Václav Bartošek; Rostislav Zezula
Stability of flat thermal flux in a slab reactor

Aplikace matematiky, Vol. 13 (1968), No. 5, 367–375

Persistent URL: `http://dml.cz/dmlcz/103183`

STABILITY OF FLAT THERMAL FLUX IN A SLAB REACTOR

VÁCLAV BARTOŠEK and ROSTISLAV ZEZULA

(Received March 17, 1967)

1. GENERAL SOLUTION OF THE EQUATION FOR FLAT THERMAL FLUX

The necessary condition for flat flux, Φ = const., of thermal neutrons in the core of a thermal reactor is given (in a two-group approximation and in slab geometry) by the following equation for the distribution of the relative fuel absorption $M(x) = \sigma_u / \sum_M^a . N_u(x)$, (the notation is the same as in the paper [1]):

(1) $$\Phi \sum_M^a (1 + M) = q_A \; ; \quad M''(x) + g[M(x)] = 0 \, ,$$

where the real function $g(z) \in C^\infty$ has the following properties:

1. $g(z)$ has only two roots $M_0^{(1)}$ and $M_0^{(2)}$.
2. $g'(z)$ has the unique real root $M_0^{(3)}$, $M_0^{(1)} < M_0^{(3)} < M_0^{(2)}$ and we have
$$g'(M_0) > 0 \quad \text{for} \quad M_0 < M_0^{(3)} \; ; \quad g'(M_0) < 0 \quad \text{for} \quad M_0 > M_0^{(3)} \, .$$
3. $G(M) = \int g(M) \, \mathrm{d}M \to \pm \infty$ for $M \to \pm \infty$.

In reactor physics $g(M)$ appears in the form:

(2) $$g(M) = \frac{1}{\tau} \{ [\eta \, p(M) - 1] \, M - 1 \} \, .$$

The dependence $p(M)$ for the heterogeneous fuel arrangement can be expressed by the relation

(3) $$p(M) = \exp(-\gamma M) \, ,$$

where γ depends on the lattice parameters. In a homogeneous reactor, the dependence $p(M)$ is expressed by a more complicated relation, its qualitative behaviour, however, remains the same so that $g(M)$ again has the three above mentioned properties.

The physical requirement of symmetry of the solution from which the initial condition

(4) $$M'(x) = 0$$

follows does not restrict the generality of the following analyses. Full generality will be reached by investigating the solution of the Eq. (1) corresponding to the initial values (the Cauchy's problem):

$$M'(x_0) = 0\,; \quad M(x_0) = M_0\,; \quad -\infty < M_0 < +\infty\,. \tag{5}$$

According to [2] we rewrite the differential equation (1) in the usual form of the system

$$\frac{\mathrm{d}M}{\mathrm{d}x} \equiv M'(x) = y(M)\,; \quad \frac{\mathrm{d}y}{\mathrm{d}x} \equiv y' = -\,g(M)\,, \tag{6}$$

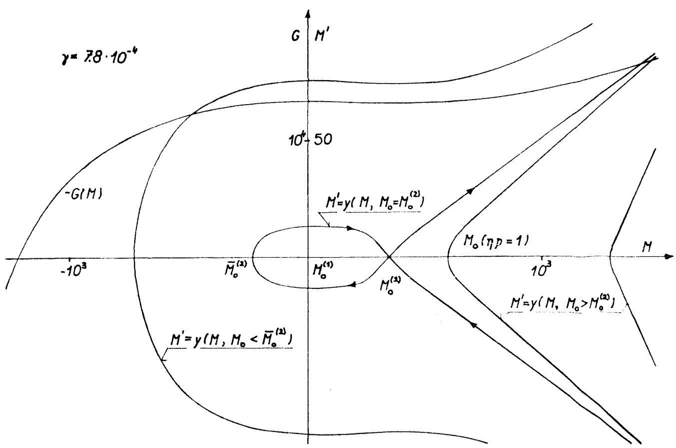

Fig. 1.

