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CONTINUOUS AND COMPACT IMBEDDINGS OF WEIGHTED SOBOLEV SPACES III

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This paper is a direct continuation of [4], [5] where fundamental concepts and notation were introduced. Continuous and compact imbeddings of weighted Sobolev spaces into weighted Lebesgue spaces on unbounded domains are investigated.

11. PRELIMINARIES

In this section we shall prove some auxiliary assertions. The main result will be a covering lemma, viz. Lemma 11.3.

11.1. Lemma. *Let $n \in \mathbb{N}$, $I = (n, +\infty)$, let $r: I \to (0, +\infty$ be a continuous non-decreasing function such that*

$$\lim_{x \to +\infty} [x - r(x)] = +\infty\,, \quad \lim_{x \to n_-} r(x) > 0\,. \tag{11.1}$$

Then there exists an increasing sequence $\{x_k\}_{k=1}^{\infty} \subset I$ with the following properties:

$$I = \bigcup_{k=1}^{\infty} I_k\,, \quad \text{where} \quad I_k = (x_k - r(x_k),\ x_k + r(x_k))\,; \tag{11.2}$$

$$I_k \cap I_l = \emptyset \quad \text{if} \quad |k - l| > 1\,. \tag{11.3}$$

Proof. Let us put $r(n) = \lim_{x \to n_-} r(x)$ and $x_0 = n$. If the points $x_0, x_1, \ldots, x_k$ are defined, we choose a point $x_{k+1} \in I$ such that

$$x_k = x_{k+1} - r(x_{k+1})\,. \tag{11.4}$$

(This is possible because the function $f(x) = x - r(x)$ is continuous on $\langle n, +\infty)$, $f(x_k) = x_k - r(x_k) < x_k$ and $\lim_{x \to +\infty} f(x) = +\infty$.) It is easy to see that the sequence $\{x_k\}_{k=1}^{\infty}$ possesses the desired properties.

11.2. Remark. Let I_k $(k \in \mathbb{N})$ be the interval from Lemma 11.1. Then

$$I_k \subset (x_{k-1}, x_{k+1})\,.\ ^{1}) \tag{11.5}$$

[1]) As in the proof of Lemma 11.1 we put $x_0 = n$.

Proof. By (11.4) we have $x_k - r(x_k) = x_{k-1}$ and further

$$x_k + r(x_k) = x_{k+1} - r(x_{k+1}) + r(x_k) \leqq x_{k+1},$$

which completes the proof.

11.3. Lemma. *Suppose* $n \in \mathbb{N}$, $I = (n, +\infty)$,

$$M = \{x \in \mathbb{R}^N;\ |x| > n\}.$$

Let $r: I \to (0, +\infty)$ *be a continuous nondecreasing function such that*

(11.6) $\quad x - r(x)$ *is nondecreasing on* I,

(11.7) $\quad \lim_{x \to +\infty} [x - r(x)] = +\infty, \quad \lim_{x \to n-} r(x) > 0.$

Then there exists a sequence $\{x_{ki}\}_{k,i=1}^{\infty} \subset M$ *with the following properties:*

(11.8) $\quad M \subset \bigcup_{k,i=1}^{\infty} B_{ki}$, *where* $B_{ki} = B(x_{ki}, r(|x_{ki}|))$;

(11.9) *there exists a number* τ *depending only on the dimension* N *such that*

$$\sum_{k,i=1}^{\infty} \chi_{B_{ki}}(z) \leqq \tau \quad \textit{for all} \quad z \in \mathbb{R}^N.$$

For $x \in \mathbb{R}$ let us denote

$$I(x) = (x - r(x),\ x + r(x)),$$
$$P(x) = \{y \in \mathbb{R}^N;\ |y| \in I(x)\}.$$

Let M, r and I be the set, the function and the interval, respectively, from Lemma 11.3. Then by Lemma 11.1 there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset I$ such that (11.2) and (11.3) hold. Further, put

$$P_k = P(x_k), \quad k \in \mathbb{N}.$$

The following lemma holds.

11.4. Lemma. *If* $x \in P_k$, $k \in \mathbb{N}$, *then*

(i) $B(x, r(|x|)) \cap (\mathbb{R}^N \setminus M) = \emptyset$ *for* $k \geqq 2$;

(ii) $B(x, r(|x|)) \cap \bigcup_{j \in \mathbb{N}, |j-k| > 2} P_j = \emptyset$;

(iii) $B(x, r(|x|)) \cap M \subset P_{k-2} \cup P_{k-1} \cup P_k \cup P_{k+1} \cup P_{k+2}$. [2])

Proof. Let $x \in P_k$. As

$$B(x, r(|x|)) \subset P(|x|),$$
$$y \in P(|x|) \Leftrightarrow |y| \in I(|x|), \quad \text{and}$$
$$y \in \mathbb{R}^N \setminus M \Leftrightarrow |y| \in \langle 0, n \rangle,$$

it is sufficient to prove the following assertion.

[2]) Here we formally put $P_j = \emptyset$ for $j \leqq 0$.

If $x \in I_k$, $k \in \mathbb{N}$, then

$$(11.10) \qquad I(x) \cap \langle 0, n \rangle = \emptyset \quad \text{for} \quad k \geqq 2 ;$$

$$(11.11) \qquad I(x) \cap \bigcup_{j \in \mathbf{N}, |j-k| > 2} I_j = \emptyset ;$$

$$(11.12) \qquad I(x) \cap I \subset I_{k-2} \cup I_{k-1} \cup I_k \cup I_{k+1} \cup I_{k+2} . \text{ }^{3})$$

Thus, suppose $x \in I_k$, $k \in \mathbb{N}$. By (11.5) we have

$$I_k \subset (x_{k-1}, x_{k+1})$$

and consequently

$$(11.13) \qquad I_k \subset (x_0, x_{k+1}) , \quad I_k \subset (x_{k-1}, +\infty) , \quad k \in \mathbb{N} .$$

Further, we have

$$(11.14) \qquad x + r(x) < x_{k+1} + r(x_{k+1}) \leqq x_{k+1} + r(x_{k+2}) =$$

$$= x_{k+2} \notin (x_{k+2}, +\infty) = \bigcup_{j=k+3}^{\infty} I_j , \quad k \in \mathbb{N} ;$$

$$(11.15) \qquad x - r(x) > x_{k-1} - r(x_{k-1}) = x_{k-2} \notin (x_0, x_{k-2}) \supset \bigcup_{j=1}^{k-3} I_j , \quad k \geqq 4 .$$

Obviously

$$(11.16) \qquad \bigcup_{j \in \mathbf{N}, |j-k| > 2} I_j = \bigcup_{j=1}^{k-3} I_j \cup \bigcup_{j=k+3}^{\infty} I_j \quad \text{for} \quad k \geqq 4 ,$$

$$(11.17) \qquad \bigcup_{j \in \mathbf{N}, |j-k| > 2} I_j = \bigcup_{j=k+3}^{\infty} I_j \quad \text{for} \quad k \in \{1, 2, 3\} ,$$

and (11.14)–(11.17) immediately yield (11.11).

From the relations

$$I(x) \cap I \subset I = \bigcup_{j=1}^{\infty} I_j$$

and (11.11) we obtain the inclusion (11.12).

If $k \geqq 2$ then (11.15) yields

$$x - r(x) > x_{k-2} \geqq x_0 ,$$

so $I(x) \subset I$, which implies (11.10) and the proof is complete.

Proof of Lemma 11.3. By Lemma 11.1 there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that (11.2) and (11.3) hold. It is easy to see that

$$(11.18) \qquad M = \bigcup_{k=1}^{\infty} P_k ,$$

$$(11.19) \qquad P_k \cap P_l = \emptyset , \quad |k - l| > 1 .$$

Fix $k \in \mathbb{N}$. By Lemma 3.3 from [4] (the Besicovitch covering lemma), where we set $A = P_k$, $\varrho(x) = r(|x|)$ for $x \in \mathbb{R}^N$, there exists a sequence $\{x_{ki}\}_{i=1}^{\infty} \subset P_k$ with the

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3) Here we formally put $I_j = \emptyset$ for $j \leqq 0$.

following properties:

(11.20) $\qquad P_k \subset \bigcup_{i=1}^{\infty} B_{ki}$, where $B_{ki} = B(x_{ki}, r(|x_{ki}|))$;

(11.21) $\qquad$ there exists a number Θ depending only on the dimension N such that

$$\sum_{i=1}^{\infty} \chi_{B_{ki}}(z) \leqq \Theta \quad \text{for all} \quad z \in \mathbb{R}^N .$$

Now, (11.18) and (11.20) imply (11.8). It remains to verify (11.9).

