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NATURAL DYNAMICAL CONNECTIONS

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1. INTRODUCTION

This paper is a continuation of the author's previous works [11], [10], [12] which try to generalize the well-known results concerning the properties and the role of various connections in the autonomous mechanics of higher-order on T^rM ([5], [2], [3], [6]) or in the non autonomous mechanics of the first order on $R \times TM$ ([4], [1]). Our approach was introduced for the time-dependent higher-order mechanics on general fibred manifolds with one-dimensional base.

Making use of the identification of the semispray distribution of type $(r - 1)$ on $J^r\pi$ with the connection of order $(r + 1)$ on π we have proved in [11] the existence and uniqueness of the so-called *characteristic* (*Euler-Lagrange*) *connection* on π whose paths are just the extremals of the given regular lagrangian or, more generally, of regular equations. The paper [10] is devoted to the description of the conditions for connections on $\pi_{r,r-1}$ to be associated to the connection mentioned above, i.e. to have the same paths. These results made it possible to give another geometrical characterization of the regular equations through the so-called *strong* and *weak horizontal distributions*.

In this paper we show the whole class of the connections on $\pi_{r,r-1}$ (and of the corresponding $f(3, -1)$ structures on $J^r\pi$) canonically associated to the given connection of order $(r + 1)$ on π as a generalization of the corresponding objects on $R \times TM$ (see [12]). As is to be expected, all structures are intrinsically related to the geometry of underlying jet bundles, more precisely to the special class of *natural affinors* (see [7] for $R \times T^rM$ and [8] for $J^r\pi$), consequently they are generated by the volume forms on the base of the fibred manifold.

The structure of this paper is as follows. In Sec. 2 we introduce the notation used. Sec. 3 sets up the known basic notions and the results of [11], [10] necessary for Sec. 4, where we present the new results. For the sake of brevity we restrict our exposition to the connections, their relation to the higher-order mechanics can be found in [11] and [10].

2. NOTATION

Throughout the paper, (Y, π, X) is a fibred manifold with $\dim X = 1$, $\dim Y = 1 + m$; $(J^r\pi, \pi_{r,s}, J^s\pi)$ and $(J^r\pi, \pi_r, X)$ are the obvious jet bundles induced by π, $J^0\pi = Y$, respectively. By (V, ψ), $\psi = (t, q^\sigma)$ we mean the fibre coordinates on $V \subset Y$, $\psi_r = (t, q^\sigma, q^\sigma_{(1)}, \ldots, q^\sigma_{(r)})$ are the adapted coordinates on $\pi_{r,0}^{-1}(V) \subset J^r\pi$, i.e.

$$q^\sigma_{(k)} = \frac{d^k q^\sigma}{dt^k}.$$

$V_{\pi_{r,s}}(J^r\pi)$ and $V_{\pi_r}(J^r\pi)$ are the $\pi_{r,s}$-vertical and π_r-vertical subbundles of $TJ^r\pi$, respectively. $S_U(\pi)$ is a module of local sections of π on U while $\mathscr{F}(U)$ is a module of local real functions on U. $J^r\gamma\colon U \to J^r\pi$ denotes the r-jet prolongation of γ and $(d/dt)\, J^r\gamma$ means the curve of tangent vectors to $J^r\gamma$. The Lie derivative of a $(1, 1)$ tensor field S with respect to ζ is denoted by $\partial_\zeta S$. Finally, all structures and mappings are supposed smooth and the summation convention is used.

3. VARIOUS CONNECTIONS AND RELATED STRUCTURES

A *connection of order* $(r + 1)$ *on* π, $r \geqq 1$, is a section

$$\Gamma\colon J^r\pi \to J^{r+1}\pi$$

of the bundle $\pi_{r+1,r}$. Using a canonical bundle imbedding $J^{r+1}\pi \hookrightarrow J^1\pi_r$ we can consider Γ as a connection on π_r. Owing to this fact the *horizontal form* of Γ is

$$h_\Gamma = \left(\frac{\partial}{\partial t} + \sum_{j=0}^{r-1} q^\sigma_{(j+1)} \frac{\partial}{\partial q^\sigma_{(j)}} + \Gamma^\sigma_{(r+1)} \frac{\partial}{\partial q^\sigma_{(r)}}\right) \otimes dt,$$

