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NOTE ON LINEAR ARBORICITY OF GRAPHS WITH LARGE GIRTH

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Harary [10] in 1970 introduced the notion of linear arboricity as one of the covering invariants of graphs. A linear forest is a graph in which each component is a path. The linear arboricity $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests whose union is G.

Akiyama, Exoo and Harary [1] introduced a conjecture which has a fundamental importance in the study of linear arboricity.

Conjecture 1. *The linear arboricity of a d-regular graph is*

$$\lceil (d + 1)/2 \rceil .$$

Although this conjecture received great attention, until now it has been proved only for the cases of $d = 2, 3, 4, 5, 6, 8$ and 10 (see [1], [2], [4], [5], [7] and [13]). As the problem of linear arboricity seems to be too difficult for complete solution, some mathematicians began to investigate partial special cases. N. Alon [3] obtained interesting results for graphs with large girth (the length of the shortest cycle in the graph). The main tool in his work was the Lovász Local Lemma (for proof, see [6] and [12]).

Lemma 1. (Lovász Local Lemma). *Let $A_1, A_2, \ldots, A_n$ be events in a probability space. A graph $T = (V(T), E(T))$ on the set of vertices $V(T) = \{1, 2, \ldots, n\}$ is called a dependency graph for $A_1, \ldots, A_n$ if for all i, A_i is mutually independent of all A_j with $(i, j) \notin E(T)$. Assume there exist n numbers $x_1, x_2, \ldots, x_n \in \langle 0, 1)$ such that*

$$\Pr(A_i) < x_i \prod_{(i,j)\in E(T)} (1 - x_j)$$

for all i, $1 \leqq i \leqq n$. Then

$$\Pr\left(\bigwedge_{i=1}^{n} \bar{A}_i\right) > \prod_{i=1}^{n} (1 - x_i) .$$

In particular, the probability that no A_i occurs is positive.

Using this nice probability tool Alon proved:

Theorem 1. *Let $H = (V, E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of V into pairwise disjoint sets. Suppose each* *set V_i is of cardinality $|V_i| \geqq 25d$. Then there is an independent set of vertices $W \subset V$ that contains at least one vertex from each V_i.*

From Theorem 1 it was not too difficult to prove two results on linear arboricity.

Theorem 2. *Let G be a d-regular graph, where d is an even integer, with girth $g \geqq 50d$. Then*

$$\text{la}(G) = (d/2) + 1 .$$

Moreover, the edges of G can be covered by $\frac{1}{2}d$ linear forests and one matching.

Theorem 3. *Let G be a d-regular graph, where d is an odd integer, with girth $g \geqq 100d$. Suppose, further, that G contains a perfect matching. Then*

$$\text{la}(G) = (d + 1)/2 .$$

The aim of this paper is to prove somewhat stronger results in a little easier way. First we improve Theorem 1.

Theorem 1A. *Let $H = (V, E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of V into pairwise disjoint sets. Suppose each set V_i is of cardinality $|V_i| \geqq 8d$. Then there is an independent set of vertices $W \subset V$ that contains at least one vertex from each V_i.*

Proof. Clearly we may assume that each set V_i is of cardinality precisely $g = 8d$ (otherwise, simply replace each V_i by a subset of cardinality g, and replace H by its induced subgraph on the union of these r new sets). Now let us pick exactly one vertex from each V_i, randomly and independently. So each particular vertex $v \in V$ is picked with probability $p = 1/8d$. Let W be the random set of all vertices picked. The goal of the proof is to show that, with positive probability, W is an independent set of vertices. For each edge $f \in H$, let A_f be the event that W contains both ends of f. Clearly, $\Pr(A_f) = p^2$ if the ends of f are in various V_i and V_j and $\Pr(A_f) = 0$ if both ends of f are in the same V_i. So we have $\Pr(A_f) \leqq p^2$.

Now we can construct such a dependency graph for the events $\{A_f : f \in E\}$ that each A_f-node is adjacent to at most $2.8d \,.\, d$ $A_{f'}$-nodes. It follows from Lemma 1 that if we can find a number $x \in \langle 0, 1)$ such that

$$\text{(1)} \qquad \Pr(A_f) \leqq p^2 = \frac{1}{(64d)^2} < x(1 - x)^{16d^2}$$

then $\Pr(\bigwedge_{f \in E} \bar{A}_f) > 0$.

One can easily check that $x = 1/32d^2$ satisfies (1). Indeed,

$$\frac{1}{32d^2}\left(1 - \frac{1}{32d^2}\right)^{16d^2} > \frac{1}{32d^2}\left(1 - \frac{16d^2}{32d^2}\right) = \frac{1}{64d^2} = p^2 \geqq \Pr(A_f) .$$

This implies that $\Pr(\bigwedge_{f \in E} \bar{A}_f) > 0$, i.e. there exists such a choice of W that W is a set of independent vertices.

