_2\eta^2) + \tfrac{1}{2}h\zeta^2 - \tfrac{1}{6}\zeta^3\right] + ku_1^0\,, \\ u_2 &= \tau\xi\zeta + a(h - \zeta)\,\nu_2\xi\eta + ku_2^0\,, \\ u_3 &= \tau\varphi(\xi, \eta) - a\left[\chi(\xi, \eta) + (h - \tfrac{1}{2}\zeta)\,\xi\zeta\right] + ku_3^0\,, \end{aligned} \tag{12}$$

where u_i^0 are complementary unknown displacements, τ and a are constants which must be determined from the end conditions. Further, $\varphi(\xi, \eta)$ is the function of torsion of the beam (9) defined by

$$M\frac{\partial^2\varphi}{\partial\xi^2} + L\frac{\partial^2\varphi}{\partial\eta^2} = 0 \quad \text{in} \quad S\,, \tag{13a}$$

$$\mathscr{D}\varphi = M\eta\cos\alpha - L\xi\cos\beta \quad \text{on} \quad \Gamma \tag{13b}$$

and $\chi(\xi, \eta)$ is the flexion function of the same beam (9) defined by

$$M\frac{\partial^2\chi}{\partial\xi^2} + L\frac{\partial^2\chi}{\partial\eta^2} + (Mv_1 + Lv_2 - E)\,\xi = 0 \quad \text{in} \quad S\,, \tag{14a}$$

$$\mathscr{D}\chi = -\tfrac{1}{2}M(v_1\xi^2 - v_2\eta^2)\cos\alpha - Lv_2\xi\eta\cos\beta \quad \text{on} \quad \Gamma\,. \tag{14b}$$

The operator $\mathscr{D}$ is given by

$$\mathscr{D} = M\cos\alpha\,\frac{\partial}{\partial\xi} + L\cos\beta\,\frac{\partial}{\partial\eta}\,. \tag{15}$$

From the displacements (12) we obtain the following components of stress

$$\begin{aligned}
\sigma_{11} &= -kG(H_1 - \tfrac{1}{2}a\xi\zeta^2) + k\tau_{11}\,,\\
\sigma_{22} &= -kF(H_1 - \tfrac{1}{2}a\xi\zeta^2) + k\tau_{22}\,,\\
\sigma_{33} &= -a(1 - k\zeta)(h - \zeta)\,E\xi - kC(H_1 - \tfrac{1}{2}a\xi\zeta^2) + k\tau_{33}\,,\\
\sigma_{12} &= k\tau_{12}\,,\\
\sigma_{23} &= L\left(\frac{\partial h_1}{\partial\eta} + \tau\xi - av_2\xi\eta\right) - kL\left[\left(\frac{\partial h_1}{\partial\eta} + \tau\xi\right)\zeta + 2av_2(h - \zeta)\,\xi\eta\right] + k\tau_{23}\,.\\
\sigma_{31} &= M\left[\frac{\partial h_1}{\partial\xi} - \tau\eta - \frac{1}{2}a(v_1\xi^2 - v_2\eta^2)\right] -\\
&\quad - kM\left\{\left[\frac{\partial h_1}{\partial\xi} - \tau\eta - a\left(h - \frac{1}{2}\zeta\right)\right]\zeta + a(h - \zeta)(v_1\xi^2 - v_2\eta^2)\right\} + k\tau_{31}\,,
\end{aligned} \tag{16}$$

where τ_{ij} are the stresses corresponding to the additional displacements u_i^0 and

$$h_1(\xi, \eta) = \tau\varphi(\xi, \eta) - a\chi(\xi, \eta)\,, \quad H_1(\xi, \eta) = \xi\frac{\partial h_1}{\partial\xi} + \eta\frac{\partial h_1}{\partial\eta}\,. \tag{17}$$

The substitution of stresses (16) into equations (3) shows that the components τ_{ij} must satisfy the equations

$$\begin{aligned}
&\frac{\partial\tau_{11}}{\partial\xi} + \frac{\partial\tau_{12}}{\partial\eta} + \frac{\partial\tau_{31}}{\partial\zeta} - (G + M)\left(\frac{\partial H_1}{\partial\xi} - \tfrac{1}{2}a\zeta^2\right) +\\
&\qquad + 2M[\tau\eta + a(v_1\xi^2 - v_2\eta^2) + a(h - \zeta)\,\zeta] = 0\,,
\end{aligned} \tag{18}$$

$$\frac{\partial \tau_{12}}{\partial \xi} + \frac{\partial \tau_{22}}{\partial \eta} + \frac{\partial \tau_{23}}{\partial \zeta} - (F + L)\frac{\partial H_1}{\partial \eta} - 2L(\tau\xi - 2av_2\xi\eta) = 0\,,$$

$$\frac{\partial \tau_{31}}{\partial \xi} + \frac{\partial \tau_{23}}{\partial \eta} + \frac{\partial \tau_{33}}{\partial \zeta} - a(2h - \zeta)(Mv_1 + Lv_2 - E)\,\xi + aC\xi\zeta = 0\,.$$