First let $x \in M$. By (11.18) there exists $k \in \mathbb{N}$ such that $x \in P_k$. Lemma 11.4 (ii) and (iii) implies that the point x is contained in no ball $B(y, r(|y|))$ provided $y \in P_j$ and $|j - k| > 2$. So the point x can be contained only in balls from the system

(11.22) $\qquad \{B_{ji};\ i = 1, 2, \ldots\}$,

where $|j - k| \leqq 2$ (not more than 5 systems from (11.22) are admissible). By (11.21), for fixed j, $|j - k| \leqq 2$, the point x is contained in at most Θ balls from the system (11.22). Hence we conclude that the point x is contained in at most 5Θ balls from the system $\{B_{ji};\ j, i \in \mathbb{N}\}$.

Now let $x \notin M$. Lemma 11.4 (i) yields that the point x can be contained only in balls from the system

$$\{B_{1i};\ i = 1, 2, \ldots\} .$$

Hence at most Θ balls of the system $\{B_{ji};\ j, i \in \mathbb{N}\}$ contain x. Consequently, (11.9) holds with $\tau = 5\Theta$.

11.5. Notation. For a domain $\Omega \subset \mathbb{R}^N$ and $n \in \mathbb{N}$ we set

(11.23) $\qquad \Omega_n = \{z \in \Omega;\ |z| < n\}$, $\quad \Omega^n = \operatorname{int}(\Omega \setminus \Omega_n)$.

11.6. Lemma. *Suppose $n_0 \in \mathbb{N}$, $I = (n_0, +\infty)$, $r: I \to (0, +\infty)$ is a function such that*

(11.24) $\qquad r(y) \leqq y/2$, $\quad y \in I$.

Let $n \geqq n_0$, $B(x, r(|x|)) \cap \Omega^{3n} \neq \emptyset$. Then $|z| > n$ for every $z \in B(x, r(|x|))$.

Proof. Let $z \in B(x, r(|x|))$, $y \in B(x, r(|x|)) \cap \Omega^{3n}$. Then

$$|x| \geqq |y| - |y - x| \geqq |y| - r(|x|) > 3n - \frac{|x|}{2} ,$$

consequently

$$|x| > 2n .$$

Further,

$$|z| \geqq |x| - |x - z| \geqq |x| - r(|x|) \geqq |x| - \frac{|x|}{2} = \frac{|x|}{2} > n$$

and the lemma is proved.

11.7. Definition. Let $I = (n, +\infty)$, $n \in \mathbb{N}$, $r: I \to (0, +\infty)$. The function r is said to have the property $V(n)$ (denoted $r \in V(n)$) if

(i) r is continuous and nondecreasing on I;

(ii) $x - r(x)$ is nondecreasing on I;

(iii) $\lim\limits_{x \to +\infty} [x - r(x)] = +\infty$, $\lim\limits_{x \to n_-} r(x) > 0$;

(iv) $r(x) \leqq x/2$ for $x \in I$;

(v) there exists a constant $c_r \geqq 1$ such that

$$c_r^{-1} \leqq \frac{r(y)}{r(x)} \leqq c_r$$

for all $x \in I$ and all $y \in I(x) \cap I$.

11.8. Remark. Let $r \in V(n)$. Then the following implication holds:

$$x \in \mathbb{R}^N, \quad |x| > n, \quad y \in B(x, r(|x|)), \quad |y| > n \Rightarrow c_r^{-1} \leqq \frac{r(|y|)}{r(|x|)} \leqq c_r. \tag{11.25}$$

12. IMBEDDING THEOREMS — THE CASE $1 \leqq p \leqq q < \infty$

In this section we suppose that $1 \leqq p \leqq q < \infty$. We will study imbeddings of weighted Soblev spaces into weighted Lebesgue spaces on unbounded domains of special types.

12.1. Theorem (sufficient conditions for the continuous imbedding). *Let Ω be a domain in $\mathbb{R}^N$, $1 \leqq p \leqq q < \infty$, $N/q - N/p + 1 \geqq 0$. Suppose that the following conditions are fulfilled*:

D1 *There exists $n_0 \in \mathbb{N}$ such that $\Omega^{n_0} = \{x \in \mathbb{R}^N;\ |x| > n_0\}$.*

D2 $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w)$, $n \geqq n_0$.

D3 *There exist positive measurable functions a_0, a_1 defined on Ω^{n_0} and a function $r \in V(n_0)$ such that for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(|x|))$ we have*

$$w(y) \leqq a_0(x), \tag{12.1}$$
$$a_1(x) \leqq v_1(y).$$

D4 *There exists a constant $K_0 > 0$ such that*

$$v_1(x)\, r^{-p}(|x|) \leqq K_0\, v_0(x) \quad \textit{for a.e.} \quad x \in \Omega^{n_0}. \tag{12.2}$$

D5 $\lim\limits_{n \to \infty} \mathscr{A}_n < \infty$, *where*

$$\mathscr{A}_n = \sup_{x \in \Omega^n} \frac{a_0^{1/q}(x)}{a_1^{1/p}(x)}\, r^{(N/q) - (N/p) + 1}(|x|). \tag{12.3}$$

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w). \tag{12.4}$$

Proof. Let us denote $X = W^{1,p}(\Omega; v_0, v_1)$. By [4], Lemma 3.1 (where we put $Q = \Omega$, $G_n = \Omega_{3n}$, $n \in \mathbb{N}$) is is sufficient to verify the condition

$$\lim_{n\to\infty} \sup_{\|u\|_X \leqq 1} \|u\|_{q,\Omega\setminus G_n,w} < \infty . \tag{12.5}$$

If we set $M = \Omega^{n_0}$ then by Lemma 11.3 there exists a sequence $\{x_{ki}\}_{k,i=1}^{\infty} \subset \Omega^{n_0}$ with the following properties:

$$\Omega^{n_0} \subset \bigcup_{k,i=1}^{\infty} B_{ki}, \quad \text{where} \quad B_{ki} = B(x_{ki}, r(|x_{ki}|)) ; \tag{12.6}$$

(12.7) there exists a number τ depending only on the dimension N such that

$$\sum_{k,i=1}^{\infty} \chi_{B_{ki}}(z) \leqq \tau \quad \text{for all} \quad z \in \mathbb{R}^N .$$

For $n \geqq n_0$ let us denote

$$\mathscr{K}_n = \{(k, i) \in \mathbb{N} \times \mathbb{N};\ B_{ki} \cap \Omega^{3n} \neq \emptyset\} .$$

By Lemma 11.6 we have $\bigcup_{(k,i)\in\mathscr{K}_n} B_{ki} \subset \Omega^n \subset \Omega^{n_0}$ and this fact enables us to use conditions **D3** and **D4** for points $y \in B_{ki}$.

Now, analogously as in the proof of Theorem 2.2 from [4] we get the estimate

$$\|u\|_{q,\Omega\setminus G_n,w} \leqq \tau^{1/p} K_1^{1/q} \mathscr{A}_n \|u\|_X \tag{12.8}$$

where $K_1 = K[\max(c_r^p K_0, 1)]^{1/p}$ (the number $K > 0$ is from (3.7)). This combined with **D5** implies (12.5).

12.2. Theorem (sufficient conditions for the compact imbedding). *Let Ω be a domain in $\mathbb{R}^N$, $1 \leqq p \leqq q < \infty$, $N/q - N/p + 1 \geqq 0$. Suppose that conditions* **D1**, **D3**, **D4** *are fulfilled and let*

D2* $\quad W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega_n; w)\,, \quad n \geqq n_0\,;$

D5* $\quad \lim_{n\to\infty} \mathscr{A}_n = 0\,,$ *where $\mathscr{A}_n$ is defined in* (12.3).

Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w) . \tag{12.9}$$

Proof. From (12.8) and **D5*** we obtain

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,\Omega,v_0,v_1} \leqq 1} \|u\|_{q,\Omega\setminus G_n,w} = 0$$

and the proof can be completed by Remark 3.2 from [4].

Necessary conditions for continuous and compact imbeddings follow from the next two theorems.