where $\Gamma^\sigma_{(r+1)} \in \mathscr{F}(J^r\pi)$ are the *components* of Γ. The dual notion to h_Γ is a *vertical form* of Γ, given by

$$v_\Gamma = I - h_\Gamma,$$

where $I = I_{TJ^r\pi}$ is the identity endomorphism. Consequently,

$$v_\Gamma = \sum_{j=0}^{r-1} \frac{\partial}{\partial q^\sigma_{(j)}} \otimes (dq^\sigma_{(j)} - q^\sigma_{(j+1)}\, dt) + \frac{\partial}{\partial q^\sigma_{(r)}} \otimes (dq^\sigma_{(r)} - \Gamma^\sigma_{(r+1)}\, dt).$$

Hence the one-dimensional π_r-horizontal distribution $\operatorname{Im} h_\Gamma = \ker v_\Gamma$ is just the semispray distribution $\Delta^r_{r-1}[\Gamma]$ generated locally by semisprays of type $(r - 1)$ on $J^r\pi$. Thus Γ yields the decomposition

$$TJ^r\pi = V_{\pi_r}J^r\pi \oplus \Delta^r_{r-1}[\Gamma].$$

The set of such connections is denoted by $\boldsymbol{\Gamma}_{r+1,r}$. A section $\gamma \in S_U(\pi)$ is called *a path* of $\Gamma \in \boldsymbol{\Gamma}_{r+1,r}$ if

$$J^{r+1}\gamma = \Gamma \circ J^r\gamma$$

on U. It turns out that γ is a path of Γ if and only if $J^r\gamma$ is an integral mapping of $\Delta^r_{r-1}[\Gamma]$.

By a *dynamical connection* on $J^r\pi$ we mean a connection Γ_d on $\pi_{r,r-1}$, i.e. a section

$$\Gamma_d\colon J^r\pi \to J^1\pi_{r,r-1}\,.$$

The *horizontal form* of Γ_d is locally given by

$$h_{\Gamma_d} = \left(\frac{\partial}{\partial t} + \Gamma^\sigma_{(r,0)} \frac{\partial}{\partial q^\sigma_{(r)}}\right) \otimes \mathrm{d}t + $$

$$+ \sum_{j=0}^{r-1} \left(\frac{\partial}{\partial q^\sigma_{(j)}} \otimes \mathrm{d}q^\sigma_{(j)} + \Gamma^\sigma_{(r,j)\lambda} \frac{\partial}{\partial q^\sigma_{(r)}} \otimes \mathrm{d}q^\lambda_{(j)}\right),$$

where $\Gamma^\sigma_{(r,0)}, \Gamma^\sigma_{(r,k)\lambda} \in \mathscr{F}(J^r\pi)$, $0 \leqq k \leqq r-1$, are the *components* of Γ_d. Consequently, Γ_d can be identified with the $(rm+1)$-dimensional $\pi_{r,r-1}$-horizontal distribution $H_{\Gamma_d} = \operatorname{Im} h_{\Gamma_d}$. A section $\gamma \in S_U(\pi)$ is called a *(dynamical) path of* Γ_d if

$$\frac{\mathrm{d}}{\mathrm{d}t} J^r\gamma \subset H_{\Gamma_d}\,.$$

An endomorphism $F\colon TJ^r\pi \to TJ^r\pi$ is called an $f(3,-1)$ *structure on* $J^r\pi$ if $F^3 - F = 0$. There is a canonical direct sum decomposition on $TJ^r\pi$ induced by any such F. The eigenspaces corresponding to the eigenvalues $0, -1, +1$ are $\operatorname{Im}(F^2 - I)$, $\operatorname{Im}(F^2 - F)$, $\operatorname{Im}(F^2 + F)$, respectively. The $f(3,-1)$ structure is called *dynamical* and is denoted by F_d if

$$F_d = \left(F^\sigma_{(r,0)} \frac{\partial}{\partial q^\sigma_{(r)}} - \sum_{j=0}^{r-1} q^\sigma_{(j+1)} \frac{\partial}{\partial q^\sigma_{(j)}}\right) \otimes \mathrm{d}t + $$

