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EVALUATION OF THE HALF-PERIODS OF THE WEIERSTRASS $\wp$-FUNCTION FOR THE ABSOLUTE INVARIANT GREATER THAN ONE

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The paper is a continuation of that on Weierstrass $\wp$-functions by the same author [3]. The solution of this mathematical problem is connected with a motion of a rigird body with one fixed point.

The half-periods of the Weierstrass $\wp$-function [1, p. 328; 13.12] may be deter mined by means of relations [1, p. 341; (9)]:

$$\omega = \frac{K}{\sqrt{(e_1 - e_3)}}; \tag{1}$$

$$\omega' = \frac{iK'}{\sqrt{(e_1 - e_3)}}, \tag{2}$$

where K, K' are the constants of the periods of Jacobi elliptic functions (complete elliptic integrals of the first type), e_1, e_3 are zero points of the Weierstrass cubic polynomial [1, p. 338; (10)] which have different real values provided that the absolute invariant [1, p. 375; (4), (5)] is greater than one:

$$J = \frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{4}{27}\frac{(1 - k^2 + k^4)^3}{k^4 k'^4} = \frac{4}{27}\frac{(1 - k^2 k'^2)^3}{k^4 k'^4} > 1\,. \tag{3}$$

Considering (3) we obtain

$$\frac{g_3^2}{g_2^3} = \frac{1}{108}\frac{(1 + k^2)^2 (1 - 2k^2)^2 (2 - k^2)^2}{(1 - k^2 k'^2)^3},$$

which yields a reciprocal equation of the sixth degree for the unknown k^2. The solutions of this equation fulfil the relation [1, p. 340; 13.16, (3)] for the six permutations of the zero points e_1, e_2, e_3 of the Weierstrass cubic polynomial [1, p. 338; (10)].

After a modification and reduction we obtain the cubic equation

$$(a-1)^3 - \frac{27}{4} J(a-1) + \frac{27}{4} J = 0 . \tag{4}$$

Let the zero points of the Weierstrass cubic polynomial [1, p. 338; (10)] fulfil the inequality

$$e_1 > e_2 > e_3 ,$$

so that according to [1, p. 342; (11)] the moduli of Jacobi elliptic functions [1, p. 340; 13.16] fulfil

$$0 < k^2 < 1 ; \quad 0 < k'^2 < 1 .$$

According to [1, p. 340; 13.16, (3)]

$$e_1 - e_3 = 3\varrho > 0$$

holds and with respect to [1, p. 332; (5)] the invariant

$$g_3 = 4e_1e_2e_3 = 4\varrho^3(2-k^2)(1-2k^2)(1+k^2) =$$
$$= 4\varrho^3(1+k'^2)(2k'^2-1)(2-k'^2) ,$$

so that

$$\operatorname{sign} g_3 = \operatorname{sign}(1-2k^2) = \operatorname{sign}(2k'^2-1) .$$

Consequently, if $g_3 > 0$, then the inequalities

$$0 < k^2 < \tfrac{1}{2} ; \quad \tfrac{1}{2} < k'^2 < 1 ; \tag{5}$$

hold; if $g_3 < 0$, then

$$\tfrac{1}{2} < k^2 < 1 ; \quad 0 < k'^2 < \tfrac{1}{2} .$$

Hence the evaluation of the most suitable value of the modulus k under the condition (3) is given by the relation

$$[k^2]_{g_3>0} = \frac{a}{2} - \sqrt{\left[\left(\frac{a}{2}\right)^2 - 1\right]} ; \tag{6}$$

$$[k^2]_{g_3<0} = 1 - \frac{a}{2} + \sqrt{\left[\left(\frac{a}{2}\right)^2 - 1\right]} , \tag{7}$$

where, using the goniometric solution of (4), we have

$$a = 1 + 3\frac{\cos\frac{1}{3}(\pi - \varphi)}{\cos\varphi} > \frac{5}{2} ; \tag{8}$$

(9) $$\cos\varphi = \frac{1}{\sqrt{J}};$$

(10) $$0 < \varphi < \frac{\pi}{2}.$$

With respect to [1, p. 340; 13.16, (3) and p. 332; (5), (6) respectively], the half-periods (1) and (2) if the Weierstrass $\wp$-functions are given by