12.3. Theorem. *Let Ω be a domain in $\mathbb{R}^N$, $1 \leqq p, q < \infty$. Let the condition* **D1** *and, moreover, the following conditions be satisfied:*

D^3 *There exist positive measurable functions $\hat{a}_0$, $\hat{a}_1$ defined on Ω^{n_0} and a function $r \in V(n_0)$ such that for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(|x|))$*

$$(12.10) \qquad w(y) \geqq \hat{a}_0(x), \quad \hat{a}_1(x) \geqq v_1(y).$$

D^4 *There exists a constant $k_0 > 0$ such that*

$$(12.11) \qquad k_0\, v_0(x) \leqq v_1(x)\, r^{-p}(|x|) \quad \textit{for a.e.} \quad x \in \Omega^{n_0}.$$

D^5 $\lim_{n \to \infty} \hat{\mathscr{A}}_n = +\infty$, *where*

$$(12.12) \qquad \hat{\mathscr{A}}_n = \sup_{x \in \Omega^n} \frac{\hat{a}_0^{1/q}(x)}{\hat{a}_1^{1/p}(x)}\, r^{(N/q)-(N/p)+1}(|x|).$$

Then the space $W_0^{1,p}(\Omega; v_0, v_1)$ is not continuously imbedded in the space $L^q(\Omega; w)$.

Proof is analogous to that of Theorem 2.4 from [4].

12.4. Theorem. *Let Ω be a domain in $\mathbb{R}^N$, $1 \leqq p, q < \infty$. Suppose conditions* **D1**, **D^3**, **D^4** *and* **D^5*** *are satisfied, where*

D^5* $\qquad \lim_{n \to \infty} \hat{\mathscr{A}}_n > 0$, *where the number $\hat{\mathscr{A}}_n$ is defined by* (12.12).

Then the space $W_0^{1,p}(\Omega; v_0, v_1)$ is not compactly imbedded in the sapce $L^q(\Omega; w)$.

Proof is analogous to that of Theorem 2.5 from [4].

From Theorems 12.1 and 12.3 (or 12.2 and 12.4, respectively) we easily obtain the following two theorems.

12.5. Theorem (continuous imbedding). *Let Ω be a domain in $\mathbb{R}^N$, $1 \leqq p \leqq q < \infty$, $N/q - N/p + 1 \geqq 0$. Suppose, in addition to* **D1**, **D2**, *that the following three conditions are fulfilled:*

D~3 *There exist positive constants $c_0 \leqq C_0$, $c_1 \leqq C_1$ and positive measurable functions a_0, a_1 defined on Ω^{n_0} and a function $r \in V(n_0)$ such that for all $x \in \Omega^{n_0}$ and for a.e. $y \in B(x, r(|x|))$ we have*

$$(12.13) \qquad c_0\, a_0(x) \leqq w(y) \leqq C_0\, a_0(x), \quad c_1\, a_1(x) \leqq v_1(y) \leqq C_1\, a_1(x).$$

D~4 *There exist positive constants $k_0 \leqq K_0$ such that*

$$(12.14) \qquad k_0\, v_0(x) \leqq v_1(x)\, r^{-p}(|x|) \leqq K_0\, v_0(x) \quad \textit{for a.e.} \quad x \in \Omega^{n_0}.$$

Then $\qquad W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$

(and also $W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$)

if and only if the condition **D5** *is satisfied.*

12.6. Theorem (compact imbedding). *Let Ω be a domain in $\mathbb{R}^N$, $1 \leqq p \leqq q < \infty$,*

$N/q - N/p + 1 \geqq 0$. *Let the conditions* **D1**, **D2***, **D˜3** *and* **D˜4** *be satisfied. Then*

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w)$$

$$(\textit{and also } W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w))$$

if and only if the condition **D5*** *is fulfilled.*

12.7. Remark. It is easy to see that the conclusion of Theorem 12.5 (or Theorem 12.6) concerning the imbedding $W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$ (or $W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega, w)$, respectively) holds for an arbitrary unbounded domain $\Omega \subset \mathbb{R}^N$.

13. EXAMPLES — THE CASE $1 \leqq p \leqq q < \infty$

From Theorems 12.5 and 12.6 we obtain the following examples.

13.1. Example. Let $\Omega = \operatorname{int}(\mathbb{R}^N \setminus \tilde{\Omega})$, where $0 \in \tilde{\Omega} \in \mathscr{C}^{0,1}$, $1 \leqq p \leqq q < \infty$, $\alpha, \beta \in \mathbb{R}$. For $x \in \Omega$ we define

$$w(x) = |x|^\alpha, \quad v_0(x) = |x|^{\beta - p}, \quad v_1(x) = |x|^\beta .$$

Then

$$W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow L^q(\Omega; |x|^\alpha)$$

or

$$W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; |x|^\alpha)$$

if and only if

$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 \leqq 0, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0$$

or

$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0, \quad \frac{N}{q} - \frac{N}{p} + 1 > 0,$$

respectively.[4])

13.2. Example. Let $\Omega = \operatorname{int}(\mathbb{R}^N \setminus \tilde{\Omega})$ where $0 \in \tilde{\Omega} \in \mathscr{C}^{0,1}$, $1 \leqq p \leqq q < \infty$, $\alpha, \beta \in \mathbb{R}$. For $x \in \Omega$ we define

$$w(x) = |x|^\alpha, \quad v_0(x) = v_1(x) = |x|^\beta .$$

Then

$$W^{1,p}(\Omega; |x|^\beta, |x|^\beta) \hookrightarrow L^q(\Omega; |x|^\alpha)$$

or

$$W^{1,p}(\Omega; |x|^\beta, |x|^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; |x|^\alpha)$$

if and only if

$$\frac{\alpha}{q} - \frac{\beta}{p} \leqq 0, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0$$

[4]) In Theorems 12.5 and 12.6 we put $r(t) = t/3$, $t \in \mathbf{R}^+$.

or $$\frac{\alpha}{q} - \frac{\beta}{p} < 0\,, \quad \frac{N}{q} - \frac{N}{p} + 1 > 0\,,$$
respectively.[5])

If a domain Ω satisfies the condition **D1**, *we write* $\Omega \in$ **D1**. *If* $\Omega \in$ **D1** *has the cone property* (*in the sense of* [1]), *we write* $\Omega \in$ **G1**.

13.3. Example. Suppose $\Omega \in$ **G1**, $1 \leqq p \leqq q < \infty$, $\alpha, \beta \in \mathbb{R}$. For $x \in \Omega$ we define
$$w(x) = e^{\alpha|x|}\,, \quad v_0(x) = v_1(x) = e^{\beta|x|}\,.$$
Then
$$W^{1,p}(\Omega; e^{\beta|x|}, e^{\beta|x|}) \hookrightarrow L^q(\Omega; e^{\alpha|x|})$$
or
$$W^{1,p}(\Omega; e^{\beta|x|}, e^{\beta|x|}) \hookrightarrow\hookrightarrow L^q(\Omega; e^{\alpha|x|})$$
if and only if
$$\frac{\alpha}{q} - \frac{\beta}{p} \leqq 0\,, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0$$
or
$$\frac{\alpha}{q} - \frac{\beta}{p} < 0\,, \quad \frac{N}{q} - \frac{N}{p} + 1 > 0\,,$$
respectively.[5])

13.4. Remark. Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$, $\emptyset \neq M \subset \overline{M} \subset \tilde{\Omega}$, $|M| = 0$. Let Ω be a domain such that $\tilde{\Omega} \setminus \overline{M} \subset \Omega \subset \tilde{\Omega}$. For $x \in \Omega$ let us put $d(x) = \operatorname{dist}(x, M)$. It is easy to see that Theorems 2.6 and 2.7 from [4] remain valid with this $d(x)$. (The proof is quite analogous.) Hence we have

13.5. Example. Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$, $1 \leqq p \leqq q < \infty$, $\alpha, \beta \in \mathbb{R}$. Let Ω be a domain in $\mathbb{R}^N$, $\tilde{\Omega} \setminus \{0\} \subset \Omega \subset \tilde{\Omega}$. For $x \in \Omega$ we put
$$w(x) = |x|^\alpha\,, \quad v_0(x) = |x|^{\beta - p}\,, \quad v_1(x) = |x|^\beta\,.$$
Then
$$W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow L^q(\Omega; |x|^\alpha)$$
or
$$W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; |x|^\alpha)$$
if and only if
$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 \geqq 0\,, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0$$
or
$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0\,, \quad \frac{N}{q} - \frac{N}{p} + 1 > 0\,,$$
respectively.

[5]) In Theorems 12.5 and 12.6 we put $r(x) \equiv 1$.