$$= \sum_{j=0}^{r-1} \frac{\partial}{\partial q^\sigma_{(j)}} \otimes \mathrm{d}q^\sigma_{(j)} - \frac{\partial}{\partial q^\sigma_{(r)}} \otimes \mathrm{d}q^\sigma_{(r)} + $$

$$+ \sum_{k=0}^{r-1} F^\sigma_{(r,k)\lambda} \frac{\partial}{\partial q^\sigma_{(r)}} \otimes \mathrm{d}q^\lambda_{(k)}$$

in any fibre coordinates. The functions $F^\sigma_{(r,0)}, F^\sigma_{(r,k)\lambda} \in \mathscr{F}(J^r\pi)$, $0 \leqq k \leqq r-1$, are called the *components* of F_d. It can be demonstrated that $\operatorname{Im}(F_d^2 - F_d) = V_{\pi_{r,r-1}} J^r\pi$. The rm- and $(rm+1)$-dimensional eigenspaces $\operatorname{Im}(F_d^2 + F_d) =: H_{F_d}$ and $H_{F_d} \oplus \oplus \operatorname{Im}(F_d^2 - I) =: H'_{F_d}$ are called *strong* and *weak horizontal*, respectively.

There is a one-one correspondence between the set of all dynamical $f(3,-1)$ structures and the set of dynamical connection on $J^r\pi$. Any such F_d and Γ_d are called *associated* if

$$H_{\Gamma_d} = H'_{F_d}$$

which locally means

$$\Gamma^\sigma_{(r,k)\lambda} = \tfrac{1}{2} F^\sigma_{(r,k)\lambda}$$

for $0 \leqq k \leqq r-1$ and

$$\Gamma^{\sigma}_{(r,0)} = F^{\sigma}_{(r,0)} + \tfrac{1}{2}\sum_{k=0}^{r-1} F^{\sigma}_{(r,k)\lambda} q^{\lambda}_{(k+1)} .$$

Let now Γ_d be a dynamical connection on $J^r\pi$ associated to F_d. The connection $\Gamma \in \boldsymbol{\Gamma}_{r+1,r}$, determined by its components

$$\Gamma^{\sigma}_{(r+1)} := \Gamma^{\sigma}_{(r,0)} + \sum_{k=0}^{r-1} \Gamma^{\sigma}_{(r,k)\lambda} q^{\lambda}_{(k+1)} = F^{\sigma}_{(r,0)} + \sum_{k=0}^{r-1} F^{\sigma}_{(r,k)\lambda} q^{\lambda}_{(k+1)} ,$$

is then called *associated* to $\Gamma_d(F_d)$. This coordinate expression globally means just

$$\Delta^r_{r-1}[\Gamma] \subset H_{\Gamma_d} ,$$

and any dynamical Γ_d associated to Γ has the same paths. In addition, Γ generates through any such Γ_d or F_d the direct sum decomposition

$$TJ^r\pi = V_{\pi_{r,r-1}} J^r\pi \oplus \Delta^r_{r-1}[\Gamma] \oplus H_{F_d} ,$$

where $\Delta^r_{r-1}[\Gamma] \oplus H_{F_d} = H'_{F_d} = H_{\Gamma_d}$.

4. NATURAL DYNAMICAL CONNECTIONS

Although our main purpose is to describe the situation in the most general case, we will first discuss the very limpid contingency of $(R \times M, \pi, R)$ with $\pi = pr_1$, where M is an arbittary m-dimensional manifold.

