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Aplikace matematiky, Vol. 16 (1971), No. 1, 46–63

Persistent URL: http://dml.cz/dmlcz/103325

VARIATIONAL FORMULATION OF THE CAUCHY PROBLEM FOR EQUATIONS WITH OPERATOR COEFFICIENTS

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(Received July 30, 1970)

PREFACE

It is an old idea, to formulate the boundary-value and mixed problems of mathematical physics in terms of equivalent variational problems. The variational formulation (principle) may be then applied in three ways: first, to the definition of weak solutions; second, to the dimensional reduction (e.g. if the three-dimensional domain under consideration has one or two prevailing dimensions); third, to other approximate methods of solution (e.g. Ritz, Galerkin, finite elements a.o.).

In the present paper, several variational principles are suggested, which are equivalent to initial-value (Cauchy) problems for equation of the first and second order in time coordinate. Their coefficients are linear operators, acting in the space $L_2(I, H)$ of square-integrable mappings of a time interval I into a Hilbert space H. In particular, the theory includes some classes of partial differential equations and of integro-differential equations. In Section 1, the "convolution scalar product" is introduced as the basic concept of the following variational principles and its properties proved. Section 2 involves three variational principles for equations of the first order in time, Section 3 another four variational principles for equations of the second order. Some kinds of a "convolution symmetry" of the operator coefficients are required in all the variational principles. In the papers [3] and [6], some of those principles were employed for the definitions of weak solutions of particular integro-differential equations.

1. CONVOLUTION SCALAR PRODUCT

Let a bounded interval $I = \langle 0, T\rangle$ and a basic real Hilbert space H be given with the scalar product (u, v) and the norm $|u| = (u, u)^{1/2}$.

Definition 1. *Let $L_2(I, H)$ denote the space of all measurable mappings $u(t)$ of I into H such that*

$$|u|_T = \left(\int_0^T |u(t)|^2 \, dt\right)^{1/2} < \infty .$$

Definition 2. *Let $L_2(I)$ denote the space of real functions, which are square-integrable on I. Let $f, g \in L_2(I)$ or $g \in L_2(I, H)$ and $u, v \in L_2(I, H)$. The function*

$$(f * g)(t) = \int_0^t f(t - \tau)\, g(\tau)\, d\tau$$

will be called the convolution of f and g.

The function

$$(u \otimes v)(t) = \int_0^t (u(t - \tau), v(\tau))\, d\tau$$

will be called the convolution scalar product of u and v.

Lemma 1. *It holds*

(1) $$|(u \otimes v)(t)| \leqq |u|_T \cdot |v|_T ,$$

(2) $$(u \otimes v)(t) = (v \otimes u)(t)$$

for every pair of $u, v \in L_2(I, H)$ and $t \in I$.

Proof. Denote $U(\tau) = u(t - \tau)$. Then we may write

$$\left|\int_0^t (u(t - \tau), v(\tau))\, d\tau\right| \leqq \int_0^t |(U(\tau), v(\tau))|\, d\tau \leqq \int_0^t |U(\tau)| \cdot |v(\tau)|\, d\tau \leqq$$
$$\leqq \left(\int_0^t |U(\tau)|^2\, d\tau\right)^{1/2} \left(\int_0^t |v(\tau)|^2\, d\tau\right)^{1/2} \leqq |u|_T \cdot |v|_T ,$$

because of the relation

$$\int_0^t |U(\tau)|^2\, d\tau = \int_0^t |u(t - \tau)|^2\, d\tau = \int_0^t |u(\xi)|^2\, d\xi \leqq |u|_T^2 ,$$

which follows from the change of variables $t - \tau = \xi$. The same transformation leads also to the formula (2).

Lemma 2. *Let $f(t) \in L_2(I)$ and $u \in L_2(I, H)$. Then also*

(3) $$(f * u)(t) = \int_0^t f(t - \tau)\, u(\tau)\, d\tau \in L_2(I, H) .$$

If $u(t)$ is continuous on I, $l(t) = 1$ then

(4) $$\frac{d}{dt}(l * u) = u(t)$$

holds for all $t \in I$ (for $t = 0$ and $t = T$ from the right and left, respectively) and $(l * u)(t)$ is continuous on I. Moreover,

(5) $$(l * (l * u))(t) = (t * u)(t)$$

holds on I.

Proof. We have (see [1], Th. 2.7)

$$\left|\int_0^t f(t-\tau)\,u(\tau)\,\mathrm{d}\tau\right| \leqq \int_0^t |f(t-\tau)| \,.\, |u(\tau)|\,\mathrm{d}\tau \leqq$$

$$\leqq \left(\int_0^t f^2(t-\tau)\,\mathrm{d}\tau\right)^{1/2} \left(\int_0^t |u(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} \leqq \|f\|_{L_2(I)}\,|u|_T\,;$$

consequently $(f * u)(t)$ is bounded on I.

In order to prove (4), denote

$$z(t) = (l * u)(t) = \int_0^t u(\tau)\,\mathrm{d}\tau\,.$$

We may write

$$\left|\frac{1}{\alpha}(z(t+\alpha) - z(t)) - u(t)\right| = \left|\frac{1}{\alpha}\int_t^{t+\alpha} [u(\tau) - u(t)]\,\mathrm{d}\tau\right| \leqq \max_{\langle t, t+\alpha\rangle} |u(\tau) - u(t)| \to 0$$

for $\alpha \to 0$, $t \in I$, $t + \alpha \in I$. Hence $\mathrm{d}z/\mathrm{d}t = u(t)$ follows for $t \in I$.