Theorems 12.5, 12.6 and Example 13.5 imply

13.6. Example. Let Ω be a domain in $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \{0\} \subset \Omega \subset \mathbb{R}^N$, let $1 \leqq p \leqq q < \infty$, $\alpha, \beta \in \mathbb{R}$. For $x \in \Omega$ we define

$$w(x) = |x|^\alpha, \quad v_0(x) = |x|^{\beta-p}, \quad v_1(x) = |x|^\beta .$$

Then

$$W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow L^q(\Omega; |x|^\alpha)$$

if and only if

$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0 .$$

The space $W^{1,p}(\Omega; |x|^\alpha)$ is not compactly imbedded in $L^q(\Omega; |x|^\alpha)$ for any $\alpha, \beta \in \mathbb{R}$. [Let us remark that in the case $\Omega = \mathbb{R}^N$ the spaces $W_0^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta)$, $W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta)$ are defined (due to the condition (1.4) from [4]) only for $p - N < \beta < N(p-1)$].

For an unbounded domain Ω, $\emptyset \neq \Omega \subset \mathbb{R}^N$ let us define

$$(13.1) \qquad {}_*a = \inf\{|x|;\ x \in \Omega\} .$$

If $\Omega \in \mathbf{D1}$, we put

$$(13.2) \qquad \bar{a} = \begin{cases} \sup\{|x|;\ x \in \mathbb{R}^N \setminus \Omega\} & \text{if } \Omega \neq \mathbb{R}^N \\ 0 & \text{if } \Omega = \mathbb{R}^N . \end{cases}$$

13.7. Example. Suppose $\Omega \in \mathbf{G1}$, ${}_*a > 1$, $1 \leqq p \leqq q < \infty$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. For $x \in \Omega$ we put

$$w(x) = |x|^\alpha \log^\gamma |x|, \quad v_0(x) = |x|^{\beta-p} \log^\delta |x|, \quad v_1(x) = |x|^\beta \log^\delta |x| .$$

I. Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$$

if and only if

$$\frac{N}{q} - \frac{N}{p} + 1 \geqq 0,$$

and

$$\frac{\alpha}{q} - \frac{\beta}{q} + \frac{N}{q} - \frac{N}{p} + 1 < 0 \quad \text{or} \quad \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0, \quad \frac{\gamma}{q} - \frac{\delta}{p} \leqq 0 .$$

II. Then $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w)$ if and only if

$$\frac{N}{q} - \frac{N}{p} + 1 > 0,$$

and

$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0 \quad \text{or} \quad \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0, \quad \frac{\gamma}{q} - \frac{\delta}{p} < 0 .$$

In this section we will study equivalent norms on the spaces $W_0^{1,p}(\Omega; v_0, v_1)$, $W^{1,p}(\Omega; v_0, v_1)$. We will assume that

$$1 \leqq p < \infty .$$

The weight functions v_i $(i = 0, 1)$ will be radial, i.e.

$$v_i(x) = \bar{v}_i(|x|) , \quad x \in \Omega ,$$

with

$$\bar{v}_i \in \mathscr{W}((_*a, \infty)) , \quad i = 0, 1 . \tag{14.1}$$

Further, we introduce the following notation. If I is an unbounded open interval in $\mathbb{R}$, then by

$$\mathscr{W}_{\mathrm{B}}(I) \quad [\text{or } \mathscr{W}_{\mathrm{C}}(I)] \tag{14.2}$$

we mean the set of all weight functions $\varrho \in \mathscr{W}(I)$ bounded from below and from above by positive constants on each bounded interval $J \subset I$ (or on each compact interval $J \subset I$, respectively).

In the proofs of theorems on equivalent norms we shall use the following lemma, the proof of which can be found in [6] and in [7].

14.1. Lemma. *Let* $1 \leqq p \leqq q \leqq \infty$, $-\infty \leqq a < b \leqq \infty$, $\omega_0, \omega_1 \in \mathscr{W}((a, b))$. *Then there exists a constant* $C > 0$ *such that the inequality*

$$\left(\int_a^b |u(t)|^q \, \omega_0(t) \, \mathrm{d}t\right)^{1/q} \leqq C\left(\int_a^b |u'(t)|^q \, \omega_1(t) \, \mathrm{d}t\right)^{1/p} \tag{14.3}$$

holds for all functions

$$u \in \mathscr{T}_1(a, b) = \{f \in AC((a, b)) ; \lim_{x \to a+} f(t) = 0\} \tag{14.4}$$

or

$$u \in \mathscr{T}_2(a, b) = \{f \in AC((a, b)) ; \lim_{x \to b-} f(t) = 0\} \tag{14.5}$$

or

$$u \in \mathscr{T}(a, b) = \mathscr{T}_1(a, b) \cap \mathscr{T}_2(a, b) \tag{14.6}$$

if and only if

$$B_1^{p,q}(a, b; \omega_0, \omega_1) = \sup_{a<x<b} \left(\int_x^b \omega_0(t) \, \mathrm{d}t\right)^{1/q} \left(\int_a^x \omega_1^{1-p'}(t) \, \mathrm{d}t\right)^{1/p'} < \infty \tag{14.7}$$

or

$$B_2^{p,q}(a, b; \omega_0, \omega_1) = \sup_{a<x<b} \left(\int_a^x \omega_0(t) \, \mathrm{d}t\right)^{1/q} \left(\int_x^b \omega_1^{1-p'}(t) \, \mathrm{d}t\right)^{1/p'} < \infty \tag{14.8}$$

or

$$B^{p,q}(a, b; \omega_0, \omega_1) = \inf_{c \in \langle a,b \rangle} \max \{B_1^{p,q}(a, c; \omega_0, \omega_1), B_2^{p,q}(c, b; \omega_0, \omega_1)\} < \infty , \tag{14.9}$$

respectively. [7])

[7]) The numbers $B_1^{p,q}(a, b; \omega_0, \omega_1)$ and $B_2^{p,q}(a, b; \omega_0, \omega_1)$ are defined for $a < b$ by (14.7) and (14.8). Further, we formally set $B_1^{p,q}(a, a; \omega_0, \omega_1) = 0 = B_2^{p,q}(b, b; \omega_0, \omega_1)$.

14.2. Remark. Condition (14.9) is equivalent to the following one (see [3]):

$$(14.9') \qquad \mathscr{B}^{p,q}(a, b; \omega_0, \omega_1) = \sup_{\substack{c,d \\ a<c<d<b}} \left(\int_c^d \omega_0(t)\,\mathrm{d}t\right)^{1/q} .$$

$$. \min\left\{\left(\int_a^c \omega_1^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'}, \left(\int_d^b \omega_1^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'}\right\} < \infty .$$

14.3. Theorem. *Let $\Omega \neq \emptyset$ be an unbounded domain in $\mathbb{R}^N$, $\Omega \subset \mathbb{R}^N \setminus \{0\}$. Suppose $1 \leqq p < \infty$ and*

$$(14.10) \qquad B^{p,p}({}_*a, \infty; \bar{v}_0(t)\, t^{N-1}, \bar{v}_1(t)\, t^{N-1}) < \infty .$$

Then there exists a constant $C > 0$ such that

$$(14.11) \qquad \|u\|_{p,\Omega,v_0} \leqq C\|\nabla u\|_{p,\Omega,v_1} \quad \forall u \in W_0^{1,p}(\Omega; v_0, v_1) .\ ^{8)}$$

Proof can be done by using spherical coordinates and Lemma 14.1 (for $q = p$).

In order to derive the analogue of Theorem 14.3 for the space $W^{1,p}(\Omega; v_0, v_1)$ we make use of the following theorem:

14.4. Theorem. *Suppose $\Omega \in$ **D1**, $1 \leqq p < \infty$, $\bar{v}_0, \bar{v}_1 \in \mathscr{W}_C(({}_*a, \infty))$. Let there exist a constant $k > 0$ and a number $t_0 \in ({}_*a, \infty)$ such that*

$$(14.12) \qquad \bar{v}_0(t) \geqq k\, \bar{v}_1(t)\, t^{-p} \quad \text{for all} \quad t \geqq t_0 .$$

Then the set

$$(14.13) \qquad C^\infty_{\mathrm{BS}}(\Omega) = \{g \in C^\infty(\Omega);\ \operatorname{supp} g \subset \bar{\Omega} \text{ is bounded}\}$$

is dense in the space $W^{1,p}(\Omega; v_0, v_1)$.

Proof. Let $u \in W^{1,p}(\Omega; v_0, v_1)$. Fix $\varepsilon > 0$. Then there exists a function

$$(14.14) \qquad u_\varepsilon \in C^\infty(\Omega) \cap W^{1,p}(\Omega; v_0, v_1)$$

such that

$$(14.15) \qquad \|u - u_\varepsilon\|_{1,p,\Omega,v_0,v_1} \leqq \varepsilon/2$$

(for the proof see [2], Section 2). Let f be a function satisfying

(i) $\quad f \in C^\infty(\mathbb{R})$,

(ii) $\quad f(t) = 1$ for $t \leqq 5/4$,

(iii) $\quad f(t) = 0$ for $t \geqq 7/4$,

(iv) $\quad 0 \leqq f(t) \leqq 1$ for $t \in \mathbb{R}$.

