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EXPLICIT EXPRESSIONS FOR THE COORDINATES OF FOCI AND VERTICES OF A QUADRIC IN DEPENDENCE ON THE COEFFICIENTS OF ITS CARTESIAN EQUATION

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The differential-geometrical investigation of trajectories in general power fields brings about the necessity of expressing explicitly the coordinates of foci of a quadric in dependence on the coefficients of its Cartesian equation

(1) $$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0\,.$$

Handbooks of analytical geometry do not include such formulae. The present note provides a list of formulae in question.

For central quadrics $(A_{33} = a_{11}a_{22} - a_{12}^2 \neq 0)$, analysis of the transformation to the normal form

(2) $$\varrho_1 x^2 + \varrho_2 y^2 + A/A_{33} = 0\,,$$

A being the discriminant of the equation of the quadric and $\varrho_{1,2}$ the roots of the equation

$$\varrho^2 - (a_{11} + a_{22})\,\varrho + A_{33} = 0,$$

yields the necessary formulae:

For the ellipse $(A_{33} > 0,\ \operatorname{sgn} a_{11} = \operatorname{sgn} a_{22} = 1\ (\varrho_{1,2} > 0)$[1]$)$ we obtain (here as well as in the sequel we suppose $a_{12} \neq 0$) [2])

(3) $$x_F = \frac{1}{A_{33}}\left\{a_{12}a_{23} - a_{13}a_{22} \pm a_{12}\sqrt{\left(\frac{2A}{a_{22} - a_{11} - \Delta}\right)}\right\},$$

(4) $$y_F = \frac{1}{A_{33}}\{a_{12}a_{13} - a_{11}a_{23} \mp \sqrt{(\tfrac{1}{2}A(a_{22} - a_{11} - \Delta))}\}\,,$$

[1]) Passing to the case $\operatorname{sgn} a_{11} = \operatorname{sgn} a_{22} = -1$ $(\varrho_{1,2} < 0)$ is obvious by changing the signs at a_{ik}.

[2]) The separate discussion of the case $a_{12} = 0$ is immediate. However, using of some relations introduced below requires a limiting process which, being carried out, yields some formal simplification of the relations.

the excentricity being

$$e = \frac{1}{A_{33}} \sqrt{(-A\Delta)}$$

where for the sake of brevity we put

$$\Delta = \sqrt{((a_{11} - a_{22})^2 + 4a_{12}^2)}\,.$$

For the hyperbola ($A_{33} < 0$, $A > 0$ [3])) there is

$$e = -\frac{1}{A_{33}} \sqrt{(A\Delta)}\,,$$

(5) $$x_F = \frac{1}{A_{33}} \{a_{12}a_{23} - a_{13}a_{22} \pm \sqrt{(\tfrac{1}{2}A(a_{11} - a_{22} + \Delta))}\}\,,$$

(6) $$y_F = \frac{1}{A_{33}} \{a_{12}a_{13} - a_{11}a_{23} \mp \sqrt{(\tfrac{1}{2}A(a_{22} - a_{11} + \Delta))}\}\,.$$

In the case of the parabola ($A_{33} = 0$, $A \neq 0$) we have instead of (2) an equation linear in one variable and instead of determining the centre we can start from the vertex of the parabola. To have a different view let us choose another way not using the normal form.

Let us recall that the tangent lines of the parabola at the end points $T_{1,2}$ of the chord passing through the focus perpendicularly to the axis o of the parabola make with o the angle $\frac{1}{4}\pi$. If we determine the coordinates (x_i, y_i) of the points T_i, $i = 1, 2$, then the required coordinates of the focus are

$$x_F = (x_1 + x_2)/2\,, \quad y_F = (y_1 + y_2)/2\,.$$

Differentiating equation (1) with respect to x we obtain

(7) $$\mathcal{A}x + \mathcal{B}x + \mathcal{C} = 0\,,$$

where

(8) $$\mathcal{A} = a_{11} + a_{12}y'\,, \quad \mathcal{B} = a_{12} + a_{22}y'\,, \quad \mathcal{C} = a_{13} + a_{23}y'\,.$$