Let us present (in accordance with [7]) all *natural affinors* (vector-valued one-forms) on $J^r\pi = R \times T^rM$. They create a linear subspace in the space of all tensors of type (1, 1) on $J^r\pi$, i.e. of all endomorphisms on $TJ^r\pi$. An arbitrary natural affinor has a form

$$\sum_{i=1}^{r} k_i J^{(r)}_i + \sum_{i=r+1}^{2r} k_i C^{(r)}_{i-r} \otimes \mathrm{d}t + k_{2r+1} I_{T^rM} + k_{2r+2} I_R ,$$

where $k_i \in \mathscr{F}(R)$; I_{T^rM} and

$$J^{(r)}_i = \sum_{j=1}^{r-i+1} j \frac{\partial}{\partial q^{\sigma}_{(i+j-1)}} \otimes \mathrm{d}q^{\sigma}_{(j-1)}$$

for $1 \leqq i \leqq r$ are the unique natural affinors on T^rM;

$$I_R = \frac{\partial}{\partial t} \otimes \mathrm{d}t ,$$

and finally

$$C^{(r)}_i = \sum_{j=1}^{r-i+1} \frac{(i+j-1)!}{(j-1)!} q^{\sigma}_{(j)} \frac{\partial}{\partial q^{\sigma}_{(i+j-1)}} \quad \text{for} \quad 1 \leqq i \leqq r$$

are the *absolute vector fields* (or *generalized Liouville vector fields*) on T^rM (see also [5]). With regard to our purpose, the following objects are of particular importance:

$$J_1^{(r)} = \sum_{j=1}^{r} j \frac{\partial}{\partial q^\sigma_{(j)}} \otimes \mathrm{d}q^\sigma_{(j-1)}$$

and

$$C_1^{(r)} = \sum_{j=1}^{r} j q^\sigma_{(j)} \frac{\partial}{\partial q^\sigma_{(j)}} .$$

Definition 1. An affinor

$$S^{(r)} = J_1^{(r)} - C_1^{(r)} \otimes \mathrm{d}t$$

will be called the *natural dynamical affinor on* $R \times T^rM$.

The meaning of this affinor is substantiated by the following assertion.

Proposition 1. *Let* ζ *be a semispray of type* $(r - 1)$ *on* $R \times T^rM$, *locally expressed by*

$$\zeta = \frac{\partial}{\partial t} + \sum_{j=0}^{r-1} q^\sigma_{(j+1)} \frac{\partial}{\partial q^\sigma_{(j)}} + \zeta^\sigma_{(r)} \frac{\partial}{\partial q^\sigma_{(r)}} ,$$

where $\zeta^\sigma_{(r)} \in \mathscr{F}(R \times T^rM)$. *Let* $\Gamma \in \Gamma_{r+1,r}$ *be the associated connection to* ζ, *i.e.*

$$h_\Gamma = \zeta \otimes \mathrm{d}t .$$

Then

$$F_d = \frac{1}{r+1} \left[(r - 1)\, v_\Gamma - 2\, \partial_\zeta S^{(r)}\right]$$

is a dynamical $f(3, -1)$ *structure on* $R \times T^rM$ *associated to* Γ.

Proof. By direct calculation in coordinates. □

Corollary 1. *Any semispray* ζ *of type* $(r - 1)$ *on* $R \times T^rM$ *generates in a canonical way the associated dynamical connection* Γ_d *on* $R \times T^rM$. *The components of this* Γ_d *are*

$$\Gamma^\sigma_{(r,k)\lambda} = \frac{k+1}{r+1} \frac{\partial \zeta^\sigma_{(r)}}{\partial q^\sigma_{(k+1)}}$$

for $0 \leqq k \leqq r - 1$, *and*

$$\Gamma^\sigma_{(r,0)} = \zeta^\sigma_{(r)} - \sum_{k=0}^{r-1} \Gamma^\sigma_{(r,k)\lambda} q^\sigma_{(k+1)} .$$

Definition 2. The $f(3, -1)$ structure F_d and the connection Γ_d from the previous assertions will be called the *natural dynamical* $f(3, -1)$ *structure* and *the natural dynamical connection* associated to ζ, respectively.

Remark that the case $r = 1$ is described in [12].