which is obviously autonomous. Graphs of the functions $M' = y(M)$ can be then plotted in the phase plane (M, M') (Fig. 1) in the following way:

Substituting the relation (3) into (2) we obtain by integration:

$$G(M) = \int g(M)\,\mathrm{d}M = \frac{-1}{\tau}\left\{M + \frac{M^2}{2} + \frac{\eta}{\gamma}\left[M + \frac{1}{\gamma}\right]\exp\left(-\gamma M\right)\right\}. \tag{7}$$

The first integral of the equation (1) has thus the form

$$\frac{(M')^2}{2} + G(M) = C\,, \tag{8}$$

where C denotes the integration constant. The equation (8) represents in the phase plane (M, M') a one-parametric system of curves $M' = y(M; C)$ which are symmetric with respect to the axis M. The solution $M(x) \equiv M(x, M_0)$ of the Cauchy's problem (1), (5) is given implicitly by the relation

$$x - x_0 = -\int_{M_0}^{M(x,M_0)} \frac{\mathrm{d}M}{|y(M, C)|}\,. \tag{9}$$

From the initial condition (4) in the Cauchy's problem (1), (5) it follows that the integration constant C is a function of the initial value M_0: $C = G(M_0)$.

Thus the curves

$$(10) \qquad M' = y(M; M_0) = \pm\sqrt{\{2[G(M_0) - G(M)]\}}\ ; \quad G(M_0) - G(M) \geqq 0$$

in the phase plane (M, M') determine the solution (9) of the Cauchy's problem (1), (5). The upper (lower) part of the curves represents the solution in the corresponding (right or left) half-part of the slab reactor.

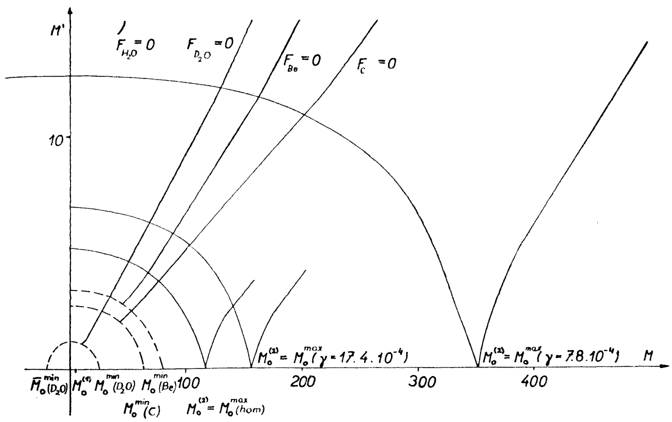

Fig. 2.

The two roots $M_0^{(1)}$ and $M_0^{(2)}$ of the equation

$$(11) \qquad y' = -g(M) \equiv \frac{-1}{\tau}\{[\eta\, p(M) - 1]\, M - 1\} = 0$$

determine in the phase plane (M, M') two singular points $(M_0^{(1)}, 0)$, $(M_0^{(2)}, 0)$, $0 < M_0^{(1)} < M_0^{(2)}$ of the system of equations (6). The singular point $(M_0^{(1)}, 0)$ is a centre-point, the point $(M_0^{(2)}, 0)$ is a saddle-point as it can be seen from the shape of the graphs in Fig. 1.

According to the qualitative theory of differential equations [3], [4], the following conclusions can be made for all functions with the properties 1.–3.:

1. The initial values $M_0 > M_0^{(2)}$ on the axis M are nonsingular points of the system of equations (6) and to every of them there corresponds a unique nonperiodic monotonously increasing (for $x > x_0$) resp. monotonously decreasing (for $x < x_0$) real solution of the Cauchy's problem (1), (5).