Fix $R > n_0$ (n_0 from the condition **D1**). For $h > 0$ we put

$$(14.16) \qquad F_h(x) = f\left(\frac{|x| - R}{h}\right), \quad x \in \mathbb{R}^N .$$

[8]) By $\|\nabla u\|_{p,\Omega,v_1}$ we mean the expression

$$\left(\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|^p_{p,\Omega,v_1}\right)^{1/p} .$$

The function F_h has the following properties:

(a) $0 \leqq F_h(x) \leqq 1\,,\quad x \in \mathbb{R}^N\,,$

(b) $F_h(x) = 1$ for $x \in \Omega_{R+5h/4} \cup \partial\Omega\,,$ [9])

(c) $\operatorname{supp} F_h \subset B(0, R + 2h)\,,$

(d) $F_h \in C^\infty(\mathbb{R}^N)\,,$

(e) there exists a constant $c_f > 0$ such that

$$\left|\frac{\partial F_h(x)}{\partial x_i}\right| \leqq c_f \cdot \frac{1}{h}\,, \quad x \in \mathbb{R}^N\,.$$

Let us now set

$$u_{\varepsilon,h}(x) = u_\varepsilon(x)\, F_h(x)\,, \quad x \in \Omega\,.$$

Then (14.14), (d), (c) and (b) yield

(14.18) $\quad u_{\varepsilon,h} \in C^\infty(\Omega)\,,$

(14.19) $\quad \operatorname{supp} u_{\varepsilon,h} \subset B(0, R + 2h)\,,$

(14.20) $\quad \operatorname{supp}(u_\varepsilon - u_{\varepsilon,h}) \subset \Omega^{R+h}\,.$

These properties together with (a) and (14.12) imply (for $h \geqq \max\{R, t_0 - R\}$)

(14.21) $\quad (\int_\Omega |u_\varepsilon - u_{\varepsilon,h}|^p\, v_0\, dx)^{1/p} \leqq (\int_{\Omega^{R+h}} |u_\varepsilon|^p\, v_0\, dx)^{1/p}\,,$

(14.22)
$$\left(\int_\Omega \left|\frac{\partial}{\partial x_j}(u_\varepsilon - u_{\varepsilon,h})\right|^p v_1\, dx\right)^{1/p} \leqq$$
$$\leqq \left(\int_{\Omega^{R+h}} \left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^p v_1\, dx\right)^{1/p} + \left(\int_{\Omega^{R+h}} |u_\varepsilon|^p \cdot \left|\frac{\partial}{\partial x_j}(1 - F_h)\right|^p v_1\, dx\right)^{1/p} \leqq$$
$$\leqq \left(\int_{\Omega^{R+h}} \left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^p v_1\, dx\right)^{1/p} + c_f \left(\int_{\Omega^{R+h} \setminus \Omega^{R+2h}} |u_\varepsilon|^p\, h^{-p}\, v_1\, dx\right)^{1/p} \leqq$$
$$\leqq \left(\int_{\Omega^{R+h}} \left|\frac{\partial u_\varepsilon}{\partial x_j}(x)\right|^p \bar{v}_1(|x|)\, dx\right)^{1/p} + 3c_f \left(\int_{\Omega^{R+h}} |u_\varepsilon(x)|^p \cdot \frac{\bar{v}_1(|x|)}{|x|^p}\, dx\right)^{1/p} \leqq$$
$$\leqq K\left(\int_{\Omega^{R+h}} \left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^p v_1\, dx + \int_{\Omega^{R+h}} |u_\varepsilon|^p\, v_0\, dx\right)^{1/p}, \quad j = 1, 2, \dots, N$$

$(K = 2^{1/p'} \max\{1, 3c_f/k^{1/p}\})\,.$

From these estimates and (14.14) we obtain that there exists $h > 0$ such that

(14.23) $\quad \|u_\varepsilon - u_{\varepsilon,h}\|_{1,p,\Omega,v_0,v_1} < \varepsilon/2\,.$

The estimates (14.15) and (14.23) imply

$$\|u - u_{\varepsilon,h}\|_{1,p,\Omega,v_0,v_1} < \varepsilon\,.$$

[9]) Similarly as in Notation 11.5 for $s > 0$ we set $\Omega_s = \{z \in \Omega;\ |z| < s\}$, $\Omega^s = \operatorname{int}(\mathbb{R}^N \setminus \Omega_s)$.

By (14.18) and (14.19) we have $u_{\varepsilon,h} \in C^{\infty}_{\mathrm{BS}}(\Omega)$ and the theorem is proved.

We shall write $\Omega \in \mathbf{D1}^{\#}$ *if the following conditions are fulfilled*:

(i) $\qquad \Omega \in \mathbf{D1}$,

(ii) $\qquad x \in \Omega\,, \quad t > 1 \Rightarrow tx \in \Omega\,.$

14.5. Theorem. *Suppose* $1 \leqq p < \infty$, $\Omega \in \mathbf{D1}^{\#}$, $\bar{v}_0, \bar{v}_1 \in \mathscr{W}_{\mathrm{C}}((_*a, \infty))$ *and let the condition* (14.12) *be fulfilled. Let*

(14.24) $$B_2^{p,p}(_*a, \infty; \bar{v}_0(t)\, t^{N-1}, \bar{v}_1(t)\, t^{N-1}) < \infty\,.$$

Then there exists a constant $C > 0$ *such that*

(14.25) $$\|u\|_{p,\Omega,v_0} \leqq C\|\nabla u\|_{p,\Omega,v_1} \quad \forall u \in W^{1,p}(\Omega; v_0, v_1)\,.$$

Proof can be done by using Theorem 14.4, Lemma 14.1 (with $p = q$) and the spherical coordinates.

14.6. Remark. Suppose $1 \leqq p \leqq \infty$.

i) Let $\Omega = \mathbb{R}^N \setminus \{x \in \mathbb{R}^N;\ |x| \leqq r\}$, $r \geqq 0$. Then it is possible to prove that the conditions (14.10) and (14.11) are equivalent.

ii) Let $\Omega = \mathbb{R}^N \setminus \{x \in \mathbb{R}^N;\ |x| \leqq r\}$, $r \geqq 0$, or $\Omega = \mathbb{R}^N$. Then one can show that the conditions (14.24) and (14.25) are equivalent.

Using Theorems 14.3, 14.5 and Remark 14.6 we can give some examples.

14.7. Example. Suppose $1 \leqq p < \infty$,

$$\bar{v}_0(t) = t^{\gamma}\,, \quad \bar{v}_1(t) = t^{\beta}\,, \quad t \in (_*a, \infty)\,.$$

(I) Let $\Omega = \operatorname{int}(\mathbb{R}^N \setminus \tilde{\Omega})$, $0 \in \tilde{\Omega} \in \mathscr{C}^{0,1}$ (thus $_*a > 0$.)
Then:

(i) the inequality (14.11) holds if

$$\beta \neq p - N\,, \quad \gamma \leqq \beta - p$$

or

$$\beta = p - N\,, \quad \gamma < -N\,;$$

(ii) the inequality (14.25) holds if

$$\beta > p - N\,, \quad \gamma \leqq \beta - p\,.$$

(II) Let $\Omega = \mathbb{R}^N \setminus \{0\}$ (thus $_*a = 0$). Then

(i) the inequality (14.11) holds if and only if

$$\beta \neq p - N\,, \quad \gamma = \beta - p\,;$$

(ii) the inequality (14.25) holds if and only if

$$\beta > p - N\,, \quad \gamma = \beta - p\,.$$

14.8. Example. Suppose $\Omega \in \mathbf{D1}$, $1 \leqq p < \infty$,

$$\bar{v}_0(t) = e^{\gamma t}\,, \quad \bar{v}_1(t) = e^{\beta t}\,, \quad t \in (_*a, \infty)\,.$$