By virtue of (4) and the continuity of $u(t)$, also $(l * u)$ is continuous on I (see [1] Th. 1.5). The formula (5) follows from the Fubini theorem, because

$$\int_0^t \mathrm{d}\tau \int_0^\tau u(s)\,\mathrm{d}s = \int_0^t \mathrm{d}s \int_s^t u(s)\,\mathrm{d}\tau = \int_0^t (t-s)\,u(s)\,\mathrm{d}s\,.$$

Lemma 3. *Let $f(t) \in L_2(I)$, and $u, v \in L_2(I, H)$. Then*

$$((f * u) \otimes v)(t) = (f * (u \otimes v))(t)$$

holds for every $t \in I$.

Proof. Changing the order of integration, we may write

$$((f * u) \otimes v)(t) = \int_0^t \left(\int_0^{t-\tau} f(t - \tau - s)\,u(s)\,\mathrm{d}s\,,\ v(\tau)\right)\mathrm{d}\tau =$$

$$= \int_0^t \left(\int_\tau^t f(t-y)\,u(y-\tau)\,\mathrm{d}y,\ v(\tau)\right)\mathrm{d}\tau = \int_0^t \mathrm{d}\tau \int_\tau^t f(t-y)\,(u(y-\tau), v(\tau))\,\mathrm{d}y =$$

$$= \int_0^t \mathrm{d}y\, f(t-y) \int_0^y (u(y-\tau), v(\tau))\,\mathrm{d}\tau = (f * (u \otimes v))|(t)\,,$$

using also the transformation $s = y - \tau$.

Lemma 4. *Let $\tilde{w} \in L_2(I, H)$ and a sequence $\{v_n\} \subset L_2(I, H)$ be such that*

$$\lim_{n \to \infty} |v_n - \tilde{w}|_T = 0 .$$

Then

$$\lim_{n \to \infty} (w \otimes v_n)(t) = (w \otimes \tilde{w})(t)$$

holds for every $w \in L_2(I, H)$ and $t \in I$.

Proof. Choose an arbitrary $t \in I$ and $w \in L_2(I, H)$. Then, by virtue of Lemma 1, we may write

$$|(w \otimes v_n)(t) - (w \otimes \tilde{w})(t)| = |(w \otimes v_n - \tilde{w})(t)| \leqq |w|_T \cdot |v_n - \tilde{w}|_T \to 0 .$$

Definition 3. *Denote $u'(t) = \mathrm{d}u/\mathrm{d}t$, $\mathscr{C}_0$ the linear manifold of continuous mappings of I into H and $\mathscr{C}_1$ the linear manifold of mappings $u(t)$ of I into H, which possess continuous derivatives $u'(t) \in \mathscr{C}_0$.*

Lemma 5. *Let $u' \in \mathscr{C}_0$ and let $v(t) \in L_2(I, H)$ be continuous for a point $t \in I$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} (u \otimes v)(t) = (u' \otimes v)(t) + (u(0), v(t))$$

holds at this point (with the derivative from the left, if $t = T$ and from the right if $t = 0$).

Proof. Note, that $u' \in \mathscr{C}_0$ yields that $u \in \mathscr{C}_0$ (see [1] Th. 15), consequently, $u(0)$ exists. We may differentiate with respect to the parameter t, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (u(t - \tau), v(\tau))\, \mathrm{d}\tau = \int_0^t (u'(t - \tau), v(\tau))\, \mathrm{d}\tau + (u(0), v(t)) .$$

Lemma 6. *Let $w \in \mathscr{C}_0$ be such that*

$$(w \otimes v)(T) = 0$$

holds for every $v \in \mathscr{M}$, and let $\mathscr{M}$ be dense in $L_2(I, H)$. Then $w(t) = \Theta$ for every $t \in I$.

Proof. Introduce a function $\tilde{w} \in \mathscr{C}_0$ by means of the relation

$$w(T - t) = \tilde{w}(t)$$

for $t \in I$. There exists a sequence $\{v_n\} \subset \mathscr{M}$ such that

$$\lim_{n \to \infty} |v_n - \tilde{w}|_T = 0 .$$

Using Lemma 4, we obtain

$$0 = \lim_{n\to\infty} (w \otimes v_n)(T) = (w \otimes \tilde{w})(T) = \int_0^T (w(T-t), \tilde{w}(t))\, \mathrm{d}t = \int_0^T |w(t)|^2\, \mathrm{d}t\,.$$

By virtue of the continuity of $|w(t)|$ on I, we conclude that $w(t)$ must vanish on I.

2. THE CAUCHY PROBLEM FOR EQUATIONS OF THE FIRST ORDER

Let us consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(Bu) + Au = f \tag{6}$$

and the initial condition

$$u(0) = u_0\,, \tag{7}$$

where A and B are linear operators in $L_2(I, H)$ such that

$$(Au \otimes v)(T) = (u \otimes Av)(T) \quad \text{for} \quad u, v \in D_A\,, \tag{8}$$

$$(Bu \otimes v)(T) = (u \otimes Bv)(T) \quad \text{for} \quad u, v \in D_B\,. \tag{9}$$

Assume that

$$(Bv)(0) = \Theta \Leftrightarrow v(0) = \Theta\,, \tag{10}$$

$$f \in \mathscr{C}_0\,, \quad u_0 \in D_B \cap D_A \quad \text{and} \quad Bu_0 \in \mathscr{C}_1\,. \tag{11}$$

Definition 4. *Let $\mathscr{K}$ denote the linear manifold of mappings $u \in \mathscr{C}_0$, for which $u \in D_A \cap D_B$, $Bu \in \mathscr{C}_1$, $Au \in \mathscr{C}_0$, $u(0) \in D_B$. Define the functional*

$$\mathscr{F}(u) = ([Bu + 1 * Au] \otimes u)(T) - 2([1 * f + (Bu_0)(0)] \otimes u)(T) \tag{12}$$

on $\mathscr{K}$.

