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THE REISSNERIAN ALGORITHMS IN THE REFINED THEORIES OF THE BENDING OF PLATES

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In this paper equations are derived for the bending of plates, that represent a certain generalization of the refined theories presented by I. BABUŠKA and M. PRÁGER [1–4]. The authors have shown that by accepting the notion of an asymptotical energetic error it is possible to find a unique algorithm (called by the authors "the Reissnerian algorithm") of the gradual refinement of Kirchhoff's technical plate theory. In the following is derived a differential equation of infinite order which is formally related to Lurje's equations [5, 6], and the boundary conditions, which in the given case correspond to the solution of the so called internal problem (the fundamental state of stress), are clarified [7, 8].

An outline of the important results obtained by the various authors is given by the survey papers [9–12].

1. In the works [1–4] it has been shown that if the displacements u, v, w are approximated in the form of finite series

$$(1.1) \qquad \begin{aligned} u_N(x, y, h\zeta) &= \sum_{i=1}^{N} (-1)^{i-1} h^{2i-1} a_i(x, y) \varphi_i^0(\zeta), \\ v_N(x, y, h\zeta) &= \sum_{i=1}^{N} (-1)^{i-1} h^{2i-1} b_i(x, y) \varphi_i^0(\zeta), \\ w_N(x, y, h\zeta) &= \sum_{i=1}^{N+1} (-1)^{i-1} h^{2i-2} c_i(x, y) \psi_i^0(\zeta), \end{aligned}$$

where $2h$ denotes the thickness of the plate, a_i, b_i and c_i are the solutions of the respective Eulerian equations, and φ_i^0 and ψ_i^0 are certain "optimum" polynomials, the energetic error of this approximate solution, when compared with the exact solution of the threedimensional equations of Lamé, will – for a whole class of functions expressing the loading – be minimum. The optimum functions φ_i^0 and ψ_i^0 represent polynomials and have been derived in [1]. Some of their first values, required further, are

$$(1.2)\quad \varphi_1^0 = -\gamma\zeta\,, \qquad \psi_1^0 = \gamma\,,$$

$$\varphi_2^0 = \frac{10-3\gamma}{10}\zeta - \frac{\gamma+2}{6}\zeta^3\,, \qquad \psi_2^0 = \frac{10+3\gamma}{10} + \frac{\gamma-2}{2}\zeta^2\,,$$

$$\varphi_3^0 = \frac{157\gamma-140}{4200}\zeta + \frac{4-3\gamma}{60}\zeta^3 - \frac{\gamma+4}{120}\zeta^5\,, \qquad \psi_3^0 = -\frac{157\gamma+140}{4200} + \frac{4+3\gamma}{20}\zeta^2 + \frac{\gamma-4}{24}\zeta^4\,,$$

$$\psi_4^0 = \frac{569\gamma-1020}{252\,000} + \frac{174-157\gamma}{8400}\zeta^2 + \frac{3\gamma-2}{240}\zeta^4 + \frac{\gamma-6}{720}\zeta^6\,,$$

where

$$(1.3)\qquad \gamma = 2(1-\nu)$$

and ν is Poisson's ratio.

In this paper we shall be concerned with a special case when the functions a_i, b_i and c_i may be expressed in the following manner

$$(1.4)\quad a_i(x,y) = \frac{\partial \Delta^{i-1}\, w_0(x,y)}{\partial x}\,, \quad b_i(x,y) = \frac{\partial \Delta^{i-1}\, w_0(x,y)}{\partial y}\,, \quad c_i(x,y) = \Delta^{i-1}\, w_0(x,y)\,,$$

where Δ denotes the twodimensional Laplace operator.

When substituting equation (1.2) and (1.3) into equation (1.1) we can see that the obtained formulas for the displacements are formally identical with the expansions, the first two members of which were determined by L. H. DONNEL in a different way [14].

It may be shown that if in the series for displacements obtained in such a manner, we consider the limit $N \to \infty$, it is then possible to sum up these series formally and we shall obtain

$$(1.5)\quad u(x,y,h\zeta) = \sum_{i=1}^{\infty}(-1)^{i-1} h^{2i-1}\varphi_i^0 \frac{\partial \Delta^{i-1} w_0}{\partial x} = \frac{4}{3}h^3\frac{\partial L_1 w_0}{\partial x}\,,$$

$$v(x,y,h\zeta) = \sum_{i=1}^{\infty}(-1)^{i-1} h^{2i-1}\varphi_i^0 \frac{\partial \Delta^{i-1} w_0}{\partial y} = \frac{4}{3}h^3\frac{\partial L_1 w_0}{\partial y}\,,$$

$$w(x,y,h\zeta) = \sum_{i=1}^{\infty}(-1)^{i-1} h^{2i-2}\psi_i^0\, \Delta^{i-1} w_0 = \frac{4}{3}h^3 L_2 w_0\,,$$

where L_1 and L_2 are partial differential operators of infinite order, which are regular in the sense of [15]

(1.6)

$$L_1(\sqrt{(\Delta)}h, \zeta) = [\sqrt{(\Delta)}\, h \sin \sqrt{(\Delta)}\, h \sin \sqrt{(\Delta)}\, h\zeta + \sqrt{(\Delta)}\, h\zeta \cos \sqrt{(\Delta)}\, h \cos \sqrt{(\Delta)}\, h\zeta + (\gamma - 1) \cos \sqrt{(\Delta)}\, h \sin \sqrt{(\Delta)}\, h\zeta] \frac{\Delta}{\sin 2\sqrt{(\Delta)}\, h - 2\sqrt{(\Delta)}\, h},$$

$$L_2(\sqrt{(\Delta)}h, \zeta) = [\sqrt{(\Delta)}\, h \sin \sqrt{(\Delta)}\, h \cos \sqrt{(\Delta)}\, h\zeta - \sqrt{(\Delta)}\, h\zeta \cos \sqrt{(\Delta)}\, h \sin \sqrt{(\Delta)}\, h\zeta - \gamma \cos \sqrt{(\Delta)}\, h \cos \sqrt{(\Delta)}\, h\zeta] \frac{\Delta \sqrt{(\Delta)}}{\sin 2\sqrt{(\Delta)}\, h - 2\sqrt{(\Delta)}\, h}.$$

Starting from equations (1.5) we derive the differential equation of the problem and we make clear the form of the corresponding boundary conditions. For this purpose it will be useful to apply the Lagrange variational principle.