(11) $$\omega = K\sqrt{2}\sqrt[4]{\left(\frac{1-k^2k'^2}{3g_2}\right)} = \\ = K\frac{\sqrt[3]{2}}{\sqrt{3}}\sqrt[6]{\left(\frac{(1+k^2)(1-2k^2)(2-k^2)}{g_3}\right)};$$

(12) $$\omega' = iK'\sqrt{2}\sqrt[4]{\left(\frac{1-k^2k'^2}{3g_2}\right)} = \\ = iK'\frac{\sqrt[3]{2}}{\sqrt{3}}\sqrt[6]{\left(\frac{(1+k^2)(1-2k^2)(2-k^2)}{g_3}\right)}$$

provided condition (3) is fulfilled.

Examples: 1. Consider the differential equation

$$\left(\frac{1}{2}\frac{dy}{dx}\right)^2 = y^3 - 24y - 16.$$

After a modification we obtain the equation

(13) $$\left(\frac{dy}{dx}\right)^2 = 4y^3 - 96y - 64,$$

which is satisfied by the Weierstrass function

(14) $$y = \wp(x + c),$$

c being a constant of integration [1, p. 332; (4)]. Let us find the half-periods (1), (2) of the Weierstrass function (14).

According to (13) $g_2 = 96$; $g_3 = 64$, hence the absolute invariant

$$J = 1{,}142857\ldots > 1.$$

Substituting this value into (9) we obtain with respect to (10)

$$\varphi = 0{,}361367123\ldots,$$

which implies with regard to (8)

$$a = 2{,}92570\ldots$$

and, according to (6) and (5)

$$k^2 = 0{,}39517\ldots,$$

Thus after the evaluation of complete integrals [2, p. 105–108]

$$K = 1{,}7741_6\,; \quad K' = 1{,}9547_9\,,$$

and with respect to (11) and (12) we obtain the half-periods (1) and (2) respectively:

$$\omega = 0{,}5688\ldots \quad \omega' = 0{,}6267\ldots i\,.$$

If it were $g_3 = -64$ in the equation (13), then according to (7) we should have $k^2 = 0{,}60482\ldots$ Hence according to (11) and (12) the half-periods (1) and (2) would be $\omega = 0{,}6267\ldots$ and $\omega' = 0{,}5688\ldots$ respectively.

2. If we have the differential equation

$$3\left(\frac{3}{2}\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 27y^3 - 117y - 92\,,$$

we modify it to

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y^3 - \frac{52}{3}\,y - \frac{368}{27}\,, \tag{15}$$

so that the invariants

$$g_2 = \frac{52}{3}\,; \quad g_3 = \frac{368}{27}\,,$$

hence the absolute invariant

$$J = 27{,}12345\ldots > 1\,.$$

After the substitution of this value into (9) we obtain with regard to (10)

$$\varphi = 1{,}377585830\ldots,$$

which implies according to (8)

$$a = 14$$

and according to (6) and (5) it is

$$k^2 = 0{,}071796\ldots.$$

Thus after the evaluation of complete elliptic integrals [2, p. 105 – 108]

$$K = 1{,}6001_8\ ; \quad K' = 2{,}7350_4\ ,$$

and with respect to (11) and (12) we have the half-periods (1) and (2) respectively:

$$\omega = 0{,}8283\ldots \quad \omega' = 1{,}4157\ldots i\ .$$

If it were $g_3 = -368/27$, then according to (7) we should have $k^2 = 0{,}92820\ldots$ Hence according to (11) and (12) the halfperiods (1) and (2) would be $\omega = 1{,}4157\ldots$ and $\omega' = 0{,}8283\ldots i$.

References

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[2] *K. Uhde:* Spezielle Funktionen der mathematischen Physik, Tafeln II. Mannheim: Bibliographisches Institut 1964, 105—108.

[3] *J. Chrapan:* Weierstrass $\wp$-function. Aplikace matematiky 4, 16 (1971).

Súhrn

VYČÍSLENIE POLPERIÓD WEIERSTRASSOVEJ $\wp$-FUNKCIE PRI ABSOLÚTNOM INVARIANTE VÄČŠOM AKO ČÍSLO 1

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V práci sú odvodené výrazy vhodné pre vyčíslenie polperiód Weierstarssovej $\wp$-funkcie pri absolútnom invariante väčšom ako číslo 1 a výpočet je ilustrovaný na dvoch numerických príkladoch.

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