2. The initial values $M_0 \neq M_0^{(1)}$ from the interval $\bar{M}_0^{(2)} < M_0 < M_0^{(2)}$ (Fig. 1) on the axis M are nonsingular points of the system of equations (6) and to every of them there corresponds a unique periodic solution of the Cauchy's problem (1), (5)

which initially decreases for $M_0 > M_0^{(1)}$ (increases for $M_0 < M_0^{(1)}$) and is represented in the phase plane (M, M') by the oval given by Eq. (10) which passes through the points $(M_0, 0)$ and $(\overline{M}_0, 0)$. These solutions represent a (conservative) space oscillation with the amplitude M_0 and with a space period $\pi(M_0)$ given by:

(12a) $$\tfrac{1}{2}\pi(M_0) = -\int_{M_0}^{\overline{M}_0} \frac{dM}{|y(M; M_0)|}$$

so that $x_0 = \pm\, n\, \pi(M_0)$ $(n = 0, 1, 2, 3, \ldots)$.

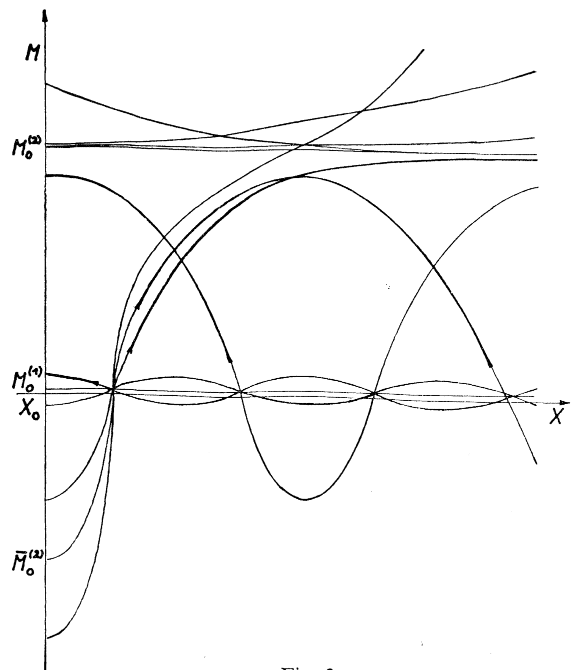

Fig. 3.

Therefore the solutions with the initial conditions $(M_0, 0)$ and $(\overline{M}_0, 0)$ are only mutually shifted by $(\pm\, n + \frac{1}{2})\, \pi(M_0)$ in the (M, x) plane (see Fig. 3).

3. Through the singular points $(M_0 = M_0^{(1)}, 0)$ resp. $(M_0 = M_0^{(2)}, 0)$ pass the singular "trivial" solutions of the Cauchy's problem

(12b) $$M(x) = M_0^{(1)} = \text{const}\,; \quad \text{resp.} \quad M(x) = M_0^{(2)} = \text{const}\,.$$

from which the first one is stable, the second one unstable. These trivial solutions are the unique solutions which assume the initial values $M = M_0^{(2)}$, $M' = 0$ for finite values x_0, and they are the limits (on an arbitrarily chosen finite interval $\langle x_0 - N, x_0 + N\rangle$) of the periodic solutions as $M_0 \to M_0^{(1)}$ resp. $M_0 \to M_0^{(2)}$. Moreover, there are further solutions which assume the values $M_0^{(2)}$, $M' = 0$ as $x_0 \to \pm \infty$ (see Fig. 3), from which one takes on the values $\overline{M}_0^{(2)}$, $M' = 0$ for finite $x = x_0$, is nonperiodic, nonmonotonous and corresponds to the closed part of the separatrix-curve $M'(M, M_0^{(2)})$ in the phase plane. The others take on the values $M_0 > M_0^{(2)}$, $M' = \pm |M'(M_0, M_0^{(2)})|$ (for finite x), are monotonously increasing resp. decreasing and correspond to the two monotonous branches of the separatrix.