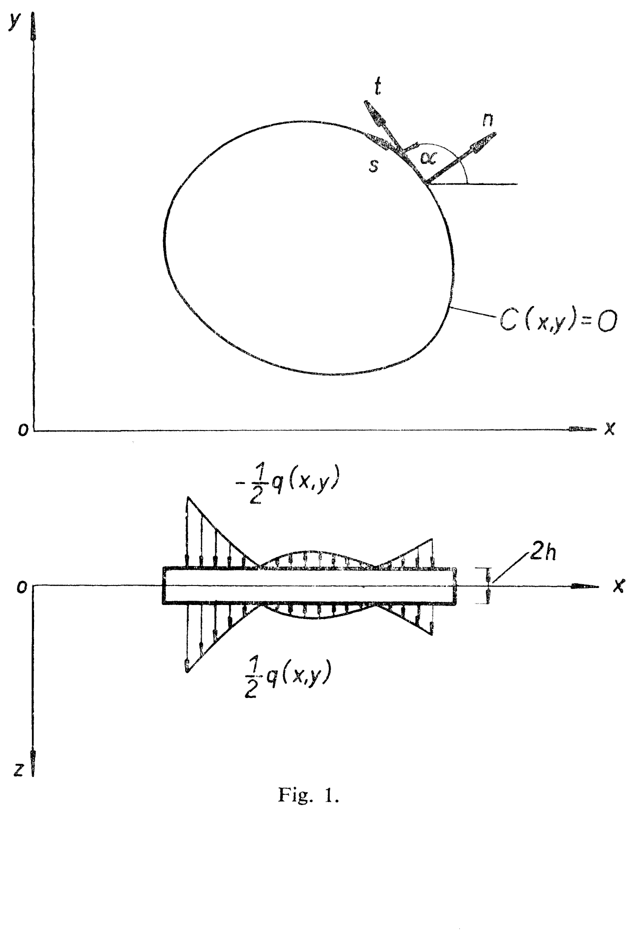

Fig. 1.

Let us assume that on the upper and the lower plane of the plate (Fig. 1), only a normal load $\pm\frac{1}{2}q(x, y)$ is acting and on the cylindrical surface of the boundary $C(x, y) = 0$ the homogeneous boundary conditions are prescribed. The potential energy of the plate will then be given by formula [16]

$$(1.7)\ V = \frac{1}{2}\iiint_{-1}^{1} \left\{\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{h\,\partial\zeta}\right)^2 + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{h\,\partial\zeta}\right)^2\right] + \right.$$
$$\left. + \mu\left[\left(\frac{\partial w}{\partial y} + \frac{\partial v}{h\,\partial\zeta}\right)^2 + \left(\frac{\partial u}{h\,\partial\zeta} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right]\right\} \mathrm{d}x\,\mathrm{d}y\,h\,\mathrm{d}\zeta -$$
$$- \iint w(x, y, h)\,.\,\tfrac{1}{2}q(x, y) + w(x, y, -h)\,.\,\tfrac{1}{2}q(x, y)]\,\mathrm{d}x\,\mathrm{d}y < \infty\,.$$

In this equation λ and μ are Lamé's constants.

From the Lagrange principle it follows that $\delta V = 0$, from which it is

$$(1.8)\qquad \iint\left\{\int_{-1}^{1}\left[\frac{\lambda}{\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{h\,\partial\zeta}\right)\left(\frac{\partial\delta u}{\partial x} + \frac{\partial\delta v}{\partial y} + \frac{\partial\delta w}{h\,\partial\zeta}\right) + \right.\right.$$
$$+ 2\left(\frac{\partial u}{\partial x}\frac{\partial\delta u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial\delta v}{\partial y} + \frac{\partial w}{h\,\partial\zeta}\frac{\partial\delta w}{h\,\partial\zeta}\right) + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{h\,\partial\zeta}\right)\left(\frac{\partial\delta w}{\partial y} + \frac{\partial\delta v}{h\,\partial\zeta}\right) +$$
$$\left. + \left(\frac{\partial u}{h\,\partial\zeta} + \frac{\partial w}{\partial x}\right)\left(\frac{\partial\delta u}{h\,\partial\zeta} + \frac{\partial\delta w}{\partial x}\right) + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\frac{\partial\delta v}{\partial x} + \frac{\partial\delta u}{\partial y}\right)\right] h\,\mathrm{d}\zeta -$$
$$\left. - \frac{1}{\mu}\,q\delta\,w(x, y, h)\right\}\mathrm{d}x\,\mathrm{d}y = 0\,.$$

When (1.5) is substituted into (1.8) and the relation

$$(1.9)\qquad \frac{\lambda}{\mu} = \frac{2-\gamma}{\gamma-1}$$

is used, we obtain

(1.10)

$$\iint\left\{\int_{-1}^{1}\left[\frac{2-\gamma}{\gamma-1}\left(L_1\Delta + \frac{\partial L_2}{h\,\partial\zeta}\right)w_0\,.\left(L_1\Delta + \frac{\partial L_2}{h\,\partial\zeta}\right)\delta w_0 + 2\,\frac{\partial L_2 w_0}{h\,\partial\zeta}\,.\,\frac{\partial L_2\,\delta w_0}{h\,\partial\zeta} + \right.\right.$$
$$+ \frac{\partial}{\partial x}\left(\frac{\partial L_1}{h\,\partial\zeta} + L_2\right)w_0\,.\,\frac{\partial}{\partial x}\left(\frac{\partial L_1}{h\,\partial\zeta} + L_2\right)\delta w_0 + \frac{\partial}{\partial y}\left(\frac{\partial L_1}{h\,\partial\zeta} + L_2\right)w_0\,.$$
$$.\,\frac{\partial}{\partial y}\left(\frac{\partial L_1}{h\,\partial\zeta} + L_2\right)\delta w_0 + 2\left(\frac{\partial^2 L_1 w_0}{\partial x^2}\,.\,\frac{\partial^2 L_1\,\delta w_0}{\partial x^2} + 2\,\frac{\partial^2 L_1 w_0}{\partial x\,\partial y}\,.\,\frac{\partial^2 L_1\,\delta w_0}{\partial x\,\partial y} + \right.$$
$$\left.\left.\left. + \frac{\partial^2 L_1 w_0}{\partial y^2}\,.\,\frac{\partial^2 L_1\,\delta w_0}{\partial y^2}\right)\right] h\,\mathrm{d}\zeta - \frac{q(x, y)}{D(1-\nu)}\,L_2(\sqrt{(\Delta)}h, 1)\,\delta w_0\right\}\mathrm{d}x\,\mathrm{d}y = 0\,,$$

where D is the flexural rigidity of the plate according to the technical theory of plates

$$(1.11) \qquad D = \frac{4}{3}\frac{\mu h^3}{1 - \nu} = \frac{8\mu h^3}{3\gamma}\,.$$