Let again (Y, π, X) be an arbitrary fibred manifold with one-dimensional base. Let Ω be a volume form on X; locally

$$\Omega = \omega\, \mathrm{d}t$$

with $\omega \in \mathscr{F}(X)$. Then one can define (according to [8]) a *natural dynamical affinor of type Ω* on $J^r\pi$, compatible with the bundle structure. This vector-valued one-form is locally expressed by

$$S_\Omega^{(r)} = \sum_{j+i=0}^{r-1} \binom{j+i+1}{i} \frac{d^j\omega}{dt^j} \frac{\partial}{\partial q^\sigma_{(j+i+1)}} \otimes (dq^\sigma_{(i)} - q^\sigma_{(i+1)}\,dt)\,,$$

where i, j are non-negative integers and $d^0\omega/dt^0 \equiv \omega$. Let $\Gamma \in \boldsymbol{\Gamma}_{r+1,r}$; (V, ψ), $\psi = (t, q^\sigma)$ any fibred chart on Y. Let $\zeta \in \Delta^r_{r-1}[\Gamma]$ be any local semispray on an open subset $W \subset \pi_{r,0}^{-1}(V)$. This means

$$\zeta = f(t)\left(\frac{\partial}{\partial t} + \sum_{j=0}^{r-1} q^\sigma_{(j+1)} \frac{\partial}{\partial q^\sigma_{(j)}} + \Gamma^\sigma_{(r+1)} \frac{\partial}{\partial q^\sigma_{(r)}}\right).$$

Then

$$-\partial_\zeta S_\Omega^{(r)} = f\omega G_\Omega^{(r)}\,,$$

where the $(1, 1)$ tensor field $G_\Omega^{(r)}$ contains derivations of ω by t, but it is independent of f, hence also of the choice of the semispray ζ.

Proposition 2. *An endomorphism*

$$F_d[\Omega] = \frac{1}{r+1}\left[(r-1)\,v_r + 2G_\Omega^{(r)}\right]$$

is the dynamical $f(3, -1)$ structure on $J^r\pi$ associated to Γ.

Corollary 2. *Any connection Γ of order $(r+1)$ on π generates in a canonical way the whole class of the associated dynamical connections on $J^r\pi$. For any volume form Ω on X, the components of $\Gamma_d[\Omega]$ are*

$$\Gamma^\sigma_{(r,k)\lambda} = \frac{1}{r+1}\left[\sum_{j=0}^{r-k-1} \binom{k+j+1}{j+1} \frac{\omega^{(j)}}{\omega} \frac{\partial \Gamma^\sigma_{(r+1)}}{\partial q^\lambda_{(k+j+1)}} - \right.$$

$$\left. - \binom{r+1}{r-k+1} \frac{\omega^{(r-k)}}{\omega} \delta^\sigma_\lambda\right]$$

for $0 \leqq k \leqq r-1$ and

$$\Gamma^\sigma_{(r,0)} = \Gamma^\sigma_{(r+1)} + \frac{1}{r+1}\,\cdot$$

$$\cdot\left[\sum_{j=1}^{r} \binom{r+1}{j+1} \frac{\omega^{(j)}}{\omega} q^\sigma_{(r+1-j)} - \sum_{k=0}^{r-1} \sum_{j=k+1}^{r} \binom{j}{k+1} \frac{\omega^{(k)}}{\omega} \frac{\partial \Gamma^\sigma_{(r+1)}}{\partial q^\lambda_{(j)}} q^\lambda_{(j-k)}\right].$$

Definition 3. The $f(3, -1)$ structure $F_d[\Omega]$ and the connection $\Gamma_d[\Omega]$ from the previous assertions will be called *the natural dynamical $f(3, -1)$ structure of type Ω* and *the natural dynamical connection of type Ω* associated to Γ, respectively.

Remarks. (i): Let $r = 1$. Then the components of the natural dynamical connection $\Gamma_d[\Omega]$ on $J^1\pi$ associated to the connection Γ of order 2 on π are

$$\Gamma^\sigma_\lambda = \frac{1}{2}\left(\frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} - \frac{\mathrm{d}\omega}{\mathrm{d}t}\frac{1}{\omega}\,\delta^\sigma_\lambda\right)$$

and

$$\Gamma^\sigma = \Gamma^\sigma_{(2)} + \frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\frac{1}{\omega}\,q^\sigma_{(1)} - \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}}\,q^\lambda_{(1)}\right),$$

which can be compared with the analogous result of Saunders in [9] and [8].

(ii): It is apparent that using a canonical volume form dt on R one obtains the situation on $R \times T^rM$; thus $S^{(r)} = S^{(r)}_{\mathrm{d}t}$ etc.

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