Theorem 1. *Let $\mathscr{K}$ be dense in $L_2(I, H)$ nad let (8) till (11) hold. Then*

$$\delta\mathscr{F}(u) = 0 \quad \text{on} \quad \mathscr{K} \tag{13}$$

if and only if $u \in \mathscr{K}$ satisfies the equation (6) on I and the initial condition (7).

Remark 1. It is sufficient to assume the density of $\mathscr{K}$ in the subset $\mathscr{C}_0 \subset L_2(I, H)$, because $\mathscr{C}_0$ is dense in $L_2(I, H)$ (see e.g. [2], Lemma IV.8.19).

Proof. In the following, we shall omit the notation (T). Denoting $\delta u = v$, we have

$$\delta\mathscr{F}(u) = ([Bv + 1 * Av] \otimes u) + ([Bu + 1 * Au] \otimes v) - \\ - 2([1 * f + (Bu_0)(0)] \otimes v).$$

Using (9), (8), (2) and Lemma 3, we obtain

$$([Bv + 1 * Av] \otimes u) = (Bu \otimes v) + 1 * (Au \otimes v) = \\ = ([Bu + 1 * Au] \otimes v).$$

Therefore we may write

$$\delta\mathscr{F}(u) = 2(w \otimes v), \tag{14}$$

where

$$w = Bu + 1 * Au - 1 * f - (Bu_0)(0). \tag{15}$$

By virtue of Definition 4 and Lemma 4, w belongs to $\mathscr{C}_0$. Consequently, we may apply Lemma 6 to obtain that (13) yields $w(t) = \Theta$ for every $t \in I$. Inserting $t = 0$, we have

$$w(0) = (Bu)(0) - (Bu_0)(0) = [B(u - u_0)](0) = \Theta.$$

From the assumption (10), $u(0) = u_0$ follows. As the element w has a continuous time derivative, we have also

$$w'(t) = \mathrm{d}(Bu)/\mathrm{d}t + Au - f = \Theta$$

for every $t \in I$. Thus we have proved, that (13) implies (6) and (7).

On the contrary, let $u \in \mathscr{K}$ satisfy (6) and (7). Integrating the equation (6), we obtain

$$(Bu)(t) - (Bu)(0) + 1 * Au = 1 * f.$$

By virtue of (10) and the initial condition (7)

$$[B(u - u_0)](0) = (Bu)(0) - (Bu_0)(0) = \Theta,$$

consequently $w(t) = \Theta$ in (14) for every $t \in I$. Then (13) holds, as follows from (14) and (1).

Remark 2. The condition (13) was employed for the definition of weak solutions in a particular case of the Cauchy problem (6), (7) in [3], where the existence, uniqueness and continuous dependence of the weak solution on f and u_0 have been proved.

Remark 3. In case of parabolic differential equations with time-independent coefficients, Theorem 1 corresponds with the "α-integral convolution principle", introduced in [4].

Theorem 2. *Let* (8), (10), (11) *and*

$$\frac{d}{dt}(Bu \otimes v)|_{t=T} = \frac{d}{dt}(Bv \otimes u)|_{t=T} \tag{16}$$

hold. Assume that the set $\mathscr{K}_0 = \{v \in \mathscr{K}, v(T) = \Theta\}$ *is dense in* $L_2(I, H)$ *and the set of* $v(T)$ *dense in* H, *if* $v \in \mathscr{K}$.

Define the functional

(17)

$$\mathscr{F}'(u) = ([(Bu)' + Au] \otimes u)(T) - 2(f \otimes u)(T) + ((Bu)(0) - 2(Bu_0)(0), u(T)).$$

Then

$$\delta\mathscr{F}'(u) = 0 \quad \text{on} \quad \mathscr{K} \tag{18}$$

if and only if $u \in \mathscr{K}$ *satisfies the equation* (6) *on* I *and the initial condition* (7).[1])

Remark 4. Again, the density of $\mathscr{K}_0$ in $\mathscr{C}_0$ is sufficient (cf. Remark 1).