Theorem 1A then implies the following results:

Theorem 2A. *Let G be a d-regular graph, where d is an even integer, with girth $g \geqq 16d$. Then*

$$\text{la}(G) = (d/2) + 1 .$$

Moreover, the edges of G can be covered by $\frac{1}{2}d$ linear forests and one matching.

Theorem 3A. *Let G be a d-regular graph, where d is an odd integer, with girth $g \geqq 32d$. Suppose, further, that G contains a perfect matching. Then*

$$\text{la}(G) = (d + 1)/2 .$$

The proof of these two theorems is the exact copy of proofs of Theorems 2 and 3 made in [3] and we will not do it again. The condition in Theorems 2 and 2A that one of the linear forests has to be a matching is a little luxurious and not necessary for a partition of a graph into linear forests. If we omit this condition we can obtain a little better result again:

Theorem 2B. *Let G be a d-regular graph, where d is an even integer, with girth $g \geqq 7d$. Then*

$$\text{la}(G) = (d/2) + 1 .$$

The proof of Theorem 2B is based on the following modification of Theorem 1A:

Theorem 1B. *Let $H = (V, E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of V into pairwise disjoint sets. Suppose each set V_i is of cardinality $|V_i| \geqq \frac{7}{2}d$. Then there exists a set of vertices $W \subset V$ containing at least one vertex from each V_i and such that the maximum degree of the induced graph $\langle W \rangle$ is 1, i.e. $\langle W \rangle$ consists only of isolated vertices and independent edges.*

Proof. This proof is very similar to the proof of Theorem 1A. First we prove the assertion for d even. Similarly as before we may assume that each set V_i is of cardinality precisely $g = \frac{7}{2}d$. Let us pick exactly one vertex from each V_i, randomly and independently. Each particular vertex v is picked with probability $p = 2/7d$. Let W be the random set of all vertices picked. Let T be the set of all triples t of vertices such that the induced graph $\langle t \rangle$ contains at least two edges. For each triple $t \in T$ let A_t be the event that W contains all three vertices of t. Clearly $\Pr(A_t) = p^3$ if all three vertices are in different sets V_i and $\Pr(A_t) = 0$ in the other case. So we have $\Pr(A_t) \leqq p^3$. Now we can construct such a dependency graph for the events $\{A_t : t \in T\}$ that each A_t-node is adjacent to at most $3 . (\frac{7}{2}d) . d . d$ other $A_{t'}$-nodes. It follows from Lemma 1 that if we can find a number $x \in \langle 0, 1)$ such that

$$\Pr(A_t) \leqq p^3 = \left(\frac{2}{7d}\right)^3 < x(1 - x)^{3.(7/2)d^3} \tag{2}$$

then $\Pr(\bigwedge_{t \in T} \bar{A}_t) > 0$. One can easiy check that $x = 1/21d^3$ satisfies (2). Indeed,

$$\frac{1}{21d^3}\left(1-\frac{1}{21d^3}\right)^{(21/2)d^3} > \frac{1}{21d^3}\left(1-\frac{\frac{21}{2}d^3}{21d^3}\right) > \frac{1}{(\frac{7}{2}d)^3} = p^3 \geqq \Pr(A_t).$$

This implies that $\Pr(\bigwedge_{t\in T} \bar{A}_t) > 0$, i.e. there exists such a choice of W that the induced graph $\langle W \rangle$ consists only of independent vertices and independent edges.

The validity of Theorem 1B for odd d clearly follows from the validity of Theorem 1B for even $d + 1$.

The proof of Theorem 2B is then quite similar to the proof of Theorem 2 given in [3]. The only difference is that the last linear forest is no more a matching but may contain also paths with two edges. In the conclusion we introduce one more modification of Theorem 1:

Theorem 1C. *Let $H = (V, E)$ be a graph with maximum degree d, and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of V into r pairwise disjoint sets. Suppose each set V_i is of cardinality $|V_i| = \frac{5}{2}d$. Then there exists a set of vertices $W \subset V$ that contains at least one vertex from each V_i and that the induced graph $\langle W \rangle$ contains no triangles.*

The last result can be proved by a small modification of the proof of Theorem 1B. We can show for $x = 2/15d^3$ that $\Pr(A_t) < x(1 - x)^{(3/2)(5/2)d^3}$ where A_t is an event that $\langle W \rangle$ contains the triangle t.

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