It is important to point out the meaning of the points in (1.10) because they indicate the limit of the action of the corresponding operator. In the equation (1.10) there are terms containing the values of operators of the variations of w_0. We, however, need the variations of the function w_0 itself and of its normal derivatives. Such an arrangement of the equation (1.10) is possible by using the formula derived by V. K. PROKOPOV [17]. According to this it holds

$$(1.12) \iint U\,.\,X(\Delta)\,V\,.\,\mathrm{d}x\,\mathrm{d}y = \iint X(\Delta)\,U\,.\,V\,.\,\mathrm{d}x\,\mathrm{d}y + \sum_{k=1}^{\infty}\int_C \left\{X^{(k)}(\Delta)\,U\,.\,\frac{\partial \Delta^{k-1} V}{\partial n} - \frac{\partial X^{(k)}(\Delta)\,U}{\partial n}\,.\,\Delta^{k-1} V\right\}\mathrm{d}s\,.$$

In this equation $X(\Delta)$ is the differential operator

$$(1.13) \qquad X(\Delta) = \sum_{j=0}^{\infty} a_j \Delta^j\,,$$

and $X^{(k)}(\Delta)$ is the so called k-th reduced operator

$$(1.14) \qquad X^{(k)}(\Delta) = \sum_{j=k}^{\infty} a_j \Delta^{j-k}\,.$$

The relation (1.12) is a generalisation of Green's formula

$$(1.15) \qquad \iint U\,.\,\Delta V\,.\,\mathrm{d}x\,\mathrm{d}y = \iint \Delta U\,.\,V\,.\,\mathrm{d}x\,\mathrm{d}y + \int_C \left(U\,.\,\frac{\partial V}{\partial n} - \frac{\partial U}{\partial n}\,.\,V\right)\mathrm{d}s\,.$$

Using, in equation (1.10), Green's generalized formula (1.12), the ordinary Green formula (1.15) and an integration by parts, we obtain

(1.16)

$$\frac{2-\gamma}{\gamma-1}\iiint_{-1}^{1}\left(\Delta L_1 + \frac{\partial L_2}{h\,\partial\zeta}\right)^2 w_0\,.\,\delta w_0\,\mathrm{d}x\,\mathrm{d}y\,h\,\mathrm{d}\zeta + \frac{2-\gamma}{\gamma-1}\int_C\int_{-1}^{1}\left\{\left(\Delta L_1 + \frac{\partial L_2}{h\,\partial\zeta}\right) w_0\,.\right.$$

$$.\,\frac{\partial L_1\,\delta w_0}{\partial n} - \frac{\partial}{\partial n}\left(\Delta L_1 + \frac{\partial L_2}{h\,\partial\zeta}\right) w_0\,.\,L_1\,\delta w_0 + \sum_{k=1}^{\infty}\left[\left(\Delta L_1^{(k)} + \frac{\partial L_2^{(k)}}{h\,\partial\zeta}\right)\left(\Delta L_1 + \frac{\partial L_2}{h\,\partial\zeta}\right) w_0\,.\right.$$

$$\left.\left..\,\frac{\partial \Delta^{k-1}\,\delta w_0}{\partial n} - \frac{\partial}{\partial n}\left(\Delta L_1^{(k)} + \frac{\partial L_2^{(k)}}{h\,\partial\zeta}\right)\left(\Delta L_1 + \frac{\partial L_2}{h\,\partial\zeta}\right) w_0\,.\,\Delta^{k-1}\,\delta w_0\right]\right\}\mathrm{d}s\,h\,\mathrm{d}\zeta\, +$$

$$+\,2\iiint_{-1}^{1}\left(\frac{\partial L_2}{h\,\partial\zeta}\right)^2 w_0\,.\,\delta w_0\,\mathrm{d}x\,\mathrm{d}y\,h\,\mathrm{d}\zeta + 2\int_C\int_{-1}^{1}\sum_{k=1}^{\infty}\left(\frac{\partial L_2^{(k)}}{h\,\partial\zeta}\,\frac{\partial L_2}{h\,\partial\zeta}\,w_0\,.\,\frac{\partial \Delta^{k-1}\,\delta w_0}{\partial n} - \right.$$

$$- \frac{\partial}{\partial n} \frac{\partial L_2^{(k)}}{h\, \partial \zeta} \frac{\partial L_2}{h\, \partial \zeta} w_0 \,.\, \Delta^{k-1}\, \delta w_0 \Big) \mathrm{d}s\, h\, \mathrm{d}\zeta - \iiint_{-1}^{1} \Delta \left(\frac{\partial L_1}{h\, \partial \zeta} + L_2 \right)^2 w_0 \,.\, \delta w_0 \,\mathrm{d}x\, \mathrm{d}y\, h\, \mathrm{d}\zeta +$$

$$+ \int_C \int_{-1}^{1} \left\{ \frac{\partial}{\partial n} \left(\frac{\partial L_1}{h\, \partial \zeta} + L_2 \right) w_0 \,.\, \left(\frac{\partial L_1}{h\, \partial \zeta} + L_2 \right) \delta w_0 - \sum_{k=1}^{\infty} \left[\Delta \left(\frac{\partial L_1^{(k)}}{h\, \partial \zeta} + L_2^{(k)} \right) . \right. \right.$$

$$. \left(\frac{\partial L_1}{h\, \partial \zeta} + L_2 \right) w_0 \,.\, \frac{\partial \Delta^{k-1}\, \delta w_0}{\partial n} - \frac{\partial}{\partial n} \Delta \,.\, \left(\frac{\partial L_1^{(k)}}{h\, \partial \zeta} + L_2^{(k)} \right) \left(\frac{\partial L_1}{h\, \partial \zeta} + L_2 \right) w_0 \,.\, \Delta^{k-1}\, \delta w_0 \Big] \Big\} .$$

$$. \,\mathrm{d}s\, h\, \mathrm{d}\zeta + 2 \iiint_{-1}^{1} \Delta^2 L_1^2 w_0 \,.\, \delta w_0 \,\mathrm{d}x\, \mathrm{d}y\, h\, \mathrm{d}\zeta + 2 \int_C \int_{-1}^{1} \left[\frac{\partial^2 L_1 w_0}{\partial n^2} \,.\, \frac{\partial L_1\, \delta w_0}{\partial n} + \right.$$

$$+ \left(\frac{\partial^2 L_1}{\partial n\, \partial s} - \frac{\partial \alpha}{\partial s} \frac{\partial L_1}{\partial s} \right) w_0 \,.\, \frac{\partial L_1\, \delta w_0}{\partial s} - \frac{\partial}{\partial n} \Delta L_1 w_0 \,.\, L_1\, \delta w_0 + \sum_{k=1}^{\infty} \Big(\Delta^2 L_1^{(k)} L_1 w_0 \,.$$