2. REDUCTION TO ALMANSI'S PROBLEM

Let us take into account the following components of stress

$$\begin{aligned}(19)\quad & \tau^*_{11} = G(H_1 - \tfrac{1}{2}a\zeta^2)\,, \quad \tau^*_{22} = F(H_1 - \tfrac{1}{2}a\zeta^2)\,, \quad \tau^*_{12} = 0\,,\\ & \tau^*_{33} = C(H_1 - \tfrac{1}{2}a\zeta^2) - 2aE(h - \zeta)\,\xi\zeta\,,\\ & \tau^*_{23} = L\left[\left(\frac{\partial H_1}{\partial \eta} + 2\tau\xi\right)\zeta + 2av_2(h - 2\zeta)\,\xi\eta)\right],\\ & \tau^*_{31} = M\left[\left(\frac{\partial H_1}{\partial \xi} - 2\tau\eta\right)\zeta + a(h - 2\zeta)(v_1\xi^2 - v_2\eta^2) - a(h - \tfrac{1}{2}\zeta)\,\zeta^2\right].\end{aligned}$$

It is easy to verify that the stresses (19) represent a solution of equations (18). Also, we can prove that the corresponding strains γ^*_{ij} satisfy the conditions of compatibility.

Let us now put

$$(20)\qquad \tau_{ij} = \tau^*_{ij} + \tilde{\tau}_{ij}\,.$$

From equations (18) we conclude that the new stress components $\tilde{\tau}_{ij}$ must verify equations

$$(21)\qquad \tilde{\tau}_{ij,j} = 0^{*)}\,.$$

If we take into account (16), (19), (20) and (13b), (14b) we find that the new components of stress $\tilde{\tau}_{ij}$ satisfy the following boundaiy conditions

$$\begin{aligned}(22)\quad & \tilde{\tau}_{11}\cos\alpha + \tilde{\tau}_{12}\cos\beta = M\left[\frac{\partial h_1}{\partial \xi} - \tau\eta - \tfrac{1}{2}a(v_1\xi^2 - v_2\eta^2)\right](\xi\cos\alpha + \eta\cos\beta)\,,\\ & \tilde{\tau}_{12}\cos\alpha + \tilde{\tau}_{22}\cos\beta = L\left[\frac{\partial h_1}{\partial \eta} + \tau\xi - av_2\xi\eta\right](\xi\cos\alpha + \eta\cos\beta)\,,\\ & \tilde{\tau}_{31}\cos\alpha + \tilde{\tau}_{23}\cos\beta = -\zeta\mathcal{D}(H_1 - 2\tau\varphi + 3a\chi) -\\ & \qquad\qquad - aE(h - \zeta)(\xi^2\cos\alpha + \xi\eta\cos\beta) \quad \text{on} \quad \Gamma\,.\end{aligned}$$

Equations (21) and (22) define Almansi's problem whose solution is known [9].

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*) Here the indices 1, 2, 3 after comma indicate partial differentiation with respect to ξ, η, ζ.

3. COMPLETE SOLUTION

Using a method similar to that used in [10] we can solve the problem defined by (21) and (22) in a simpler way.

To this purpose let us consider the following representation of the stresses $\tilde{\tau}_{ij}$:

$$\tag{23} \begin{aligned} \tilde{\tau}_{11} &= \frac{\partial^2 \phi}{\partial \eta^2} - M(\omega_1 + \theta_1), \\ \tilde{\tau}_{22} &= \frac{\partial^2 \phi}{\partial \xi^2} - L(\omega_1 + \theta_2), \\ \tilde{\tau}_{12} &= -\frac{\partial^2 \phi}{\partial \xi \, \partial \eta}, \\ \tilde{\tau}_{33} &= \Omega + \frac{1}{2}\left(\frac{b_1}{\nu_2}\xi + \frac{a_1}{\nu_1}\eta\right)\zeta^2, \\ \tilde{\tau}_{23} &= L\left[\left(\frac{\partial \omega_1}{\partial \eta} + \frac{\partial \theta_2}{\partial \eta}\right)\zeta + \frac{\partial \omega_0}{\partial \eta}\right], \\ \tilde{\tau}_{31} &= M\left[\left(\frac{\partial \omega_1}{\partial \xi} + \frac{\partial \theta_1}{\partial \xi}\right)\zeta + \frac{\partial \omega_0}{\partial \xi}\right], \end{aligned}$$

where ϕ, ω_1, ω_0 are unknown functions which will be defined below,

$$\tag{24} \begin{aligned} \Omega = E\left[\omega_1 + \frac{1}{2}(\theta_1 + \theta_2) + \frac{1}{6}\left(\frac{\nu_1}{\nu_2} a_1 \xi^3 + \frac{\nu_2}{\nu_1} b_1 \eta^3\right)\right] + \\ + \nu_1\left[\frac{\partial^2 \phi}{\partial \eta^2} - M(\omega_1 + \theta_1)\right] + \nu_2\left[\frac{\partial^2 \phi}{\partial \xi^2} - L(\omega_1 + \theta_2)\right], \end{aligned}$$

$$\theta_1 = b_1 \xi \eta^2 - c_1 \xi \eta, \quad \theta_2 = a_1 \xi^2 \eta + c_1 \xi \eta .$$

a_1, b_1, c_1 being some constants which will be chosen in such a way to guarantee the existence of the functions ω_1 and ϕ.