(I) Let ${}_*a > 0$. Then

(i) the inequality (14.11) holds if

$$\gamma \leqq \beta, \quad [\gamma, \beta] \neq [0, 0];$$

(ii) the inequality (14.25) holds if $\Omega \in \mathbf{D1}^{\#}$ and

$$\beta > 0, \quad \gamma \leqq \beta$$

or

$$\beta = 0, \quad \gamma < 0, \quad 1 < p < N$$

or

$$\beta = 0, \quad \gamma < 0, \quad p = 1.$$

(III) Let ${}_*a = 0$. Then

(i) the inequality (14.11) holds if $0 \notin \Omega$ and

$$\gamma < 0, \quad \gamma \leqq \beta, \quad p > N$$

or

$$\beta > 0, \quad \gamma \leqq \beta$$

or

$$\beta = 0, \quad \gamma < 0, \quad 1 < p < N$$

or

$$\beta = 0, \quad \gamma < 0, \quad p = 1;$$

(ii) the inequality (14.25) holds if $\Omega \in \mathbf{D1}^{\#}$ and

$$\beta > 0, \quad \gamma \leqq \beta$$

or

$$\beta = 0, \quad \gamma < 0, \quad 1 < p < N$$

or

$$\beta = 0, \quad \gamma < 0, \quad p = 1.$$

14.9. Example. Suppose $\Omega \in \mathbf{D1}$, $1 \leqq p < \infty$,

$$\bar{v}_0(t) = t^{\gamma} \log^{\delta} t, \quad \bar{v}_1(t) = t^{\beta} \log^{\eta} t, \quad t \in ({}_*a, \infty), \quad {}_*a > 1.$$

Then

(i) the inequality (14.11) holds if

$$\beta \neq p - N,$$

and

$$\gamma < \beta - p \quad \text{or} \quad \gamma = \beta - p, \quad \delta \leqq \eta$$

or

$$\beta = p - N,$$

and

$$\gamma < -N \quad \text{or} \quad \gamma = -N, \quad \eta \neq p - 1, \quad \delta \leqq \eta - p \quad \text{or}$$
$$\gamma = -N, \quad \eta = p - 1, \quad \delta < -1.$$

(ii) the inequality (14.25) holds if $\Omega \in \mathbf{D1}^*$,

$$\beta > p - N ,$$

and

$$\gamma < \beta - p \quad \text{or} \quad \gamma = \beta - p , \quad \delta \leqq \eta ;$$

or

$$\beta = p - N ,$$

and

$$\gamma < -N , \quad \eta > p - 1 , \quad \text{or} \quad \gamma = -N , \quad \delta \leqq \eta - p .$$

15. IMBEDDING THEOREMS — THE CASE $1 \leqq q < p < \infty$

In this section we assume that

$$1 \leqq q < p < \infty$$

and that the weight functions w, v_0, v_1 are radial:[10])

$$(15.1) \qquad w(x) = \bar{w}(|x|) , \quad v_i(x) = \bar{v}_i(|x|) , \quad i = 0, 1 , \quad x \in \Omega .$$

The following lemma plays the principal role in the proofs of imbedding theorems with $1 \leqq q < p < \infty$.

15.1. Lemma. *Let* $1 \leqq q < p \leqq \infty$, $1/r = 1/q - 1/p$, $-\infty \leqq a < b \leqq \infty$, $\omega_0, \omega_1 \in \mathscr{W}((a, b))$. *Then there exists a positive costant* C *such that the inequality*

$$(15.2) \qquad (\int_a^b |u(t)|^q \, \omega_0(t) \, dt)^{1/q} \leqq C(\int_a^b |u'(t)|^p \, \omega_1(t) \, dt)^{1/p}$$

holds for all functions

$$(15.3) \qquad u \in \mathscr{T}_1(a, b) = \{f \in AC((a, b)); \ \lim_{t \to a^+} f(t) = 0\}$$

or

$$(15.4) \qquad u \in \mathscr{T}_2(a, b) = \{f \in AC((a, b)); \ \lim_{t \to b^-} f(t) = 0\}$$

or

$$(15.5) \qquad u \in \mathscr{T}(a, b) = \mathscr{T}_1(a, b) \cap \mathscr{T}_2(a, b)$$

if and only if

$$(15.6) \qquad A_1^{p,q}(a, b; \omega_0, \omega_1) = \\ = [\int_a^b ((\int_x^b \omega_0(t) \, dt)^{1/q} (\int_a^x \omega_1^{1-p'}(t) \, dt)^{1/q'})^r \, \omega_1^{1-p'}(x) \, dx]^{1/r} < \infty$$

or

$$(15.7) \qquad A_2^{p,q}(a, b; \omega_0, \omega_1) = \\ = [\int_a^b ((\int_a^x \omega_0(t) \, dt)^{1/q} (\int_x^b \omega_1^{1-p'}(t) \, dt)^{1/q'})^r \, \omega_1^{1-p'}(x) \, dx]^{1/r} < \infty$$

---

[10]) If $v_0 \equiv v_1$, then we write v instead of v_0 and v_1.

or

$$(15.8) \qquad A^{p,q}(a, b; \omega_0, \omega_1) = = \inf_{c\in\langle a,b\rangle} \max\{A_1^{p,q}(a, c; \omega_0, \omega_1),\ A_2^{p,q}(c, b; \omega_0, \omega_1)\} < \infty\,, \text{ [11]}$$

respectively.

For the proof see [6] and [7].

15.2. Remark. (i) If $C > 0$ is the least constant such that the inequality (15.2) holds on the class $\mathscr{T}_i(a, b)$ $(i = 1, 2)$ or $\mathscr{T}(a, b)$ then

$$(15.9) \qquad q^{1/q}\left(\frac{p'q}{r}\right)^{1/q'} A_i^{p,q}(a, b; \omega_0, \omega_1) \leqq C \leqq q^{1/q}(p')^{1/q'} A_i^{p,q}(a, b; \omega_0, \omega_1)$$

$(i = 1, 2)$ (see [6])

or

$$2^{-1/p}q^{1/q}\left(\frac{p'q}{r}\right)^{1/q'} A^{p,q}(a, b; \omega_0, \omega_1) \leqq C \leqq 2^{1/r}q^{1/q}(p')^{1/q'} A^{p,q}(a, b; \omega_0, \omega_1)\,, \text{ [12]}$$

respectively (see [7]).

(ii) Assume in addition that $\omega_0, \omega_1 \in \mathscr{W}_C((a, b))$. Then checking the proofs of necessity of conditions (15.6)–(15.8) we can see that these conditions are necessary for the inequality (15.2) to hold on the (smaller) classes $\mathscr{T}_1^*(a, b)$, $\mathscr{T}_2^*(a, b)$, $\mathscr{T}^*(a, b)$, respectively, where

$$\mathscr{T}_1^*(a, b) = \{u \in C^\infty(a, b);\ a \notin \operatorname{supp} u\}\,,$$
$$\mathscr{T}_2^*(a, b) = \{u \in C^\infty(a, b);\ b \notin \operatorname{supp} u\}\,,$$
$$\mathscr{T}^*(a, b) = \mathscr{T}_1^*(a, b) \cap \mathscr{T}_2^*(a, b) = C_0^\infty((a, b))\,.$$

15.3. Lemma. *Let $R > 0$. Then there exists a partition of unity $\Phi^R = \{\Phi_1^R, \Phi_2^R\}$ with the following properties:*

(i) $\Phi_1^R, \Phi_2^R \in C^\infty(\mathbb{R}^N)$;

(ii) $\operatorname{supp} \Phi_1^R \subset B(0, R + 4)$;

(iii) $\operatorname{supp} \Phi_2^R \subset \mathbb{R}^N \setminus \operatorname{cl}(B(0, R))$;

(iv) $0 \leqq \Phi_1^R, \Phi_2^R \leqq 1$ *on* $\mathbb{R}^N$;

(v) $\Phi_1^R(x) + \Phi_2^R(x) = 1,\ x \in \mathbb{R}^N$;

(vi) *there exists a constant $k > 0$ (independent of R) such that*

$$\left|\frac{\partial \Phi_j^R}{\partial x_i}(x)\right| \leqq K \quad \textit{for} \quad j = 1, 2\,, \quad i = 1, 2, \ldots, N\,, \quad x \in \mathbb{R}^N\,.$$

Proof is standard and is left to the reader.

---

[11]) The numbers $A_1^{p,q}(a, b; \omega_0, \omega_1)$ and $A_2^{p,q}(a, b; \omega_0, \omega_1)$ are defined for $a < b$ by (15.6) and (15.7). Further, we formally set $A_1^{p,q}(a, a; \omega_0, \omega_1) = 0 = A_2^{p,q}(b, b; \omega_0, \omega_1)$.

[12]) Moreover, $C \leqq q^{1/q}(p')^{1/q} A^{p,q}(a, b; \omega_0, \omega_1)$ if $A^{p,q}(a, b; \omega_0, \omega_1) = A_i^{p,q}(a, b; \omega_0, \omega_1)$ for some $i \in \{1, 2\}$.