$$. \frac{\partial \Delta^{k-1}\, \delta w_0}{\partial n} - \frac{\partial}{\partial n} \Delta^2 L_1^{(k)} L_1 w_0 \,.\, \Delta^{k-1}\, \delta w_0 \Big) \Big] \mathrm{d}s\, \mathrm{d}h\, \mathrm{d}\zeta - \frac{2}{\gamma D} \left\{ \iiint_{-1}^{1} L_2(\sqrt{(\Delta)}\, h, 1)\, q \,. \right.$$

$$. \,\delta w_0 \,\mathrm{d}x\, \mathrm{d}y\, h\, \mathrm{d}\zeta + \int_C \int_{-1}^{1} \sum_{k=1}^{\infty} \left[L_2^{(k)}(\sqrt{(\Delta)}\, h, 1)\, q \,.\, \frac{\partial \Delta^{k-1}\, \delta w_0}{\partial n} - \right.$$

$$\left. \left. - \frac{\partial}{\partial n} L_2^{(k)}((\sqrt{})\Delta\, h, 1)\, q \,.\, \Delta^{k-1}\, \delta w_0 \right] \mathrm{d}s\, h\, \mathrm{d}\zeta \right\} = 0 \,.$$

In this equation the operators L_1 and L_2 are given in their finite form by equation (1.6). For further calculations, their form in series following from equation (1.5) is also advantageous:

$$L_1(\sqrt{(\Delta)}\, h, \zeta) = \frac{3}{4h^2} \sum_{j=0}^{\infty} (-1)^j (\sqrt{(\Delta)}\, h)^{2j} \varphi_{j+1}^0(\zeta) \,, \tag{1.17}$$

$$L_2(\sqrt{(\Delta)}\, h, \zeta) = \frac{3}{4h^3} \sum_{j=0}^{\infty} (-1)^j (\sqrt{(\Delta)}\, h)^{2j} \psi_{j+1}^0(\zeta) \,.$$

The reduced operators $L_1^{(k)}$ and $L_2^{(k)}$, according to the definition (1.14) will be

$$L_1^{(k)}(\sqrt{(\Delta)}\, h, \zeta) = \tfrac{3}{4} h^{2k-2} \sum_{j=0}^{\infty} (-1)^{j+k} (\sqrt{(\Delta)} h)^{2j} \varphi_{j+k+1}^0(\zeta) \,, \tag{1.18}$$

$$L_2^{(k)}(\sqrt{(\Delta)}\, h, \zeta) = \tfrac{3}{4} h^{2k-3} \sum_{j=0}^{\infty} (-1)^{j+k} (\sqrt{(\Delta)} h)^{2j} \psi_{j+k+1}^0(\zeta) \,.$$

Further in equation (1.16) α is the angle between the tangent to the curve $C(x, y) = 0$, and the axis x (Fig. 1), and $\partial \alpha / \partial s$ is the curvature of the boundary C. The Laplace operator in the line integrals is

$$\Delta = \frac{\partial^2}{\partial n^2} + \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial n^2} + \frac{\partial \alpha}{\partial s} \frac{\partial}{\partial n} + \frac{\partial^2}{\partial s^2} \,. \tag{1.19}$$

With regard to the arbitrariness of δw_0 the members with the volume integrals in (1.16) will be equal to zero

(1.20)

$$\iint\left\{\int_{-1}^{1}\left[\frac{2-\gamma}{\gamma-1}\left(L_1\Delta+\frac{\partial L_2}{h\,\partial\zeta}\right)^2+2\left(\frac{\partial L_2}{h\,\partial\zeta}\right)^2-\left(\frac{\partial L_1}{h\,\partial\zeta}+L_2\right)^2\Delta+2L_1^2\,\Delta\right]w_0\,.\,h\,\mathrm{d}\zeta-\right.$$

$$\left.-\frac{2}{\gamma D}L_2(\sqrt{(\Delta)}\,h,1)\,q\right\}\delta w_0\,\mathrm{d}x\,\mathrm{d}y=0\,.$$

When substituting for L_1 and L_2 the expressions (1.5), and taking Δ as a number, the expressions in (1.20) can be integrated with respect to ζ. We are not giving here cumbersome calculations, we only present the final result

(1.21)

$$\iint\left[-2\gamma\frac{\Delta\sqrt{(\Delta)}\cos^2\sqrt{(\Delta)}h}{\sin 2\sqrt{(\Delta)}\,h-2\sqrt{(\Delta)}h}\,\Delta^2 w_0+\frac{\gamma}{(1-\nu)\,D}\,\frac{\Delta\sqrt{(\Delta)}\cos^2\sqrt{(\Delta)}h}{\sin 2\sqrt{(\Delta)}\,h-2\sqrt{(\Delta)}\,h}\,q\right].$$

$$.\,\delta w_0\,\mathrm{d}x\,\mathrm{d}y=0\,.$$

With regard to the arbitrariness of δw_0 the expression under the integral must be equal to zero. From the latter we obtain the basic differential equation of a variant of the refined Babuška-Prâger theory

$$\frac{\Delta\sqrt{(\Delta)}\cos^2\sqrt{(\Delta)}\,h}{\sin 2\sqrt{(\Delta)}\,h-2\sqrt{(\Delta)}\,h}\left[\Delta^2 w_0-\frac{q}{2(1-\nu)\,D}\right]=0\,. \tag{1.22}$$

The transcendental differential operator standing before the bracket is then $-(1/\gamma)\,L_2(\sqrt{(\Delta)}\,h,1)$ and thus using (1.17) we can write the equation (1.22) in the form of a differential equation of infinite order

$$\sum_{j=1}^{\infty}\frac{1}{\gamma}\,\psi_j^0(1)\,(-1)^{j-1}\,h^{2j-2}\Delta^{j-1}\left[\Delta^2 w_0-\frac{q}{2(1-\nu)\,D}\right]=0\,, \tag{1.23}$$

where the values of ψ_j^0 are given in (1.2). The first three terms of this equation (1.23) are

$$\left(1-\frac{4}{5}h^2\,\Delta+\frac{27}{175}\,h^4\Delta^2-\ldots\right)\left[\Delta^2 w_0-\frac{q}{2(1-\nu)\,D}\right]=0\,. \tag{1.24}$$

This equation is identical, up to the terms of the order h^4, with the equation derived in [18] directly, provided that we consider in equations (1.1) and (1.4) $N=2$.

It is worth noticing that the biharmonic solution of the equations (1.22) and (1.23) corresponding to Kirchhoff's theory, represents a particular integral of these equations.