4. To the initial values $M_0 < \overline{M}_0^{(2)}$ on the axis M there corresponds the unique nonperiodic solution with a nonmonotonous second derivative (see Fig. 3).

2. PARTICULAR SOLUTIONS FOR THE PHYSICALLY REALIZABLE BOUNDARY CONDITIONS

From the physical meaning of M as the relative absorption in fuel the requirement $M(x, M_0) > 0$ follows for physically realizable part of the solution, independently of the boundary conditions for the investigated problem. It is as well evident from

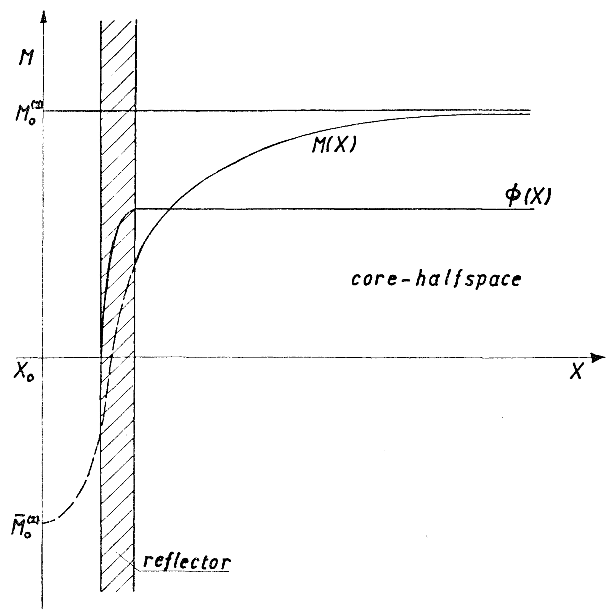

Fig. 4.

the physical reasons that the singular "asymptotic" solutions corresponding to the singular points $(M_0^{(1)}, 0)$ and $(M_0^{(2)}, 0)$ describe the critical state of the core of infinite dimensions so that they may fulfil only the trivial boundary conditions, i.e. finite value of the flux and slowing-down density and their derivatives in the infinity.

Solutions corresponding to the initial value $M_0 > M_0^{(2)}$, resp. $M_0 < \overline{M}_0^{(2)}$ are monotonously increasing for $x > x_0$ and can obey e.g. the boundary conditions which describe neutron sources on the boundary. For $M_0 \leqq \overline{M}_0^{(2)} < 0$ the reactor must obviously have an inner reflector (see Fig. 4).

The finite critical core could appear in the interval of the initial values $M_0^{(2)} > M_0(\mp M_0^{(1)}) > \overline{M}_0^{(2)}$, for which the usual conditions of continuity on the interface of the core and reflector could be fulfilled as well as the conditions of zero value of the flux and slowing-down density on the extrapolated boundary of the reflector. The mentioned conditions of continuity on the interface b of the core (A) and the reflector (R) have the form (for reactor with a flattened flux):

$$\text{(13a)} \qquad \Phi_R(b) = \Phi_A(b) = \Phi\,; \quad \Phi'_R(b) = 0\,,$$

$$\text{(13b)} \quad q_R(b) = \zeta\,.\, q_A(b, M_0)\,; \quad q'_R(b) = q'_A(b, M_0)\,\tau_A/\tau_R\,; \quad (\zeta = (\xi\textstyle\sum_s)_R/(\xi\sum_s)_A)\,.$$

From the boundary conditions (13a), (13b) we determine four constants in the well known general solution of the usual two-group equations in the reflector.