15.4. Theorem. Suppose $\Omega \in \mathbf{G1}$, $1 \leqq q < p < \infty$, $\bar{w}, \bar{v} \in \mathscr{W}_B((_*a, \infty))$. *Let there exist* $R \in \langle_* a, \infty)$ *such that*

(15.10) $$A_1^{p,q}(R, \infty; \bar{w}(t)\, t^{N-1}, \bar{v}(t)\, t^{N-1}) < \infty .$$

Then

(15.11) $$W^{1,p}(\Omega; v, v) \hookrightarrow\hookrightarrow L^q(\Omega; w) .$$

Proof. Using [2] we obtain

(15.12) $$W^{1,p}(\Omega; v, v) = \overline{\mathscr{V}}^{\|\cdot\|_{1,p,\Omega,v,v}}$$

where $\mathscr{V} = \{u \in C^\infty(\Omega);\ \|u\|_{1,p,\Omega,v,v} < \infty\}$. Hence, by Remark 3.2 from [4], it suffices to prove that

(15.13) $$\limsup_{n\to\infty} \{\|u\|_{q,\Omega\setminus G_n,w};\ u \in \mathscr{V},\ \|u\|_{1,p,\Omega,v,v} \leqq 1\} = 0 ,$$

where we set $G_n = \Omega_{n+5}$.

Now we fix $n \in \mathbb{N}$, $n > \max\{R, \bar{a}\}$, and take the partition of unity $\{\Phi_1^n, \Phi_2^n\}$ from Lemma 15.3. Let $u \in \mathscr{V}$, $\|v\|_{1,p,\Omega,v,v} \leqq 1$. Then

15.14 $$u = u_1 + u_2 , \quad \text{where} \quad u_i = u\Phi_i^n , \quad i = 1, 2, \ldots ;$$
$$\operatorname{supp} u_1 \subset B(0, n + 4) ;$$
$$\operatorname{supp} u_2 \subset \mathbb{R}^N \setminus \operatorname{cl}(B(0, n)) .$$

Further we have

(15.15) $$\begin{aligned}\|u\|_{q,\Omega\setminus G_n,w}^q &= \int_{\mathbb{R}^N\setminus B(0,n+5)} |u(x)|^q\, w(x)\, dx = \\ &= \int_{\mathbb{R}^N\setminus B(0,n+5)} |u_2(x)|^q\, w(x)\, dx \leqq \int_{\mathbb{R}^N\setminus B(0,n)} |u_2(x)|^q\, w(x)\, dx = \\ &= \int_{S_1} \int_n^\infty |u_2(t, \Theta)|^q\, \bar{w}(t)\, t^{N-1}\, dt\, d\Theta ,\end{aligned}$$

where $\Theta = x/|x|$ is a point on the unit sphere $S_1 = \{x \in \mathbb{R}^N;\ |x| = 1\}$. By the definition of u_2 we have $u_2(\cdot, \Theta) \in C^\infty((n, \infty))$, $u_2(n, \Theta) = 0$ (for fixed Θ) and so Lemma 15.1 and Remark 15.2 imply

(15.16) $$\int_n^\infty |u_2(t, \Theta)|^q\, \bar{w}(t)\, t^{N-1}\, dt \leqq \mathscr{A}_n^q \left(\int_n^\infty \left|\frac{\partial u_2}{\partial t}(t, \Theta)\right|^p \bar{v}(t)\, t^{N-1}\, dt\right)^{p/q}$$

with

(15.17) $$\mathscr{A}_n = q^{1/q}(p')^{1/q'}\, A_1^{p,q}(n, \infty; \bar{w}(t)\, t^{N-1}, \bar{v}(t)\, t^{N-1}) .$$

From (15.15) and (15.16) by virtue of the Hölder inequality we obtain

(15.18) $$\begin{aligned}\|u\|_{q,\Omega\setminus G_n,w}^q &\leqq \mathscr{A}_n^q \int_{S_1} \left(\int_n^\infty \left|\frac{\partial u_2}{\partial t}(t, \Theta)\right|^p \bar{v}(t)\, t^{N-1}\, dt\right)^{q/p} d\Theta \leqq \\ &\leqq |S_1|^{(p-q)/p}\, \mathscr{A}_n^q \left(\int_{S_1}\int_n^\infty \left|\frac{\partial}{\partial t}(\Phi_2^n \cdot u)(t, \Theta)\right|^p \bar{v}(t)\, t^{N-1}\, dt\, d\Theta\right)^{q/p} \leqq \\ &\leqq |S_1|^{(p-q)/p}\, \mathscr{A}_n^q N^q \left(N^p K^p \int_\Omega |u(x)|^p\, v(x)\, dx + \int_\Omega |\nabla u(x)|^p\, v(x)\, dx\right)^{q/p} \leqq c_1^q \mathscr{A}_n^q,\end{aligned}$$

where $c_1^q = N^q|S_1|^{(p-q)/p} \max\{N^q K^p, 1\}$ and $|S_1|$ is the $(N-1)$-dimensional measure of the unit sphere S_1. By (15.17), (15.6) and (15.10) it easily follows that $\lim_{n\to\infty} \mathscr{A}_n = 0$, so (15.18) implies (15.13) and the theorem is proved.

15.5. Theorem. *Suppose* $\Omega \in \mathbf{G1}$, $1 \leqq q < p < \infty$, $\bar{w}, \bar{v}_0, \bar{v}_1 \in \mathscr{W}_B((_*a, \infty))$ *and* (14.12). *Let there exist* $R \in \langle _*a, \infty)$ *such that*

$$(15.19)\qquad A_2^{p,q}(R, \infty; \bar{w}(t)\, t^{N-1}, \bar{v}_1(t)\, t^{N-1}) < \infty\,.$$

Then

$$(15.20)\qquad W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w)\,.$$

Proof can be done in a way similar to that used for proving Theorem 15.4. Only instead of Lemma 15.3 we use Theorem 14.4. Details are left to the reader.

15.6. Theorem. *Let* Ω *be an unbounded domain in* $\mathbb{R}^N$, $1 \leqq q < p < \infty$, $\bar{w}, \bar{v}_0, \bar{v}_1, \bar{\lambda} \in \mathscr{W}_B((_*a, \infty))$ *and*

$$(15.21)\qquad A^{p,q}(_*a, \infty; \bar{w}(t)\, t^{N-1}, \bar{v}_1(t)\, t^{N-1}) < \infty\,.$$

Let the function $\bar{\lambda}$ *satisfy*

$$(15.22)\qquad \bar{\lambda} \text{ is decreasing in an interval } (s, \infty) \subset (_*a, \infty)\,;$$

$$(15.23)\qquad \lim_{t\to\infty} \bar{\lambda}(t) = 0\,.$$

Then

$$(15.24)\qquad W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w\,.\,\lambda)\,,$$

where $\lambda(x) = \bar{\lambda}(|x|)$, $x \in \Omega$.

Proof. Using Lemma 15.1 (the condition (15.8)) one can prove

$$(15.25)\qquad X = W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$$

if the assumptions of Theorem 15.6 are satisfied.

In virtue of Remark 3.2 from [4] it is sufficient to verify that

$$(15.26)\qquad \lim_{n\to\infty} \sup_{\|u\|_X \leqq 1} \|u\|_{q,\Omega^n,w\lambda} = 0\,.$$

Take $u \in X$, $n > s$. Applying (15.22) and (15.25) we get

$$(15.27)\qquad \|u\|^q_{q,\Omega^n,w\lambda} = \int_{\Omega^n} |u(x)|^q\, w(x)\, \bar{\lambda}(|x|)\, dx \leqq$$
$$\leqq \bar{\lambda}(n) \int_{\Omega^n} |u(x)|^q\, w(x)\, dx \leqq \bar{\lambda}(n)\,.\, K^q \|u\|_X^q\,,$$

where K is the norm of the imbedding operator (15.25). Then (15.26) is a consequence of (15.27) and the assumption (15.23).

15.7. Theorem. *Let* Ω *be an unbounded domain in* $\mathbb{R}^N$, $1 \leqq q < p < \infty$ *and* $\bar{w}, \bar{v}, \bar{\lambda} \in \mathscr{W}_B((_*a, \infty))$. *Let the following conditions be fulfilled:*

(i) *there exists* $R \in \langle _*a, \infty)$ *such that*

$$(15.28)\qquad A^{p,q}(R, \infty; \bar{w}(t)\, t^{N-1}, \bar{v}(t)\, t^{N-1}) < \infty\,;$$

(ii) *the function* $\bar{\lambda}$ *satisfies* (15.22) *and* (15.23).