The stresses (23) satisfy equations (21) if the functions ω_0 and ω_1 satisfy the equations

$$\tag{25} M\frac{\partial^2 \omega_0}{\partial \xi^2} + L\frac{\partial^2 \omega_0}{\partial \eta^2} = 0 \quad \text{in} \quad S$$

and

$$\tag{26} M\frac{\partial^2 \omega_1}{\partial \xi^2} + L\frac{\partial^2 \omega_1}{\partial \eta^2} + E\left(\frac{b_1}{\nu_2}\xi + \frac{a_1}{\nu_1}\eta\right) = 0 \quad \text{in} \quad S.$$

The third equation (22) yields the following boundary conditions for the functions ω_0 and ω_1:

$$\tag{27} \mathscr{D}\omega_0 = -ahE(\xi^2 \cos\alpha + \xi\eta \cos\beta) \quad \text{on} \quad \Gamma$$

and

$$\mathscr{D}\omega_1 = -\mathscr{D}(H_1 - 2\tau\varphi + 3a\chi) + aE(\xi^2 \cos\alpha + \xi\eta \cos\beta) - \tag{28}$$
$$- M \frac{\partial\theta_1}{\partial\xi} \cos\alpha - L \frac{\partial\theta_2}{\partial\eta} \cos\beta \quad \text{on} \quad \Gamma.$$

Making use of (11) we can prove that the conditions of existence for the functions ω_0 and ω_1 are satisfied with arbitrary a_1, b_1, c_1.

All compatibility conditions (7) will be satisfied by the strains $\tilde{\gamma}_{ij}$ if the function ϕ satisfies the equation

$$\beta_{22} \frac{\partial^4\phi}{\partial\xi^4} + (2\beta_{12} + \beta_{33}) \frac{\partial^4\phi}{\partial\xi^2\,\partial\eta^2} + \beta_{11} \frac{\partial^4\phi}{\partial\eta^4} = \tag{29}$$
$$= (L\beta_{22} + M\beta_{12} + \nu_2) \frac{\partial^2\omega_1}{\partial\xi^2} + (L\beta_{12} + M\beta_{11} + \nu_1) \frac{\partial^2\omega_1}{\partial\eta^2} +$$
$$+ 2[(M\beta_{11} + \nu_1)\, b_1\xi + (L\beta_{22} + \nu_2)\, a_1\eta] \quad \text{in} \quad S.$$

The first two equations (8) require that

$$\frac{\partial^2\phi}{\partial\eta^2} \cos\alpha - \frac{\partial^2\phi}{\partial\xi\,\partial\eta} \cos\beta = M(\omega_1 + \theta_1) \cos\alpha + \tag{30}$$
$$+ M \left[\frac{\partial h_1}{\partial\xi} - \tau\eta - \frac{1}{2} a(\nu_1\xi^2 - \nu_2\eta^2)\right] (\xi \cos\alpha + \eta \cos\beta),$$
$$- \frac{\partial^2\phi}{\partial\xi\,\partial\eta} \cos\alpha + \frac{\partial^2\phi}{\partial\eta^2} \cos\beta = L(\omega_1 + \theta_2) \cos\beta +$$
$$+ L \left(\frac{\partial h_1}{\partial\eta} + \tau\xi - a\nu_2\xi\eta\right) (\xi \cos\alpha + \eta \cos\beta) \quad \text{on} \quad \Gamma,$$

where

$$\beta_{ij} = \frac{\nu_{ij} - \nu_i\nu_j}{E}, \quad (i, j = 1, 2), \quad \beta_{33} = \frac{1}{D}.$$

From (30) it is obvious that we can obtain the function ϕ in a similar way to that corresponding to Airy's function for the plane problem of orthotropic bodies. It follows that we can choose the constants a_1, b_1, c_1 in such a way to guarantee the existence of the function ϕ.

As shown in [10], we can find the constants a_1, b_1, c_1 before we know the function ω_1.

The solution given here will satisfy all equations and boundary conditions except the end conditions.

Therefore it still remains to correct the end conditions by superposition of solutions of some adequate problems for the cylindrical beam (9) but we can solve all needed additional problems [8], [11].

S o m e r e m a r k s – Making $a = 0$ we obtain the solution concerning the problem of torsion.

– A similar method can be developed when $\theta = \zeta^2$ in the equation (2).

– The above results can be extended to the case when the material of the beam is anisotropic with one plane of elastic symmetry perpendicular to the axis x_3.

– The results can be generalized also to composite beams.

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Souhrn

ORTOTROPNÍ SKORO CYLINDRICKÉ NOSNÍKY: OHYB PŘÍČNÝM ZATÍŽENÍM

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V práci je řešen problém ohybu příčným zatížením pro ortotropní skoro cylindrický nosník převedením na Almansiho problém. Předložená metoda je značně jednodušší než dosud známé řešení.

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