The conditions on the external (extrapolated) boundary of the reflector a:

$$\text{(14)} \qquad \Phi_R(\pm a) = 0\,; \quad q_R(\pm a) = 0$$

determine the relations among the critical core size b, the thickness of reflector $t = a - b$ and the initial relative fuel absorption M_0. From the conditions (14) we obtain a linear system for $M(b, M_0)$ and $M' = y[M(b, M_0), M_0]$ which has the following solution:

$$\text{(15a)} \qquad M(b, M_0) = \frac{1}{\zeta}\frac{\sum_R^a}{\sum_M^a}\,\frac{(1 - L_R^2/\tau_R)}{1 - \dfrac{L_R}{\sqrt{\tau_R}}\,\dfrac{\tanh t/L_R}{\tanh t/\sqrt{\tau_R}}} - 1 \equiv \vartheta_1(t)\,,$$

$$\text{(15b)} \quad M' = y[M(b, M_0), M_0] = \frac{-\zeta}{\sqrt{\tau_R}}\,\frac{\tau_R}{\tau_A}\,\frac{1}{\tanh t/\sqrt{\tau_R}}\,[M(b, M_0) + 1] \equiv \vartheta_2(t)\,.$$

The relations (15a), (15b) give us a parametric representation of a "critical" curve in the phase plane (M, M') and determine two of the three values M_0, b, t.

Eliminating the thickness of the reflector t from the equations (15a,b) we obtain the equation of the "critical" curve in the form of an implicit dependence between the values $M(b, M_0)$ and $M' = y[M(b, M_0), M_0]$ which we denote as $F\{M(b, M_0), M' = y[M(b, M_0), M_0]\} = 0$. On the other hand, mutual dependence of M and M' in the core is given by the first integral (8) of the Equation (1) which forms an oval for $\overline{M}_0^{(2)} < M_0(\mp M_0^{(1)}) < M_0^{(2)}$. The intersection of the oval with the curve $F = 0$ determines the points

$$\pm b \pm n\,\pi(M_0) \text{ for } M_0^{(2)} > M_0 > M_0^{(1)} \text{ resp. } \pi(M_0)\,(\pm n + \tfrac{1}{2}) \mp b \text{ for } \overline{M}_0^{(2)} < M_0 < M_0^{(1)}$$
$$(n = 0, 1, 2, \ldots)\,,$$

where the conditions of continuity of fluxes and normal currents (13a,b) on the interface of the core and reflector in a critical reactor with a flattened thermal flux in the core are fulfilled. By choosing M_0, the intersection point $M = M[t(M_0)]$; $M' = y[t(M_0)]$, of the oval going through M_0 with the curve $F = 0$ in phase plane is given so that the reflector thickness $t = t(M_0)$ can be calculated from (15b). The halfthickness $b = b(M_0)$ of the critical core can be then determined from (9) and (12a).

The form of the curve $F = 0$ depends on the physical qualities of the core and reflector and is plotted in Fig. 2 for the core moderated by D_2O with the reflector formed by H_2O, D_2O, C, Be. The improper integral (9) exists (for $M_0 \neq M_0^{(2)}, M_0^{(1)}$) because the integrand has the singularity of the type $1/\sqrt{Z}$.

In the phase plane the function $F = 0$ attains values $M = \vartheta_1(\infty)$, $M' = \vartheta_2(\infty)$ for $t = \infty$ to which, as it is evident from Fig. 2, there corresponds, according to (10), the minimal initial value of the relative absorption density $M_0^{\min} \geqq M_0^{(1)}$ uniquely determined by the inequality $M_0^{(1)} < M_0^{\min} < M_0^{(2)}$ and by the relation (10) from which it follows that

$$G(M_0^{\min}) = \tfrac{1}{2}[\vartheta_2(\infty)]^2 + G[\vartheta_1(\infty)]\,; \tag{16}$$

then (9) yields the critical dimension of the core $b_{\min}$. For initial values $\overline{M}_0^{\min} < M_0 < M_0^{\min}$ the boundary conditions (13a,b) and (14) obviously cannot be fulfilled. The point of intersection of the curve $F = 0$ with the separatrix-oval (10), corresponding to the initial value $M_0^{(2)}$ corresponds to the minimal thickness of the reflector