2. It remains to determine the boundary conditions for the differential equation (1.22). For this purpose it is necessary to substitute in the equation (1.16) for the operator L_1, L_2, $L_1^{(k)}$ and $L_2^{(k)}$ their expansions (1.17), and (1.18), and to integrate with respect to ζ. No elementary operations and arrangements are given here, only the result is pressented in the form

$$(2.1)\quad \frac{9}{8h^3}\int_C \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^{j+k}\,h^{2j+2k-6}\left\{\frac{2-\gamma}{\gamma-1}\left[h^2A_{j,0,k}\,\Delta^j w_0\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-\right.\right.$$

$$-\,h^2A_{j,0,k}\frac{\partial\Delta^j w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0+B_{j,k}\,\Delta^{j-1}w_0\,\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-B_{j,k}\frac{\partial\Delta^{k-1}w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0+$$

$$+\,h^2C_{j,k}\,\Delta^j w_0\,\frac{\partial\Delta^{k-1}w_0}{\partial n}-h^2C_{j,k}\,\frac{\partial\Delta^j w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0+(-1)^l\,h^{2l+2}D_{j,k,l}\,\Delta^{j+l}w_0\,.$$

$$\left.\cdot\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-(-1)^l\,h^{2l+2}D_{j,k,l}\,\frac{\partial\Delta^{j+l}w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0\right]+$$

$$+\,2\left[h^2E_{j,0,k}\,\Delta^j w_0\,\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-h^2E_{j,0,k}\,\frac{\partial\Delta^j w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0+\right.$$

$$\left.+\,(-1)^l\,h^{2l+2}E_{j,k,l}\,\Delta^{j+l}w_0\,\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-(-1)^l\,h^{2l+2}E_{j,k,l}\,\frac{\partial\Delta^{j+l}w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0\right]+$$

$$+\,h^4F_{j,1,k}\frac{\partial\Delta^j w_0}{\partial n}\,\Delta^k\,\delta w_0+(-1)^l\,h^{2l+2}\,F_{j,k,l}\,\Delta^{j+l}w_0\,\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-$$

$$-\,(-1)^l\,h^{2l+2}F_{j,k,l}\,\frac{\partial\Delta^{j+l}w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0+2\left[h^2A_{j,0,k}\,\frac{\partial^2\Delta^{j-1}w_0}{\partial n^2}\,\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}+\right.$$

$$+\,h^2A_{j,0,k}\left(\frac{\partial^2\Delta^{j-1}w_0}{\partial n\,\partial s}-\frac{\partial\alpha}{\partial s}\,\frac{\partial\Delta^{j-1}w_0}{\partial s}\right)\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial s}-h^2A_{j,0,k}\,\frac{\partial\Delta^j w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0+$$

$$\left.+\,(-1)^l\,h^{2l+2}A_{j,k,l}\,\Delta^{j+l}w_0\,\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-(-1)^l\,h^{2l+2}A_{j,k,l}\,\frac{\partial\Delta^{j+l}w_0}{\partial n}\,\Delta^{k-1}\,\delta w_0\right]+$$

$$\left.+\,\frac{4h^4}{3\gamma D}\,\psi^0_{j+k}(1)\left[\Delta^{j-1}q\,\frac{\partial\Delta^{k-1}\,\delta w_0}{\partial n}-\frac{\partial\Delta^{j-1}q}{\partial n}\,\Delta^{k-1}\,\delta w_0\right]\right\}\mathrm{d}s=0\,,$$

where we denote

$$(2.2)\qquad A_{j,k,l}=\int_0^1\varphi^0_{j+k}\,\varphi^0_l\,\mathrm{d}\zeta\,,$$

$$B_{j,k}=\int_0^1\frac{\mathrm{d}\psi^0_j}{\mathrm{d}\zeta}\,\varphi^0_k\,\mathrm{d}\zeta\,,$$

$$C_{j,k}=\int_0^1\left(\frac{\mathrm{d}\psi^0_{j+1}}{\mathrm{d}\zeta}-\varphi^0_j\right)\frac{\mathrm{d}\psi^0_{k+1}}{\mathrm{d}\zeta}\,\mathrm{d}\zeta\,,$$

$$D_{j,k,l} = \int_0^1 \left(\frac{d\psi^0_{j+k+1}}{d\zeta} - \varphi^0_{j+k}\right)\left(\frac{d\psi^0_{l+1}}{d\zeta} - \varphi^0_l\right) d\zeta\,,$$

$$E_{j,k,l} = \int_0^1 \frac{d\psi^0_{j+k+1}}{d\zeta}\,\frac{d\psi^0_{l+1}}{d\zeta}\, d\zeta\,,$$

$$F_{j,k,l} = \int_0^1 \left(\frac{d\varphi^0_{j+k}}{d\zeta} + \psi^0_{j+k}\right)\left(\frac{d\varphi^0_{l+1}}{d\zeta} + \psi^0_{l+1}\right) d\zeta\,.$$

Some first values of the coefficients $A_{j,k,l} - F_{j,k,l}$ are presented in Tab. 1.

Further we substitute these coefficients into (2.1) and we arrange the individual terms according to the variations $\partial\Delta^k\,\delta w_0/\partial n$, $\partial\Delta^k\,\delta w_0/\partial s$ and $\Delta^k\,\delta w_0$. In the calculations the terms containing $\Delta^k q$ and $\partial\Delta^k q/\partial n$ must be eliminated by using the equation (1.23). We obtain