Proof. Denoting $\delta u = v$, we have

$$\delta\mathscr{F}'(u) = ([Bv)' + Av] \otimes u)(T) + ([(Bu)' + Au] \otimes v)(T) - \\ - 2(f \otimes v)(T) + ((Bv)(0), u(T)) + ((Bu)(0) - 2(Bu_0)(0), v(T)).$$

Using Lemma 5, we derive

$$((Bv)' \otimes u)(T) + ((Bv)(0), u(T)) = \frac{d}{dt}(Bv \otimes u)|_{t=T},$$

$$((Bu)' \otimes v)(T) + ((Bu)(0), v(T)) = \frac{d}{dt}(Bu \otimes v)|_{t=T}.$$

Therefore we may write, making use of (8) and (16),

$$\delta\mathscr{F}'(u) = 2\{([(Bu)' + Au - f] \otimes v)(T) + ((Bu)(0) - (Bu_0)(0), v(T))\}. \tag{19}$$

Denoting

$$w = (Bu)' + Au - f, \tag{20}$$

we have

$$\delta\mathscr{F}'(u) = 2(w \otimes v)(T)$$

for every $v \in \mathscr{K}_0$. Obviously, $w \in \mathscr{C}_0$, consequently (18) and Lemma 6 yield that $w(t) = \Theta$ for $t \in I$, i.e., (6) is satisfied. Inserting this result into (19), we obtain

$$\delta\mathscr{F}'(u) = 2((Bu)(0) - (Bu_0)(0), v(T)) = 0$$

for every $v \in \mathscr{K}$. From the density of $v(T)$ in H,

$$(21) \qquad (Bu)(0) - (Bu_0)(0) = \Theta$$

follows and (10) yields the initial condition (7).

On the contrary, let $u \in \mathscr{K}$ satisfy (6) and (7). Then (21) holds by virtue of (10), and (18) follows from (19). The proof is complete.

Restricting the domain of the functional $\mathscr{F}(u)$ to the functions, satisfying the initial condition (7) a priori, we are led to a modified

Theorem 3. *Let* (8), (10), (11) *and* (16) *hold. Assume that the set* $\mathscr{K}_1 = \{v \in \mathscr{K}, v(0) = \Theta\}$ *is dense in* $L_2(I, H)$. *Denote* $\mathscr{K}_2 = u_0 \oplus \mathscr{K}_1$ *and define the functional*

$$(22) \qquad \mathscr{F}'_1(u) = ([(Bu)' + Au - 2f] \otimes u)(T) - ((Bu_0)(0), u(T)).$$

Then

$$(23) \qquad \delta\mathscr{F}'_1(u) = 0 \quad \text{on} \quad \mathscr{K}_2,$$

if and only if $u \in \mathscr{K}_2$ *satisfies the equation* (6) *on* I.

Proof. In the same way, as previously, we derive (for any $u \in \mathscr{K}_2$ and $v \in \mathscr{K}_1$)

$$\delta\mathscr{F}'_1(u) = ([(Bv)' + Av] \otimes u)(T) + ([(Bu)' + Au] \otimes v)(T) - 2(f \otimes v)(T) - \\ - ((Bu_0)(0), v(T)),$$

$$((Bv)' \otimes u)(T) = \frac{d}{dt}(Bv \otimes u)|_{t=T},$$

$$((Bu)' \otimes v)(T) = \frac{d}{dt}(Bu \otimes v)|_{t=T} - ((Bu_0)(0), v(T)).$$

using (8) and (10). By virtue of (16),

$$((Bv)' \otimes u)(T) = ((Bu)' \otimes v)(T) + ((Bu_0)(0), v(T))$$

and consequently, using also (8), we obtain

$$(24) \qquad \delta\mathscr{F}'_1(u) = 2([(Bu)' + Au - f] \otimes v)(T).$$

If we denote again by w the expression in (20), $w \in \mathscr{C}_0$ and therefore the equation (6) on I follows from (23) with the use of Lemma 6. On the contrary, let $u \in \mathscr{K}_2$ satisfy (6). Then (23) holds, because of (24).

[1]) In the case of parabolic equations with time-independent coefficients, Theorem 2. corresponds with the "β-differential convolution principle", introduced in [4].

Remark 5. Let us denote

$$v(T-t) = \delta u(T-t) = \varphi(t),$$

so that $\varphi(T) = v(0) = \Theta$ in Theorem 3. Assume that $B = I$ (identity operator),

$$(Au)(t) = A(t)\,u(t), \quad A(T-t) = A^*(t) \quad \text{for every} \quad t \in I,$$

where $A^*(t)$ denotes the operator adjoint of $A(t)$. Then the symmetry (8) holds and

$$(25) \qquad (Au \otimes v)(T) = (u \otimes Av)(T) = \int_0^T (A(T-t)\,v(T-t), u(t))\,\mathrm{d}t = \\ = \int_0^T (u(t), A^*(t)\,\varphi(t))\,\mathrm{d}t\,.$$

With regard to (24) and (25),

$$\int_0^T \{(u'(t), \varphi(t)) + (u(t), A^*(t)\,\varphi(t)) - (f(t), \varphi(t))\}\,\mathrm{d}t = 0$$

follows from the condition (23) for every $\varphi(t)$ such that $\varphi(T-t) \in \mathscr{K}$, $\varphi(T) = \Theta$. This condition corresponds with the definition of the generalized Problem 2.1 in [5], because

$$\varphi(T-t) \in \mathscr{K} \Rightarrow \{\varphi \in \mathscr{C}_1, \varphi(t) \in D_{A^*(t)}, \; A^*(t)\,\varphi(t) \in \mathscr{C}_0\}$$

for every $t \in I$. Integrating the first term by parts, and inserting the initial condition, we derive the relation

$$(26) \quad \int_0^T \{(u(t), A^*(t)\,\varphi(t)) - (u(t), \varphi'(t))\}\,\mathrm{d}t = \int_0^T (f(t), \varphi(t)\,\mathrm{d}t + (u_0, \varphi(0))\,,$$

which corresponds with the generalized Problem 2.2 in [5] (except for the condition $u \in \mathscr{K}_2$). In case of differential operators A, the product $(u(t), A^*(t)\,\varphi(t))$ may often be extended continuously to a bilinear form $a(t; u(t), \varphi(t))$ (see [5] p. 44 and [4]).