$$t_{\min} = \sqrt{\tau_R}\,\operatorname{artanh}\left[\frac{1}{\sqrt{\tau_R}}\frac{\tau_R}{\tau_A}\,\zeta\,\frac{M[t(M_0^{(2)})] + 1}{y[t(M_0^{(2)})]}\right] \tag{17}$$

for which the boundary conditions are fulfilled for the finite value of the coordinate $x - x_0$ (see Fig. 4):

$$x - x_0 = \tfrac{1}{2}\pi(M_0^{(2)}) - b_{\max} = \int_{M_0^{(2)}}^{M[t(M_0^{(2)})]} \frac{\mathrm{d}M}{|y(M, M_0^{(2)})|} \tag{18}$$

while the half-period

$$\frac{\pi}{2}(M_0^{(2)}) = \lim_{M_0 \to M_0^{(2)}-} \tfrac{1}{2}\pi(M_0) = +\infty \tag{19a}$$

and the core halfthickness

$$b_{\max} = \lim_{M_0 \to M_0^{(2)}-} b(M_0) = +\infty \tag{19b}$$

so that in this case the reactor core has the form of a halfspace $(\tfrac{1}{2}\pi^{(M_0^{(2)})} - b_{\max}, \infty)$ [5]; the corresponding reflector lies in the interval $(\tfrac{1}{2}\pi^{(M_0^{(2)})} - b_{\max} - t_{\min}, \tfrac{1}{2}\pi^{(M_0^{(2)})} - b_{\max})$. The fuel distribution in the halfspace-core is given implicitly by the formula

$$(\tfrac{1}{2}\pi(M_0^{(2)}) - b_{\max}) + x - x_0 = \int_{M[t(M_0^{(2)})]}^{M(x, M_0^{(2)})} \frac{\mathrm{d}M}{|y(M, M_0^{(2)})|}\,. \tag{20}$$

Moreover, it is clear that $M_0^{(2)} = M_0^{\max}$ is the maximal physically allowable initial fuel concentration, in the critical core with flattened thermal flux.

Results of the calculations of $t_{\min}$ (by (17)), $M_0^{\min}$ (by (16)), and $M_0^{\max} = M_0^{(2)}$ (by (11)) are given in Table 1. Values of the concentrations $M = M_0^{\min}$, $M = M_0^{\max}$ are normalized, as usual [1], with the help of the relation $N_U = (\sum_M^a/\sigma_U)\, M$.

Table 1

$N_u = 0{\cdot}0472 \,.\, 10^{24}$ atoms/cm^3 (for pure natural uranium)

γ	reflector	D_2O	Be	C
$7{\cdot}8 \,.\, 10^{-4}$	$t_{\min}$ [cm] $N_u^{\max}(\gamma)$ $N_u^{\min}$	20,0 $0{\cdot}00455 \,.\, 10^{24}$ $0{\cdot}000366 \,.\, 10^{24}$	12,1 $0{\cdot}00455 \,.\, 10^{24}$ $0{\cdot}001035 \,.\, 10^{24}$	25·1 $0{\cdot}00455 \,.\, 10^{24}$ $0{\cdot}000827 \,.\, 10^{24}$
$17{\cdot}4 \,.\, 10^{-4}$	$t_{\min}$ [cm] $N_u^{\max}(\gamma)$ $N_u^{\min}$	35·0 $0{\cdot}00201 \,.\, 10^{24}$ $0{\cdot}000366 \,.\, 10^{24}$	19·1 $0{\cdot}00201 \,.\, 10^{24}$ $0{\cdot}001035 \,.\, 10^{24}$	33·0 $0{\cdot}00201 \,.\, 10^{24}$ $0{\cdot}000827 \,.\, 10^{24}$
homog.	$t_{\min}$ [cm] $N_u^{\max}(\gamma)$ $N_u^{\min}$	53·5 $0{\cdot}00150 \,.\, 10^{24}$ $0{\cdot}000366 \,.\, 10^{24}$	26,0 $0{\cdot}00150 \,.\, 10^{24}$ $0{\cdot}001035 \,.\, 10^{24}$	49,5 $0{\cdot}00150 \,.\, 10^{24}$ $0{\cdot}000827 \,.\, 10^{24}$

