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ON THE ESTIMATE OF THE ERROR OF QUADRATURE FORMULAE

MILOŠ ZLÁMAL

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Let $f(x)$ be a function defined on the interval $\langle 0, 1 \rangle$ and let

$$I(f) = \sum_{k=1}^{m} c_k f(x_k) \quad (0 \leqq x_1 < x_2 < \ldots < x_m \leqq 1) \tag{1}$$

be any quadrature formula which is exact for every polynomial of degree less or equal n, $n \leqq 2m - 1$.[1]) Our aim is to investigate the error

$$E(f) = \int_0^1 f(x)\,\mathrm{d}x - I(f) \tag{2}$$

in the class $W_2^{(r)}$ with

$$r \leqq n + 1\,. \tag{3}$$

By $W_2^{(r)}$ we denote the class of functions $f(x)$ such that $f^{(r-1)}(x)$ is absolutely continuous on the interval $\langle 0, 1 \rangle$ and $f^{(r)}(x) \in L_2(0, 1)$. We will find an explicit expression for

$$E^* = \sup_{\substack{f \in W_2^{(r)} \\ \|f^{(r)}\|_{L_2} \leqq 1}} |E(f)| \tag{4}$$

so that for the error $E(f)$ it holds the following exact estimate

$$|E(f)| \leqq E^* \|f^{(r)}\|_{L_2}\,. \tag{5}$$

Let $P_{r-1}(x)$ be a polynomial of degree $r - 1$ such that

$$P_{r-1}(0) = f(0)\,,\ P'_{r-1}(1) = f'(1)\,,\ P''_{r-1}(0) = f''(0)\,,\ \ldots,$$
$$P^{(r-1)}_{r-1}(0) = f^{(r-1)}(0) \text{ if } r \text{ is odd}\,,$$
$$P^{(r-1)}_{r-1}(1) = f^{(r-1)}(1) \text{ if } r \text{ is even}\,.$$

[1]) The necessary and sufficient condition is $\sum_{k=1}^{m} c_k x_k^i = (1/i + 1)$, $i = 0, 1, \ldots, n$.

As $r - 1 \leqq n$ we have $E(P_{r-1}) = 0$ and if we set $f(x) - P_{r-1}(x) = \varphi(x)$ it holds $E(f) = E(\varphi)$. The function $\varphi(x)$ satisfies the boundary conditions

$$(6) \qquad 0 = \varphi(0)\,,\; 0 = \varphi'(1)\,,\; 0 = \varphi''(0)\,,\; \ldots\,,\; 0 = \begin{cases} \varphi^{(r-1)}(0) \text{ if } r \text{ is odd}\,, \\ \varphi^{(r-1)}(1) \text{ if } r \text{ is even}\,. \end{cases}$$

Now let us denote by ${}^0W_2^{(r)}$ the space of functions from $W_2^{(r)}$ which satisfy the boundary conditions (6). The expression

$$(\varphi, \psi) = \int_0^1 \varphi^{(r)}(x) \,.\, \psi^{(r)}(x)\,\mathrm{d}x$$

is a scalar product and ${}^0W_2^{(r)}$ with the norm

$$\|\varphi\| = (\varphi, \varphi)^{1/2} = \left\{\int_0^1 [\varphi^{(r)}(x)]^2\,\mathrm{d}x\right\}^{1/2}$$

is a separable Hilbert space. The functional $E(\varphi)$ is linear in ${}^0W_2^{(r)}$ so that we have $|E(\varphi)| \leqq E \,.\, \|\varphi\|$ where E is the norm of the functional $E(\varphi)$. As $E(\varphi) = E(f)$ and $\|\varphi\| = \|f^{(r)}\|_{L_2}$ we have $|E(f)| \leqq E\,\|f^{(r)}\|_{L_2}$ and obviously

$$E^* = E\,.$$

To find E let us consider the sequence

$$\varphi_j(x) = \sqrt{2}\left(\frac{2}{\pi}\right)^r \frac{1}{(2j-1)^r} \sin(2j-1)\frac{\pi}{2}x\,, \quad j = 1, 2, \ldots\,.$$

This sequence is complete and orthonormal in ${}^0W_2^{(r)}$. For this reason it holds the well known formula

$$E^{*2} = E^2 = \sum_{j=1}^{\infty} [E(\varphi_j)]^2\,.$$

Now it is sufficient to compute $E(\varphi_j)$ and to sum. We have

$$E(\varphi_j) = \sqrt{2}\left(\frac{2}{\pi}\right)^r \frac{1}{(2j-1)^{2r}}\left[\frac{2}{\pi}\frac{1}{2j-1} - \sum_{k=1}^{m} c_k \sin(2j-1)\pi\frac{x_k}{2}\right]$$

and

$$(E^*)^2 = 2\left(\frac{2}{\pi}\right)^{2r} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2r}}\left[\frac{2}{\pi}\frac{1}{2j-1} - \sum_{k=1}^{m} c_k \sin(2j-1)\pi\frac{x_k}{2}\right]^2 =$$