3. THE CAUCHY PROBLEM FOR EQUATIONS OF THE SECOND ORDER

Let us consider the equation

$$(27) \qquad \frac{\mathrm{d}}{\mathrm{d}t}(Cu') + Bu' + Au = f$$

with the initial conditions

$$(28) \qquad u(0) = u_0\,, \quad u'(0) = v_0\,,$$

where A, B and C are linear operators in $L_2(I, H)$ such that

$$(29) \qquad (Bv)(0) = \Theta \Leftarrow v(0) = \Theta ,$$

$$(30) \qquad (Cv)(0) = \Theta \Leftrightarrow v(0) = \Theta .$$

Furthermore, assume that $f \in \mathscr{C}_0$, $u_0 \in D_C \cap D_B$, $v_0 \in D_C$, $Cu_0 \in \mathscr{C}_0$ and $Bu_0 \in \mathscr{C}_0$.

Definition 5. *Let $\mathscr{K}$ denote the linear manifold of functions $u \in \mathscr{C}_1$, for which $u \in D_A \cap D_B \cap D_C$; $u' \in D_B \cap D_C$; $Bu' \in \mathscr{C}_0$; $Cu' \in \mathscr{C}_1$; $Au, Bu, Cu \in \mathscr{C}_0$.*

Theorem 4. *Let $\mathscr{K}$ be dense in $L_2(I, H)$ and assume that* (29), (30),

$$(31) \qquad (t * (Av \otimes u))(T) = (t * (Au \otimes v))(T) \quad for \quad u, v \in D_A ,$$

$$(32) \qquad (1 * (Bv \otimes u))(T) = (1 * (Bu \otimes v))(T) \quad for \quad u, v \in D_B ,$$

$$(33) \qquad (1 * (Cv \otimes u))(T) = (1 * (Cu \otimes v))(T) \quad for \quad u, v \in D_C ,$$

$$(34) \qquad (Bv \otimes u)(T) = (Bu \otimes v)(T) \quad for \quad u, v \in D_B ,$$

$$(35) \qquad (Cv \otimes u)(T) = (Cu \otimes v)(T) \quad for \quad u, v \in D_C ,$$

$$(36) \qquad \frac{\mathrm{d}}{\mathrm{d}t}(Bu \otimes v)|_{t=T} = \frac{\mathrm{d}}{\mathrm{d}t}(u \otimes Bv)|_{t=T} \quad for \quad u, v \in D_B ,$$

$$(37) \qquad \frac{\mathrm{d}}{\mathrm{d}t}(Cu \otimes v)|_{t=T} = \frac{\mathrm{d}}{\mathrm{d}t}(u \otimes Cv)|_{t=T} \quad for \quad u, v \in D_C$$

hold. Define the functional

$$\mathscr{F}(u) = ([Cu + 1 * Bu + t * Au] \otimes u)(T) - \\ - 2([t * f + Cu_0 + 1 * Bu_0 + t(Cv_0)(0)] \otimes u)(T) .$$

Then

$$(38) \qquad \delta\mathscr{F}(u) = 0 \quad on \quad \mathscr{K}$$

if and only if $u \in \mathscr{K}$ satisfies the equation (27) *and the initial conditions* (28).

Remark 6. Obviously, (31) till (37) hold, if

$$(Av \otimes u)(t) = (Au \otimes v)(t) ,$$
$$(Bv \otimes u)(t) = (Bu \otimes v)(t) ,$$
$$(Cv \otimes u)(t) = (Cu \otimes v)(t)$$

hold for every $t \in I$ and $u, v \in D_A$, D_B, D_C, respectively, and the derivatives in (36) (37) exist. Again, the density of $\mathscr{K}$ in $\mathscr{C}_0$ would be sufficient.

Proof of Theorem 4. Setting $\delta u = v \in \mathscr{K}$, omitting (T) and using (31), (32), (35), we may write

$$\delta \mathscr{F}(u) = ([Cv + 1 * Bv + t * Av] \otimes u) + ([Cu + 1 * Bu + t * Au] \otimes v) - \\ - 2([t * f + Cu_0 + 1 * Bu_0 + t(Cv_0)(0)] \otimes v) = 2(w \otimes v),$$

where

$$w = Cu + 1 * Bu + t * Au - t * f - Cu_0 - 1 * Bu_0 - t(Cv_0)(0).$$

From (38) $w(t) = \Theta$ on I follows, with the use of Lemma 6 and the continuity of w. Inserting $t = 0$, we obtain

$$(Cu)(0) - (Cu_0)(0) = [C(u - u_0)](0) = \Theta,$$

which yields

(39) $$u(0) = u_0,$$

because of (30). As $w(t)$ vanishes everywhere on I, we have

$$(w \otimes v)(t) = 0$$

for every $t \in I$ and $v \in \mathscr{K}$, consequently

$$\frac{d}{dt}(w \otimes v)|_{t=T} = 0.$$

Using (37), Lemma 5, (39) and (35), we derive

(40) $$\frac{d}{dt}([Cu - Cu_0] \otimes v|_{t=T} = \frac{d}{dt}(C(u - u_0) \otimes v)|_{t=T} = \\ = \frac{d}{dt}(u - u_0 \otimes Cv)|_{t=T} = \\ = (u' \otimes Cv)(T) + (u(0) - u_0, (Cv)(T)) = (Cu' \otimes v)(T).$$