$$(2.3)\quad \int_c \Bigg\{\Bigg[2\gamma\left(\frac{\partial^2 w_0}{\partial n^2} + \nu\,\frac{\partial\alpha}{\partial s}\,\frac{\partial w_0}{\partial n} + \nu\,\frac{\partial^2 w_0}{\partial s^2}\right) - \frac{4}{5}\,\gamma\nu h^2\left(\frac{\partial^2\Delta w_0}{\partial s^2} + \frac{\partial\alpha}{\partial s}\,\frac{\partial\Delta w_0}{\partial n}\right) -$$
$$- \frac{4}{525}\,\gamma\nu h^4\left(\frac{\partial^2\Delta^2 w_0}{\partial s^2} + \frac{\partial\alpha}{\partial s}\,\frac{\partial\Delta^2 w_0}{\partial n}\right) - \ldots\Bigg]\frac{\partial\delta w_0}{\partial n} +$$
$$+ \Bigg[\frac{8}{5}\,\nu h^2\left(\frac{\partial^2 w_0}{\partial n^2} + \nu\,\frac{\partial\alpha}{\partial s}\,\frac{\partial w_0}{\partial n} + \nu\,\frac{\partial^2 w_0}{\partial s^2}\right) + \frac{4}{525}\,(\gamma + 2)\,h^4\,\Delta^2 w_0 -$$
$$- \frac{h^4}{525}\,(85\gamma^2 - 332\gamma + 340)\left(\frac{\partial^2\Delta w_0}{\partial s^2} + \frac{\partial\alpha}{\partial s}\,\frac{\partial\Delta w_0}{\partial n}\right) - \ldots\Bigg]\frac{\partial\Delta\,\delta w_0}{\partial n} +$$
$$+ \Bigg[\frac{8}{525}\,\nu h^4\left(\frac{\partial^2 w_0}{\partial n^2} + \nu\,\frac{\partial\alpha}{\partial s}\,\frac{\partial w_0}{\partial n} + \nu\,\frac{\partial^2 w_0}{\partial s^2}\right) + \ldots\Bigg]\frac{\partial\Delta^2\,\delta w_0}{\partial n} + \ldots +$$
$$+ \Bigg[\gamma^2\left(\frac{\partial^2 w_0}{\partial n\,\partial s} - \frac{\partial\alpha}{\partial s}\,\frac{\partial w_0}{\partial s}\right) + \frac{2}{5}\,h^2\,\gamma(2-\gamma)\left(\frac{\partial^2\Delta w_0}{\partial n\,\partial s} - \frac{\partial\alpha}{\partial s}\,\frac{\partial\Delta w_0}{\partial s}\right) +$$
$$+ \frac{2}{525}\,h^4\,\gamma(2-\gamma)\left(\frac{\partial^2\Delta^2 w_0}{\partial n\,\partial s} - \frac{\partial\alpha}{\partial s}\,\frac{\partial\Delta^2 w_0}{\partial s}\right) + \ldots\Bigg]\frac{\partial\delta w_0}{\partial s} +$$
$$+ \Bigg[\frac{2}{5}\,h^2\,\gamma(2-\gamma)\left(\frac{\partial^2 w_0}{\partial n\,\partial s} - \frac{\partial\alpha}{\partial s}\,\frac{\partial w_0}{\partial s}\right) + \frac{h^4}{525}\,(85\gamma^2 - 332\gamma + 340)\,.$$
$$.\left(\frac{\partial^2\Delta w_0}{\partial n\,\partial s} - \frac{\partial\alpha}{\partial s}\,\frac{\partial\Delta w_0}{\partial s}\right) + \ldots\Bigg]\frac{\partial\Delta\,\delta w_0}{\partial s} +$$
$$+ \Bigg[\frac{2}{525}\,\gamma(2-\gamma)\,h^4\left(\frac{\partial^2 w_0}{\partial n\,\partial s} - \frac{\partial\alpha}{\partial s}\,\frac{\partial w_0}{\partial s}\right) + \ldots\Bigg]\frac{\partial\Delta^2\,\delta w_0}{\partial s} + \ldots +$$
$$+ \left[-2\gamma\,\frac{\partial\Delta w_0}{\partial n}\right]\delta w_0 + \left[\frac{4}{5}\,h^2(\gamma+2)\,\frac{\partial\Delta w_0}{\partial n} + h^4\,\frac{4(2-\gamma)}{525}\,\frac{\partial\Delta^2 w_0}{\partial n} + \ldots\right]\Delta\delta w_0 +$$
$$+ \left[\frac{4}{525}\,h^4(\gamma+2)\,\frac{\partial\Delta w_0}{\partial n} + \ldots\right]\Delta^2\,\delta w_0 + \ldots\Bigg\}\,ds = 0\,.$$

Table 1

$$A_{1,0,1} = \frac{1}{3}\gamma^2 , \quad A_{1,1,1} = A_{2,0,1} = A_{1,0,2} = \frac{2\gamma}{15}(\gamma - 2) ,$$

$$A_{1,1,2} = A_{2,0,2} = \frac{1}{1575}(85\gamma^2 - 332\gamma + 340) , \quad A_{1,2,1} = A_{2,1,1} = A_{3,0,1} =$$

$$= A_{1,0,3} = \frac{2\gamma}{1575}(2 - \gamma) ,$$

$$B_{1,k} = 0 , \quad B_{2,1} = \frac{1}{3}\gamma(2 - \gamma) , \quad B_{3,1} = -\frac{2}{15}\gamma^2 , \quad B_{2,2} = -\frac{2}{15}(2 - \gamma)^2 ,$$

$$B_{4,1} = \frac{2\gamma^2}{1575} , \quad B_{3,2} = \frac{1}{1575}(8 + 170\gamma - 85\gamma^2) , \quad B_{2,3} = \frac{2}{1575}(\gamma - 2)^2 ,$$

$$C_{1,1} = \frac{2}{3}(\gamma - 1)(\gamma - 2) , \qquad C_{2,1} = \frac{4}{15}(\gamma - 1)(\gamma - 2) , \qquad C_{1,2} = \frac{4}{15}\gamma(\gamma - 1) ,$$

$$C_{3,1} = \frac{4}{1575}(\gamma - 1)(2 - \gamma) , \quad C_{2,2} = \frac{2}{1575}(\gamma - 1)(85\gamma - 4) , \quad C_{1,3} = \frac{4\gamma}{1575}(1 - \gamma) ,$$

$$D_{1,1,1} = \frac{8}{15}(\gamma - 1)^2 , \quad D_{2,1,1} = D_{1,2,1} = -\frac{8}{1575}(\gamma - 1)^2 , \quad D_{1,1,2} = \frac{68}{315}(\gamma - 1)^2 ,$$

$$E_{1,0,1} = \frac{1}{3}(\gamma - 2)^2 , \quad E_{2,0,1} = E_{1,0,2} = E_{1,1,1} = \frac{2}{15}\gamma(\gamma - 2) ,$$

$$E_{3,0,1} = E_{1,0,3} = E_{1,2,1} = E_{2,1,1} = \frac{2\gamma}{1575}(2 - \gamma) , \quad E_{1,1,2} = E_{2,0,2} =$$

$$= \frac{1}{1575}(85\gamma^2 - 8\gamma + 16) ,$$

$$F_{1,1,1} = F_{2,0,1} = \frac{32}{15} , \quad F_{2,1,1} = F_{1,1,2} = F_{1,2,1} = -\frac{32}{1575} ,$$

$$G_{2,1} = \frac{4}{3}\gamma , \quad G_{3,1} = 0 , \quad G_{2,2} = \frac{8}{15}(\gamma + 2) , \quad G_{3,2} = \frac{8}{1575}(\gamma - 2) ,$$

$$G_{2,3} = -\frac{8}{1575}(\gamma + 2) .$$

Let us show further that it is possible to give a physical interpretation to the terms in the brackets. We shall, therefore, introduce polymoments and shearing polyforces in the following manner: The expression

$$M_x^k = -\frac{h^2}{\gamma}\int_{-1}^{1} \sigma_x \varphi_{k+1}^0 \, \mathrm{d}\zeta \tag{2.4}$$

will be called the k-th bending polymoment.