$$= 2\left(\frac{2}{\pi}\right)^{2r}\left\{\frac{4}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^{2r+2}} - \frac{4}{\pi}\sum_{k=1}^{m} c_k \sum_{j=1}^{\infty}\frac{\sin(2j-1)\pi x_k/2}{(2j-1)^{2r+1}} + \right.$$

$$+ \sum_{k,l=1}^{m} c_k c_l \sum_{j=1}^{\infty}\frac{1}{2}\frac{1}{(2j-1)^{2r}}\left[\cos(2j-1)\pi\tfrac{1}{2}(x_k - x_l) - \right.$$

$$\left.\left. - \cos(2j-1)\pi\tfrac{1}{2}(x_k + x_l)\right]\right\}.$$

If we use Euler polynomials $E_r(x)$ (see [1], pp. 23–29 and 66) the final result can be written in this way:

$$(7)\qquad (E^*)^2 = (-1)^r \frac{2^{2r-2}}{(2r-1)!} \left\{ \sum_{k,l=1}^{m} c_k c_l [E_{2r-1}(\tfrac{1}{2}|x_k - x_l|) - E_{2r-1}(\tfrac{1}{2}(x_k + x_l))] - \right.$$
$$\left. - \frac{4}{r} \sum_{k=1}^{m} c_k \, E_{2r}(\tfrac{1}{2}x_k) - \frac{4}{r(2r+1)} E_{2r+1}(0) \right\}.$$

We do not apply (7) to known quadrature formulae to get numerical results as the reader can find these values in [2] where they were computed in a special way. Special cases of the formula (7) ($r = 1, 2, 3$) are proved in [3].

Bibliography

[1] *N. E. Nörlund:* Vorlesungen über Differenzenrechnung. Berlin 1944.

[2] *С. М. Никольский:* Квадратурные формулы (Quadrature Formulae). Москва 1958.

[3] *В. Ф. Кузютин:* Об оценке ошибки квадратурной формулы (On the Estimate of the Error of Quadrature Formulae). Методы вычислений, выпуск II, Ленинград 1963.

Souhrn

O ODHADU CHYBY KVADRATURNÍCH FORMULÍ

MILOŠ ZLÁMAL

Nechť $f(x)$ je funkce definovaná na intervalu $\langle 0, 1 \rangle$ a

$$I(f) = \sum_{k=1}^{m} c_k f(x_k), \quad 0 \leqq x_1 < x_2 < \ldots < x_m \leqq 1$$

libovolná kvadraturní formule, která je přesná pro každý polynom stupně menšího nebo rovného n, $n \leqq 2m - 1$. Cílem článku je vyšetřit chybu

$$E(f) = \int_0^1 f(x) \, dx - I(f)$$

ve třídě funkcí $W_2^{(r)}$ ($r \leqq n + 1$). Pod $W_2^{(r)}$ rozumíme třídu funkcí $f(x)$ takových, že $f^{(r-1)}(x)$ je absolutně spojitá na intervalu $\langle 0, 1 \rangle$ a $f^{(r)}(x) \in L_2(0, 1)$. Je nalezen explicitní výraz (7) pro

$$E^* = \sup_{\substack{f \in W_2^{(r)} \\ \|f^{(r)}\|_{L_2} \leqq 1}} |E(f)|,$$

takže pro chybu $E(f)$ platí přesný odhad

$$|E(f)| \leqq E^* \, . \, \|f^{(r)}\|_{L_2} \, .$$

Резюме

ОБ ОЦЕНКЕ ОШИБКИ КВАДРАТУРНЫХ ФОРМУЛ

МИЛОШ ЗЛАМАЛ (Miloš Zlámal)

Пусть $f(x)$ – функция, определенная на промежутке $\langle 0, 1 \rangle$ и

$$I(f) = \sum_{k=1}^{m} c_k f(x_k), \quad 0 \leqq x_1 < x_2 < \ldots < x_m \leqq 1$$

произвольная квадратурная формула, которая является точной для каждого многочлена степени меньшей или равной n, $n \leqq 2m - 1$. Цель работы – исследовать ошибку

$$E(f) = \int_0^1 f(x)\,\mathrm{d}x - I(f)$$

в классе функций $W_2^{(r)}$ $(r \leqq n + 1)$. Под $W_2^{(r)}$ мы подразумеваем класс функций $f(x)$ таких, что $f^{(r-1)}(x)$ абсолютно непрерывна в промежутке $\langle 0, 1 \rangle$ и $f^{(r)}(x) \in$ $\in L_2(0, 1)$. Найдено явное выражение (7) для

$$E^* = \sup_{\substack{f \in W_2^{(r)} \\ \|f^{(r)}\|_{L_2} \leqq 1}} |E(f)|\,,$$

так что для ошибки $E(f)$ имеет место точная оценка

$$|E(f)| \leqq E^* \, \|f^{(r)}\|_{L_2}\,.$$

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