By virtue of (40), Lemma 5 and Lemma 2, we have

$$\frac{d}{dt}(w \otimes v)|_{t=T} = (w_1 \otimes v)(T) \quad \text{for every} \quad v \in \mathscr{K},$$

where

(41) $$w_1 = Cu' + Bu + 1 * Au - 1 * f - Bu_0 - (Cv_0)(0).$$

As $w_1 \in \mathscr{C}_0$, we may apply again Lemma 6 to obtain $w_1(t) = \Theta$ on I. Consequently

(42) $$w_1(0) = (Cu')(0) - (Cv_0)(0) + (Bu)(0) - (Bu_0)(0) = \\ = [C(u' - v_0)](0) + [B(u - u_0)](0) = \Theta.$$

The second term vanishes because of (39) and (29). Using (30), we obtain

$$u'(0) = v_0 . \tag{43}$$

Repeating the consideration, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_1 \otimes v)\big|_{t=T} = 0$$

for every $v \in \mathscr{K}$. By virtue of Lemma 3, (36), (39) and (34), we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}([Bu - Bu_0] \otimes v)\big|_{t=T} = \frac{\mathrm{d}}{\mathrm{d}t}(B(u - u_0) \otimes v)\big|_{t=T} = \tag{44}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(u - u_0 \otimes Bv)\big|_{t=T} = (u' \otimes Bv)(T) = (Bu' \otimes v)(T) .$$

Then using Lemma 5, Lemma 2, (44), (30) and (43), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_1 \otimes v)\big|_{t=T} = (w_2 \otimes v)(T) = 0$$

for every $v \in \mathscr{K}$, where

$$w_2 = (Cu')' + Bu' + Au - f \in \mathscr{C}_0 . \tag{44'}$$

Hence $w_2(t) = \Theta$ on I follows with the use of Lemma 6. Thus (38) yields both (27) and (28).

On the contrary, let $u \in \mathscr{K}$ satisfy (27) and (28). Then $(w_2 \otimes v)(t)$ vanishes for every $t \in I$, consequently

$$(1 * (w_2 \otimes v))(T) = 0$$

for every $v \in \mathscr{K}$. Lemma 3, (32), (34) and (28) yield that

$$((1 * Bu') \otimes v)(T) = ([Bu - Bu] \otimes v)(T) . \tag{45}$$

Using (30), we derive

$$((1 * (Cu')') \otimes v)(T) = ([Cu' - (Cv_0)(0)] \otimes v)(T) . \tag{46}$$

Therefore

$$(1 * (w_2 \otimes v))(T) = (w_1 \otimes v)(T) = 0$$

for every $v \in \mathscr{K}$, consequently $w_1(t) = \Theta$ on I. Then

$$(1 * (w_1 \otimes v))(T) = 0 .$$

By virtue of Lemma 3, (33), (35) and (28), we have

$$((1 * Cu') \otimes v)(T) = ([Cu - Cu_0] \otimes v)(T) ,$$

therefore

$$(1 * (w_1 \otimes v))(T) = (w \otimes v)(T) = \tfrac{1}{2}\delta\mathscr{F}(u) = 0$$

and the proof is complete.

Theorem 5. *Let* (29), (30), (32), (34), (35), (36), (37) *and*

$$(1 * (Av \otimes u))(T) = (1 * (Au \otimes v))(T) \tag{47}$$

hold. Assume that the set $\mathscr{K}_0 = \{v \in \mathscr{K}, (Cv)(T) = \Theta\}$ *is dense in* $L_2(I, H)$ *and the set of* $(Cv)(T)$, *where* $v \in \mathscr{K}$, *is dense in* H. *Define the functional*

$$\begin{aligned}\mathscr{F}'(u) = ([Cu' + Bu + 1 * Au] \otimes u)(T) - \\ - 2([1 * f + Bu_0 + (Cv_0)(0)] \otimes u)(T) + (u(0) - 2u_0, (Cu)(T)).\end{aligned} \tag{48}$$

Then

$$\delta\mathscr{F}'(u) = 0 \quad on \quad \mathscr{K} \tag{49}$$

if and only if $u \in \mathscr{K}$ *satisfies* (27) *and* (28).

Remark 7. Obviously, (32), (34), (35), (36), (37) and (47) hold, if the conditions of Remark 6 are satisfied.

Remark 8. Note, that

$$\mathscr{F}'(u) = d\mathscr{F}(u)/\mathrm{d}T$$

follows from Lemma 5 and Lemma 2, because of (37), (35) and the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_0 \otimes Cu)|_{t=T} = (u_0, (Cu)(T)) = ((Cu_0)' \otimes u) + ((Cu_0)(0), u(T)).$$

Proof of Theorem 5. Denote $\delta u = v \in \mathscr{K}$. Using (35), (37), we derive

$$(Cv' \otimes u)(T) + (v(0), (Cu)(T)) = (Cu' \otimes v)(T) + (u(0), (Cv)(T)). \tag{50}$$