Substituting a chosen numerical value k in (8) for y', equations (1) and (7) reduce to a system by means of which the coordinates of the tangent point of the parabola (1) and the tangent line with the slope k may be determined. The condition for the directions given by the values $y'_{1,2}$ to make angles $\frac{1}{4}\pi$ with the axis of the parabola (whose slope is $-a_{11}/a_{22}$) yields

$$y'_{1,2} = \frac{\pm a_{12} - a_{11}}{a_{12} \pm a_{11}}$$

[3]) Passing to the case $A < 0$ is obvious.

and determines the corresponding system of values $\mathscr{A}_{1,2}$, $\mathscr{B}_{1,2}$, $\mathscr{C}_{1,2}$. Using them we find

$$x_{1,2} = \frac{a_{22}\mathscr{C}_{1,2} - 2a_{23}\mathscr{B}_{1,2}\mathscr{C}_{1,2} + a_{33}\mathscr{B}_{1,2}^2}{2\mathscr{B}_{1,2}(a_{23}\mathscr{A}_{1,2} - a_{13}\mathscr{B}_{1,2})},$$

$$y_{1,2} = \frac{\mathscr{A}_{1,2}x_{1,2} + \mathscr{C}_{1,2}}{-\mathscr{B}_{1,2}},$$

so that the coordinates of the focus of the parabola is

$$x_F = \frac{a_{22}[a_{13}^2 - a_{23}^2 + (a_{11} + a_{22})\,a_{33}] - 2a_{12}a_{13}a_{23}}{2(a_{11} + a_{22})\,(a_{12}a_{23} - a_{22}a_{13})}, \tag{9}$$

$$y_F = \frac{a_{12}[a_{13}^2 - a_{23}^2 - (a_{11} + a_{22})\,a_{33}] + 2a_{22}a_{13}a_{23}}{2(a_{11} + a_{22})\,(a_{12}a_{23} - a_{22}a_{13})}; \tag{10}$$

or, in a more "symmetrical" form

$$x_F = \frac{a_{22}[a_{13}^2 - a_{23}^2 + (a_{22} - a_{11})\,a_{33}] - 2a_{12}(a_{13}a_{23} - a_{12}a_{33})}{2(a_{11} + a_{22})\,(a_{12}a_{23} - a_{22}a_{13})}, \tag{11}$$

$$y_F = \frac{a_{12}[a_{13}^2 - a_{23}^2 + (a_{22} - a_{11})\,a_{33}] + 2a_{22}(a_{13}a_{23} - a_{12}a_{33})}{2(a_{11} + a_{22})\,(a_{12}a_{23} - a_{22}a_{13})}. \tag{12}$$

Let us add the expressions for the vertices of the quadrics:

The principal and the secondary vertices of the ellipse:

$$x_{VH} = \frac{1}{A_{33}}\left\{a_{12}a_{23} - a_{13}a_{22} \pm a_{12}\sqrt{\left(\frac{2AA_{33}}{\Delta(a_{22}(a_{22} - a_{11} - \Delta) + 2a_{12}^2)}\right)}\right\}, \tag{13}$$

$$y_{VH} = \frac{1}{A_{33}}\left\{a_{12}a_{13} - a_{11}a_{23} \mp \right.$$
$$\left. \mp \sqrt{\left(\frac{A(a_{11} + a_{22} + \Delta)\,(2a_{12}^2 + (a_{22} - a_{11})\,\Delta - \Delta^2)}{2\Delta(a_{11} - a_{22} + \Delta)}\right)}\right\}, \tag{14}$$