Similarly the k-th twisting polymoment will be

$$M_{xy}^k = -\frac{h^2}{\gamma}\int_{-1}^{1} \tau_{xy} \varphi_{k+1}^0 \, \mathrm{d}\zeta \tag{2.5}$$

and the k-th shearing polyforce will be

$$Q_x^k = \frac{h}{\gamma}\int_{-1}^{1} \tau_{xz} \psi_{k+1}^0 \, \mathrm{d}\zeta \, . \tag{2.6}$$

In these relations σ_x, τ_{xy} and τ_{xz} are normal and tangential stresses acting in the cross-section $x = \text{const.}$

When for the stresses we substitute the corresponding terms from Hooke's law, we obtain

(2.7)

$$M_x^k = -\frac{2h^2}{\gamma}\frac{4\mu h^3}{3}\int_0^1 \left[\frac{2-\gamma}{\gamma-1}\left(\Delta L_1 w_0 + \frac{\partial L_2 w_0}{h\,\partial\zeta}\right) + 2L_1 \frac{\partial^2 w_0}{\partial x^2}\right] \varphi_{k+1}^0 \mathrm{d}\zeta =$$

$$= -\frac{2\mu h^3}{\gamma}\sum_{i=1}^{\infty}(-1)^{i-1} h^{2i-2}\left[2A_{i,0,k+1}\frac{\partial^2 \Delta^{i-1} w_0}{\partial x^2} + \right.$$

$$\left. + \frac{2-\gamma}{\gamma-1}\left(A_{i,0,k+1} - B_{i+1,k+1}\right)\Delta^i w_0\right],$$

$$M_{xy}^k = -\frac{2h^2}{\gamma}\frac{8\mu h^3}{3}\int_0^1 \frac{\partial^2 L_1 w_0}{\partial x\,\partial y} \,.\, \varphi_{k+1}^0 \, \mathrm{d}\zeta = \tag{2.8}$$

$$= -\frac{4\mu h^3}{\gamma}\sum_{i=1}^{\infty}(-1)^{i-1} h^{2i-2} A_{i,0,k+1}\frac{\partial^2 \Delta^{i-1} w_0}{\partial x\,\partial y},$$

$$Q_x^k = \frac{2h}{\gamma}\frac{4\mu h^3}{3}\int_0^1 \frac{\partial}{\partial x}\left(L_2 + \frac{\partial L_1}{h\,\partial\zeta}\right) w_0 \,.\, \psi_{k+1}^0 \, \mathrm{d}\zeta = \tag{2.9}$$

$$= -\frac{2\mu h^3}{\gamma}\sum_{i=1}^{\infty}(-1)^{i-1} h^{2i-2} G_{i+1,k+1}\frac{\partial \Delta^i w_0}{\partial x},$$

where the values

$$G_{i,k} = \int_0^1 \left(\frac{d\varphi_i^0}{d\zeta} + \psi_i^0\right)\psi_k^0\, d\zeta \tag{2.10}$$

are also given in table 1.

When we consider that on the edge (Fig. 1) it holds [19]

$$\frac{\partial w_0}{\partial t} = \frac{\partial w_0}{\partial s}, \tag{2.11}$$

$$\frac{\partial^2 w_0}{\partial n\, \partial t} = \frac{\partial^2 w_0}{\partial n\, \partial s} - \frac{\partial \alpha}{\partial s}\frac{\partial w_0}{\partial s}$$

and the operator Δ is given by the expression (1.19), then we can determine the values M_n^k, M_{ns}^k, Q_n^k on the edge by a transformation of the moments and the shearing forces given in rectangular coordinates. We shall give here the three first polymoments and shearing forces up to the order h^4 only

(2.12)

$$M_n^0 = -\frac{8}{3}\mu h^3\left[\frac{\partial^2 w_0}{\partial n^2} + \nu\frac{\partial \alpha}{\partial s}\frac{\partial w_0}{\partial n} + \nu\frac{\partial^2 w_0}{\partial s^2} - \frac{2}{5}\nu h^2\left(\frac{\partial^2 \Delta w_0}{\partial s^2} + \frac{\partial \alpha}{\partial s}\frac{\partial \Delta w_0}{\partial n}\right) - \right.$$
$$\left. - \frac{2}{525}\nu h^4\left(\frac{\partial^2 \Delta^2 w_0}{\partial s^2} + \frac{\partial \alpha}{\partial s}\frac{\partial \Delta^2 w_0}{\partial n}\right) - \ldots\right],$$

$$M_n^1 = \frac{4\mu h^3}{3\gamma}\left[\frac{8}{5}\nu\left(\frac{\partial^2 w_0}{\partial n^2} + \nu\frac{\partial \alpha}{\partial s}\frac{\partial w_0}{\partial n} + \nu\frac{\partial^2 w_0}{\partial s^2}\right) + \frac{4}{525}(\gamma + 2)\, h^2 \Delta^2 w_0 - \right.$$
$$\left. - \frac{h^2}{525}(85\gamma^2 - 332\gamma + 340)\left(\frac{\partial^2 \Delta w_0}{\partial s^2} + \frac{\partial \alpha}{\partial s}\frac{\partial \Delta w_0}{\partial n}\right) - \ldots\right],$$

$$M_n^2 = -\frac{32}{1575}\nu\frac{\mu h^3}{\gamma}\left[\frac{\partial^2 w_0}{\partial n^2} + \nu\frac{\partial \alpha}{\partial s}\frac{\partial w_0}{\partial n} + \nu\frac{\partial^2 w_0}{\partial s^2} + \ldots\right],$$