With the use of (34), (47), and (50), we obtain

$$\delta\mathscr{F}'(u) = 2(w_1 \otimes v)(T) + 2(u(0) - u_0, (Cv)(T)), \tag{51}$$

where w_1 is defined in (41). Hence (49) and Lemma 6 yield (for $\mathscr{M} = \mathscr{K}_0$) that $w_1(t)$ vanishes on I. Consequently

$$\delta\mathscr{F}'(u) = 2(u(0) - u_0, (Cv)(T)) = 0 \tag{52}$$

for $v \in \mathscr{K}$ and $u(0) = u_0$ follows from the density of $(Cv)(T)$ in H. Inserting $t = 0$, we obtain

$$\begin{aligned}w_1(0) = (Cu')(0) - (Cv_0)(0) + (Bu)(0) - (Bu_0)(0) = \\ = [C(u' - v_0)](0) + [B(u - u_0)](0) = \Theta.\end{aligned}$$

Using (29), (52) and (30), we conclude that

$$u'(0) = v_0 \,. \tag{53}$$

As $w_1(t)$ vanishes on I, we have

$$(w_1 \otimes v)(t) = 0$$

for $t \in I$ and consequently

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_1 \otimes v)|_{t=T} = 0$$

for every $v \in \mathscr{K}$. By virtue of (36), (52), (34) and Lemma 5, (44) holds. Then using Lemma 5, Lemma 2, (44), (30), and (53), we obtain

$$(w_2 \otimes v)(T) = 0$$

for every $v \in \mathscr{K}$, where $w_2 \in \mathscr{C}_0$ is defined in (44'). Lemma 6 yields that w_2 vanishes on I, i.e., (27) holds. Thus from (49) both (27) and (28) follow.

On the contrary, let $u \in \mathscr{K}$ satisfy (27) and (28). Then $(w_2 \otimes v)(t)$ vanishes for every $t \in I$, consequently

$$(1 * (w_2 \otimes v))(T) = 0$$

for every $v \in \mathscr{K}$. Using Lemma 3, (32), (28), (34), (30) and (51), we derive (see also the Proof of Theorem 4)

$$(1 * (w_2 \otimes v))(T) = (w_1 \otimes v)(T) = \tfrac{1}{2}\delta\mathscr{F}'(u)\,.$$

Hence the variation vanishes on $\mathscr{K}$, if (27) and (28) hold.

Restricting the domain of the functional $\mathscr{F}'(u)$ to the functions, satisfying the first initial condition (28) a priori, we are led to a modified

Theorem 6. Let (29), (30), (32), (34), (35), (36), (37) and (47) hold. Assume that the set $\mathscr{K}_1 = \{v \in \mathscr{K},\ v(0) = \Theta\}$ is dense in $L_2(I, H)$. Denote $\mathscr{K}_2 = u_0 \oplus \mathscr{K}_1$ and define the functional

$$\mathscr{F}_1'(u) = ([Cu' + Bu + 1 * Au] \otimes u)(T) - \\ - 2([1 * f + Bu_0 + (Cv_0)(0)] \otimes u)(T) - (u_0, (Cu)(T))\,. \tag{54}$$

Then

$$\delta\mathscr{F}_1'(u) = 0 \quad \text{on} \quad \mathscr{K}_2 \tag{55}$$

if and only if $u \in \mathscr{K}_2$ satisfies (27) and (28).

Remark 9. The condition (55) was employed for the definition of weak solutions in a particular case of the Cauchy problem (27), (28) in [3], where the existence, uniqueness and continuous dependence of the weak solution on f, u_0 and v_0 have been proved. Using the relation (45), the functional $\mathscr{F}'_1(u)$ of (54) can be modified easily into that of [3].

Proof of Theorem 6. We have $\delta u = v \in \mathscr{K}_1$. From (35), (37) and the definition of $\mathscr{K}_2$, the relation

$$(Cv' \otimes u)(T) = (Cu' \otimes v)(T) + (u_0, (Cv)(T)) \tag{56}$$

follows. Making use of (34), (47) and (56), we obtain

$$\delta\mathscr{F}'_1(u) = 2(w_1 \otimes v)(T),$$

where w_1 is defined in (41). Hence (55) and Lemma 6 (for $\mathscr{M} = \mathscr{K}_1$) yield that w_1 vanishes on I. Inserting $t = 0$ and making use of (29), we obtain

$$w_1(0) = [C(u' - v_0)](0) = \Theta,$$

consequently, by virtue of (30),

$$u'(0) = v_0. \tag{57}$$

Next we have

$$(w_1 \otimes v)(t) = 0$$

for $t \in I$ and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_1 \otimes v)|_{t=T} = 0$$

for every $v \in \mathscr{K}_1$. From (34), (36) and Lemma 5, (44) follows. Then using Lemma 5, Lemma 2, (44), (30) and (57), we obtain

$$(w_2 \otimes v)(T) = 0$$

for every $v \in \mathscr{K}_1$, where $w_2 \in \mathscr{C}_0$ is defined in (44'). Hence $w_2(t) = \Theta$ on I follows with the use of Lemma 6. Thus (55) yields both (27) and (28).

On the contrary, let $u \in \mathscr{K}_2$ satisfy (27) and (28). Then $(w_2 \otimes v)(t)$ vanishes for every $t \in I$, consequently

$$(1 * (w_2 \otimes v))(T) = 0$$

for every $v \in \mathscr{K}_1$. Using Lemma 3, (32), (34) and (28), we obtain (45) and using (30), we derive (46). Hence

$$(1 * (w_2 \otimes v))(T) = (w_1 \otimes v)(T) = \tfrac{1}{2}\delta\mathscr{F}'_1(u) = 0$$

and the proof is complete.