Globally, it can be stated that for the initial fuel concentration M_0 from the interval $M_0^{\min} \leqq M_0 < M_0^{\max}$ the Cauchy's problem (1), (5), (13a,b), together with the relations (10) and (15a,b) is equivalent with the problem of eigenvalues for the same equation as considered in (Bartošek and Zezula, [1]) and describes physically the critical reactor with the thickness $n\,\pi(M_0) - b(M_0)$, $n\,\pi(M_0) + b(M_0)$, $(n = 0, 1, 2, \ldots)$ of the core with the flattened flux of the thermal neutrons and with the thickness of reflector $t(M_0)$.

Thus, this means that in contrast to the reactor with the constantly distributed fuel, where to the given multiplication coefficient of the core there corresponds an infinite number of pairs of the dimension of the core and reflector, in the case of the reactor with a flattened flux, just as a consequence of the requirement of flattening of thermal flux, to the given initial fuel concentration M_0 (and thus also to the determined initial multiplication coefficient) there corresponds only one pair of the dimension of the core and reflector.

Acknowledgement: The authors would like to express their thanks to M. Hron, V. Lelek, I. Marek, J. Roček and Z. Vorel for valuable discussions.

References

[1] *V. Bartošek* and *R. Zezula:* Flat Flux in a Slab Reactor with Natural Uranium. Journal of Nucl. Energy Pt A/B *20*, 129—134, (1966).
[2] *E. F. Beckenbach:* Modern Mathematics for the Engineer. McGraw-Hill Book Comp. New York — Toronto — London.
[3] *S. Lefshetz:* Differential Equations: Geometric Theory. Interscience Publishers, London 1957.
[4] *R. Bellman:* Stability Theory of Differential Equations. London 1953.
[5] *J. Roček*: Private communication, 1966.

Souhrn

STABILITA VYROVNANÉHO TOKU TEPELNÝCH NEUTRONŮ V DESKOVÉM REAKTORU

VÁCLAV BARTOŠEK a ROSTISLAV ZEZULA

S použitím kvalitativní teorie diferenciálních rovnic dokazují autoři ekvivalenci okrajové úlohy a Cauchyho úlohy pro deskový reaktor. Je dokázána existence intervalu počátečních (tj. pro $x = 0$) koncentrací paliva, pro které Cauchyho problém (se dvěma algebraickými podmínkami) popisuje kritický deskový reaktor, zatímco řešení mimo tento interval popisují podkritické systémy nebo nenásobící prostředí.

Резюме

УСТОЙЧИВОСТЬ ПОСТОЯННОГО ПОТОКА ТЕПЛОВЫХ НЕЙТРОНОВ В ДОСКОВОМ РЕАКТОРЕ

Васлав Бартошек и Ростислав Зезула (VÁCLAV BARTOŠEK a ROSTISLAV ZEZULA)

Используя качественную теорию дифференциальных уравнений показана эквивалентность краевой задачи и задачи Коши для начальных значений в случае доскового реактора с постоянным потоком тепловых нейтронов. Показано, что существует интервал начальных (т. е. центральных) концентраций горючего, для которых задача о начальных значениях (вместе с двумя алгебраическими условиями) описывает критический реактор с уравновешенным (стабилизированным) током, в то время как решение лежащее во внешности этого промежутка описывает подкритические системы, или неумножающую среду.

Authors' addresses: Dr. *Václav Bartošek* CSc., Ústav jaderného výzkumu ČSAV, Řež u Prahy. Dr. Ing. *Rostislav Zezula* CSc., Matematický ústav Karlovy university, Praha 8, Sokolovská 83.