(2.13)

$$M_{ns}^0 = -\frac{4}{3}\mu h^3\left[\gamma\left(\frac{\partial^2 w_0}{\partial n\,\partial s} - \frac{\partial \alpha}{\partial s}\frac{\partial w_0}{\partial s}\right) + \frac{2}{5}(2 - \gamma)\, h^2\left(\frac{\partial^2 \Delta w_0}{\partial n\,\partial s} - \frac{\partial \alpha}{\partial s}\frac{\partial \Delta w_0}{\partial s}\right) + \right.$$
$$\left. + \frac{2}{525}(2 - \gamma)\, h^4\left(\frac{\partial^2 \Delta^2 w_0}{\partial n\,\partial s} - \frac{\partial \alpha}{\partial s}\frac{\partial \Delta^2 w_0}{\partial s}\right) + \ldots\right],$$

$$M_{ns}^1 = \frac{4\mu h^3}{3\gamma}\left[\frac{2}{5}\gamma(\gamma - 2)\left(\frac{\partial^2 w_0}{\partial n\,\partial s} - \frac{\partial \alpha}{\partial s}\frac{\partial w_0}{\partial s}\right) + \frac{h^2}{525}(85\gamma^2 - 332\gamma + 340)\, . \right.$$
$$\left. . \left(\frac{\partial^2 \Delta w_0}{\partial n\,\partial s} - \frac{\partial \alpha}{\partial s}\frac{\partial \Delta w_0}{\partial s}\right) + \ldots\right],$$

$$M_{ns}^2 = -\frac{4\mu h^3}{3\gamma}\left[\frac{2}{525}\gamma(\gamma - 2)\left(\frac{\partial^2 w_0}{\partial n\,\partial s} - \frac{\partial \alpha}{\partial s}\frac{\partial w_0}{\partial s}\right) + \ldots\right],$$

(2.14)

$$Q_n^0 = -\frac{8}{3}\,\mu h^3 \frac{\partial \Delta w_0}{\partial n},$$

$$Q_n^1 = -\frac{4\mu h^3}{3\gamma}\left[\frac{4}{5}(\gamma + 2)\frac{\partial \Delta w_0}{\partial n} - \frac{4}{525}(\gamma - 2)\,h^2 \frac{\partial \Delta^2 w_0}{\partial n} + \dots\right],$$

$$Q_n^2 = \frac{4\mu h^3}{3\gamma}\left[\frac{4}{525}(\gamma + 2)\frac{\partial \Delta w_0}{\partial n} + \dots\right].$$

When comparing the expressions in the brackets of the equation (2.3) to the equations of the above polymoments and polyforces (2.12) to (2.14), we find that equation (2.3) can be written in the compact form

$$(2.15)\qquad \frac{3\gamma}{4\mu h^3}\int_C\left(- M_n^0 \frac{\partial \delta w_0}{\partial n} + h^2 M_n^1 \frac{\partial \Delta \delta w_0}{\partial n} - h^4 M_n^2 \frac{\partial \Delta^2\, \delta w_0}{\partial n} + \dots -\right.$$

$$- M_{ns}^0 \frac{\partial \delta w_0}{\partial s} + h^2 M_{ns}^1 \frac{\partial \Delta \delta w_0}{\partial s} - h^4 M_{ns}^2 \frac{\partial \Delta^2 \delta w_0}{\partial s} + \dots +$$

$$\left. + Q_n^0\, \delta w_0 - h^2 Q_n^1\, \Delta \delta w_0 + h^4 Q_n^2\, \Delta^2 \delta w_0 - \dots\right) \mathrm{d}s = 0\,.$$

Equation (2.15) can be arranged into the form

$$(2.16)\qquad \int_C \sum_{i=0}^{\infty} (-1)^{i+1}\, h^{2i}\left[M_n^i \frac{\partial \Delta^i\, \delta w_0}{\partial n} - \left(Q_n^i + \frac{\partial M_{ns}^i}{\partial s}\right)\Delta^i\, \delta w_0\right]\mathrm{d}s = 0\,.$$

From this equation we obtain the static boundary conditions in the form representing a generalization of Kirchhoff's conditions

$$(2.17)\qquad M_n^i = 0\,,\quad Q_n^i + \frac{\partial M_{ns}^i}{\partial s} = 0\,,\quad (i = 0, 1, 2, \dots)\,.$$

We can see that apart from fulfilling the ordinary boundary conditions of Kirchhoff, which formally correspond to the case $i = 0$, (of course when neglecting the terms of order h^2 and higher in the equations (2.12 to 2.14)) it is required that all, the i-th bending polymoments and the i-th generalized shearing polyforces, should be equal to zero at the edge. It is obvious that the introduction of one function $w_0(x, y)$ enables to satisfy for all $|\zeta| \leqq 1$ two boundary conditions only and, therefore, on the free edge there remain the nonvanishing stresses τ_{nz} and τ_{ns}. These will represent the boundary effects of the Saint-Venant type and thus the solution corresponds to the fundamental state of stress [20]. However, the above solution is more general than that corresponding to the biharmonic solution only.

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Súhrn

REISSNEROVSKÉ ALGORITMY V SPRESNENÝCH TEÓRIACH OHYBU DOSÁK

ALEXANDER HANUŠKA

V článku sa vychádza z teórie Reissnerovských algoritmov, predložených I. Babuškom a M. Prágerom. Pre špeciálny prípad jednej neznámej funkcie $w_0(x, y)$ sú vyjadrené posuny u, v, w pomocou diferenciálnych operátorov nekonečného rádu rovnicami (1.5). Pri aplikácii Lagrangeovho princípu je treba použiť zobecnené Greenove formuly (1.12) odvodené V. K. Prokopovom. Použitím týchto formúl a integrovaním podľa ζ obdrží sa diferenciálna rovnica nekonečného rádu (1.22). Je ukázané ďalej, že príslušným okrajovým podmienkam je možné dať fyzikálnu interpretáciu zavedením polymomentov a polysíl podľa (2.4—2.6). Obdržané okrajové podmienky (2.17) predstavujú zobecnenie Kirchhoffových okrajových podmienok a tedy uvažovaný prípad jednej funkcie w_0 zodpovedá len základnému stavu napätosti.

Резюме

РЕЙСНЕРОВСКИЕ АЛГОРИФМЫ В УТОЧНЕННЫХ ТЕОРИЯХ ИЗГИБА ПЛАСТИНОК

АЛЕКСАНДЕР ГАНУШКА (ALEXANDER HANUŠKA)

Статья исходит из теории Рейснеровских алгорифмов, предложенных И. Бабушком и М. Прагером. Для специального случая одной неизвестной функции $w_0(x, y)$ выражаются перемещения u, v, w при ппмощи дифференциальных операторов бесконечного порядка уравнениями (1.5). При применении принципа Лагранжа надо пользоваться обобщенными формулами Грина (1.12) выведенными В. К. Прокоповым. Использованием этих формул и интегрированием по ζ получено дифференциальное уравнение бесконечного порядка (1.22). В дальнейшем показано, что соответствующие граничные условия можно интерпретировать посредством введения полимоментов и полисил следуя уравнениям (2.4—2.6). Полученные граничные условия (2.17) представляют обобщение граничных условий Кирхгофа и рассматриваемый случай одной функции соответствует только основному состоянию напряжения.

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