Theorem 7. *Let* (29), (30), (34), (35), (36), (37) *and*

$$(Au \otimes v)(T) = (Av \otimes u)(T) \tag{58}$$

hold. Assume that the set $\mathscr{K}_3 = \{v \in \mathscr{K}, v(0) = v(T) = \Theta\}$ *is dense in* $L_2(I, H)$ *and the set of* $v(T)$, *where* $v \in \mathscr{K}_1$, *is dense in* H. *Define the functional*

$$\mathscr{F}_1''(u) = ([(Cu')' + Bu' + Au - 2f] \otimes u)(T) + \\ + ((Cu')(0) - 2(Cv_0)(0), u(T)) - (u_0, (Bu)(T) + (Cu)'(T)). \tag{59}$$

Then

$$\delta\mathscr{F}_1''(u) = 0 \quad on \quad \mathscr{K}_2 = u_0 \oplus \mathscr{K}_1 \tag{60}$$

if and only if $u \in \mathscr{K}_2$ *satisfies* (27) *and* (28).

Proof. Denote $\delta u = v \in \mathscr{K}_1$. Using Lemma 5, (34), (35), (36) and (58), we obtain

$$\delta\mathscr{F}_1''(u) = 2(w_2 \otimes v)(T) + 2((Cu')(0) - (Cv_0)(0), v(T)),$$

where $w_2 \in \mathscr{C}_0$ is defined in (44'). From (60), $\mathscr{K}_3 \subset \mathscr{K}_1$ and Lemma 6 (for $\mathscr{M} = \mathscr{K}_3$), $w_2(t) = \Theta$ on I follows, i.e. (27). Then

$$\delta\mathscr{F}_1''(u) = 2((Cu')(0) - (Cv_0)(0), v(T))$$

for every $v \in \mathscr{K}_1$. The assumption of the density of $v(T)$ yields

$$(Cu')(0) - (Cv_0)(0) = [C(u' - v_0)](0) = \Theta \tag{61}$$

and using (30), we obtain $u'(0) = v_0$. As $u \in \mathscr{K}_2$, $u(0) = u_0$ holds.

On the contrary, let $u \in \mathscr{K}_2$ satisfy (27) and (28). Then obviously

$$(w_2 \otimes v)(T) = 0$$

for every $v \in \mathscr{K}_1$ and using (30), we conclude that also (61) holds. Consequently, the variation $\delta\, \mathscr{F}_1''(u)$ vanishes and the proof is complete.

Remark 10. Note that

$$\mathscr{F}_1''(u) = d\, \mathscr{F}_1'(u)/dT$$

follows from Lemma 5, (34) and the definition of $\mathscr{K}_2$.

Remark 11. The condition (60) was employed for the definition of weak solutions in a particular case of the Cauchy problem (27), (28) in [6], where the existence, uniqueness and continuous dependence of the weak solution on f, u_0 and v_0 have been proved.

4. GENERALIZATION OF THE CONVOLUTION SCALAR PRODUCT

The convolution scalar product $(u \otimes v)(t)$ in the preceding Theorems may be replaced by any bilinear (with respect to the elements u, v) function $b(u, v; t)$, which possesses the following properties:

(I) $b(u, v; t) \leqq c_0 |u|_T \, |v|_T$ (with a constant c_0 independent of u, v, t),

(II) $b(u, v; t) = b(v, u; t)$,

(III)
$$b(l * u, v; t) = (l * b(u, v; t))(t),$$
$$b(t * u, v; t) = (t * b(u, v; t))(t),$$

(IV)
$$\frac{\mathrm{d}}{\mathrm{d}t} b(u, v; t) = b(u', v; t) + b(u\delta, v; t),$$

where the relation (I), (II) and (III) hold for every $u, v \in L_2(I, H)$ and $t \in I$, (IV) for $u \in \mathscr{C}_1$ and $v(t)$ continuous at a point $t \in I$, δ is the Dirac function

(V) an operator $\mathscr{B}$ in $\mathscr{C}_0$ exist such that

$$b(u, \mathscr{B}u; T) = 0 \Rightarrow u = \Theta$$

holds for every $u \in \mathscr{C}_0$.

Then the Lemmas 1 till 6 and their proofs hold again, if we substitute $(u \otimes v)(t)$ by $b(u, v; t)$, $\tilde{w}(t) = w(T - t)$ by $(\mathscr{B}w)(t)$. Modifying Theorems 1 till 7 and their proofs in the same way, we are led to a wider class of Cauchy problems, which can also be formulated by means of a variational approach.

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Výtah

VARIAČNÍ FORMULACE CAUCHYHO PROBLÉMU PRO ROVNICE S OPERÁTOROVÝMI KOEFICIENTY

IVAN HLAVÁČEK

V článku je navrženo několik variačních principů, které jsou ekvivalentní počátečním (Cauchyho) problémům pro rovnice prvního a druhého řádu v časové souřadnici. Koeficienty rovnic jsou lineární operátory v prostoru $L_2(I, H)$ zobrazení časového intervalu I do jistého Hilbertova prostoru H, integrovatelných s kvadrátem. Teorie zahrnuje některé třídy parciálních diferenciálních a integro-diferenciálních rovnic. Základem všech uvedených variačních principů je pojem „konvolučního skalárního součinu". Na operátorové koeficienty jsou pak kladeny podmínky jisté symetrie ve smyslu tohoto součinu. Některé z principů byly použity autorem k definici slabých řešení integro-diferenciálních rovnic [3], [6].

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