$$x_{VV} = \frac{1}{A_{33}}\left\{a_{12}a_{23} - a_{13}a_{22} \pm \right.$$
$$\left. \pm \sqrt{\left(\frac{A(a_{11} + a_{22} - \Delta)\,(2a_{12}^2 + (a_{22} - a_{11})\,\Delta - \Delta^2)}{2\Delta(a_{11} - a_{22} + \Delta)}\right)}\right\},$$

$$y_{VV} = \frac{1}{A_{33}}\left\{a_{12}a_{13} - a_{11}a_{23} \pm a_{12}\sqrt{\left(\frac{2AA_{33}}{\Delta(a_{11}(a_{22} - a_{11} - \Delta) - 2a_{12}^2)}\right)}\right\},$$

the vertices of the hyperbola

$$(15)\quad x_V = \frac{1}{A_{33}}\left\{a_{12}a_{23} - a_{13}a_{22} \pm \frac{1}{2}\sqrt{\left(\frac{A(a_{22} + a_{11} - \Delta)(a_{22} - a_{11} - \Delta)}{\Delta}\right)}\right\},$$

$$(16)\quad y_V = \frac{1}{A_{33}}\left\{a_{12}a_{13} - a_{11}a_{23} \mp \frac{1}{2}\sqrt{\left(\frac{A(a_{11} + a_{22} - \Delta)(a_{11} - a_{22} - \Delta)}{\Delta}\right)}\right\}$$

and the vertex of the parabola

$$(17)\quad x_V = \frac{(a_{11}a_{13} + a_{12}a_{23})(a_{12}a_{13} - a_{22}a_{23} - 2a_{11}a_{23}) + a_{12}a_{33}(a_{11} + a_{22})^2}{2(a_{11} + a_{22})^2(a_{11}a_{23} - a_{12}a_{13})},$$

$$(18)\quad y_V = \frac{(a_{11}a_{13} + a_{12}a_{23})(a_{12}a_{23} - a_{11}a_{13} - 2a_{22}a_{13}) + a_{11}a_{33}(a_{11} + a_{22})^2}{-2(a_{11} + a_{22})^2(a_{11}a_{23} - a_{12}a_{13})}.$$

Remark. Variants offer for the solution of the problem (leading to different forms of the results), e.g.: Compare the equation of the quadric written for general foci (x_{F_1}, y_{F_1}), (x_{F_2}, y_{F_2}) and a general length of the semiaxis with (1), determine from (1) the coordinates of vertices and hence the foci, transform (1) to the form corresponding to the focus definition etc. As for the last suggestion, cf. III § 3 in [3]. The limiting process $a_{11}a_{22} - a_{12}^2 \to 0$ in expressions (3) to (6) and (13) to (16) requires evaluation of indetermined expressions and provides not very advantageous tool for obtaining formulae (9) to (12) and (17), (18).

References

[1] *Bydžovský B.* Úvod do analytické geometrie, Praha 1923 resp. 1946.

[2] *Kasner E.* The Trajectories of Dynamics, Transactions of the American Mathematical Society 7, 1906, 401.

[3] *Mišoň K.* Diferenciálně geometrické a kinematicko-geometrické modely v dynamice silových polí — Strojnický časopis SAV, XXI, 1970, 630.

Souhrn

EXPLICITNÍ VYJÁDŘENÍ SOUŘADNIC OHNISEK A VRCHOLŮ KUŽELOSEČKY V ZÁVISLOSTI NA SOUČINITELÍCH JEJÍ KARTÉZSKÉ ROVNICE

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Je podáno explicitní vyjádření souřadnic ohnisek a vrcholů kuželoseček v závislosti na součinitelích jejich rovnic v kartézských souřadnicích. Uvedené výrazy se omezují na případ kuželoseček, jejichž osy nejsou rovnoběžny s osami souřadnic ($a_{12} \neq 0$). Pro $a_{12} = 0$ vyžadují některé získané výrazy provedení limitního procesu.

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