Then

$$(15.29)\qquad W_0^{1,p}(\Omega; v, v) \hookrightarrow\hookrightarrow L^q(\Omega; w \,.\, \lambda)\,,$$

where $\lambda(x) = \bar{\lambda}(|x|)$, $x \in \Omega$.

Proof. First of all we prove that

$$(15.30)\qquad X = W_0^{1,p}(\Omega; v, v) \hookrightarrow L^q(\Omega; w)\,.$$

By Lemma 3.1 from [4] it suffices to verify

$$(15.31)\qquad \lim_{n\to\infty} \sup_{\|u\|_X \leqq 1} \|u\|_{q,\Omega\setminus G_n,w} < \infty\,,$$

where $G_n = \Omega_{n+5}$, $n \in \mathbb{N}$.

Let R be the number from the assumption (i) and let $\{\Phi_1^R, \Phi_2^R\}$ be the partition of unity from Lemma 15.3. Take $u \in X$, $\|u\|_X \leqq 1$, and $n \in \mathbb{N}$, $n > R$. Then (15.15) holds. By Lemma 15.1 and Remark 15.2 we have

$$(15.32)\qquad \int_n^\infty |u_2(t,\Theta)|^q \,\bar{w}(t)\, t^{N-1}\,\mathrm{d}t \leqq \int_R^\infty |u_2(t,\Theta)|^q \,\bar{w}(t)\, t^{N-1}\,\mathrm{d}t \leqq$$
$$\leqq \left[2^{1/r} q^{1/q}(p')^{1/q'} A^{p,q}(R,\infty; \bar{w}(t)\,t^{N-1}, \bar{v}(t)\,t^{N-1})\right]^q \,.$$
$$\cdot \left(\int_R^\infty \left|\frac{\partial u_2}{\partial t}(t,\Theta)\right|^p \bar{v}(t)\, t^{N-1}\,\mathrm{d}t\right)^{q/p}.$$

Now, similarly as in (15.18) we obtain

$$\|u\|_{q,\Omega\setminus G_n,w} \leqq c\, A^{p,q}(R,\infty; \bar{w}(t)\,t^{N-1}, \bar{v}(t)\,t^{N-1})$$

with c independent of $u \in X$. This implies (15.31) and so (15.30) holds.

By Remark 3.2 from [4] the proof will be complete if we verify (15.26). This can be done in the same way as in the proof of Theorem 15.6 (cf. (15.27)).

Sufficient conditions for non-existence of imbeddings are given by the following theorem.

15.8. Theorem. *Suppose* $\Omega \in \mathbf{D1}$, $1 \leqq q < p < \infty$, $\bar{v}_0 \in \mathcal{W}((_*a, \infty))$, $\bar{w}, \bar{v}_1 \in \mathcal{W}_C((_*a, \infty))$. *Let there exist* $R \geqq \bar{a}$ *such that*

$$(15.33)\qquad B^{p,p}(R,\infty; \bar{v}_0(t)\,t^{N-1}, \bar{v}_1(t)\,t^{N-1}) < \infty\,,$$

$$(15.34)\qquad A^{p,q}(R,\infty; \bar{w}(t)\,t^{N-1}, \bar{v}_1(t)\,t^{N-1}) = \infty\,.$$

Then the space $W_0^{1,p}(\Omega; v_0, v_1)$ *is not continuously imbedded into the space* $L^q(\Omega; w)$.

Proof. By (15.34), Lemma 15.1 and Remark 15.2 (ii) there exists a sequence of functions $\{z_n\} \subset C_0^\infty((R,\infty))$ such that

$$(15.35)\qquad \int_R^\infty |z_n(t)|^q \,\bar{w}(t)\, t^{N-1}\,\mathrm{d}t \to \infty \quad \text{for} \quad n \to \infty\,,$$

$$(15.36)\qquad \int_R^\infty |z_n'(t)|^p \,\bar{v}_1(t)\, t^{N-1}\,\mathrm{d}t = 1\,,\quad n \in \mathbb{N}\,.$$

For $n \in \mathbb{N}$ let us put

$$u_n(x) = \begin{cases} z_n(|x|)\,, & \text{if} \quad x \in \mathbb{R}^N \setminus \mathrm{cl}\,(B(0,R))\,,\\ 0\,, & \text{if} \quad x \in \Omega \cap \mathrm{cl}\,(B(0,R))\,. \end{cases}$$

Then we have

$$\{u_n\} \subset C_0^\infty(\mathbb{R}^N \setminus \operatorname{cl}(B(0, R)))\,,$$

(15.37) $$\int_\Omega |u_n(x)|^q\, w(x)\, dx = \int_{S_1} \int_R^\infty |z_n(t)|^q\, \bar{w}(t)\, t^{N-1}\, dt\, d\Theta = \\ = |S_1| \cdot \int_R^\infty |z_n(t)|^q\, \bar{w}(t)\, t^{N-1}\, dt \to \infty \quad \text{for} \quad n \to \infty\,,$$

(15.38) $$\int_\Omega |\nabla u_n(x)|^p\, v_1(x)\, dx = \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}(x)\right|^p \bar{v}_1(|x|)\, dx \leqq \\ \leqq N|S_1| \cdot \int_R^\infty |z_n'(t)|^p\, \bar{v}_1(t)\, t^{N-1}\, dt = N|S_1|\,, \quad n \in \mathbb{N}$$

($\Theta = x/|x|$ is a point on the unit sphere $S_1 = \{x \in \mathbb{R}^N;\ |x| = 1\}$).
Using (15.33) and Theorem 14.3 we arrive at

$$\|u_n\|_{p,\Omega,v_0} \leqq C\left(\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p,\Omega,v_1}^p\right)^{1/p}$$

and by (15.38) we obtain

$$\|u_n\|_{p,\Omega,v_0} \leqq C(N|S_1|)^{1/p}\,, \quad n \in \mathbb{N}\,.$$

The last estimates (15.37), (15.38) and the fact $u_n \in C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega; v_0, v_1)$ imply that the space $W_0^{1,p}(\Omega; v_0, v_1)$ is not continuously imbedded into $L^q(\Omega; w)$.

16. EXAMPLES — THE CASE $1 \leqq q < p < \infty$

From Theorems 15.4 – 15.8 we obtain

16.1. Example. Suppose $1 \leqq q < p < \infty$.

I. Let $\Omega \in \mathbf{D1}$, $_*\alpha > 0$, $\beta \neq p - N$. Then the following three conditions are equivalent:

(i) $W_0^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; |x|^\alpha)$,

(ii) $W_0^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow L^q(\Omega; |x|^\alpha)$,

(iii) $\dfrac{\alpha}{q} - \dfrac{\beta}{p} + \dfrac{N}{q} - \dfrac{N}{p} + 1 < 0\,.$

II. Let $\Omega \in \mathbf{G1}$, $_*\alpha > 0$, $\beta > p - N$. Then the following three conditions are equivalent:

(i) $W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; |x|^\alpha)$,

(ii) $W^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta) \hookrightarrow L^q(\Omega; |x|^\alpha)$,

(iii) $\dfrac{\alpha}{q} - \dfrac{\beta}{p} + \dfrac{N}{q} - \dfrac{N}{p} + 1 < 0\,.$

III. Let $\Omega = \mathbb{R}^N \setminus \{0\}$ or $\Omega = \mathbb{R}^N$, $\beta \neq p - N$. Then the space $W_0^{1,p}(\Omega; |x|^{\beta-p}, |x|^\beta)$ is continuously imbedded into the space $L^q(\Omega; |x|^\alpha)$ for no $\alpha \in \mathbb{R}$.

16.2. Example. Suppose $_*\alpha > 1$, $1 \leqq q < p < \infty$. For $x \in \Omega$ we define

$$w(x) = |x|^\alpha \log^\gamma |x| , \quad v_0(x) = |x|^{\beta - p} \log^\delta |x| , \quad v_1(x) = |x|^\beta \log^\delta |x| .$$

I. Let $\Omega \in \mathbf{D1}$, $\beta \neq p - N$. Then the following conditions are equivalent:

(i) $W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w)$,

(ii) $W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$,

(iii) $\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0$ or

$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0 , \quad \frac{\gamma}{q} - \frac{\delta}{p} + \frac{1}{q} - \frac{1}{p} < 0 .$

II. Let $\Omega \in \mathbf{G1}$, $\beta > p - N$. Then the following conditions are equivalent:

(i) $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w)$,

(ii) $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$,

(iii) $\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0$ or

$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0 , \quad \frac{\gamma}{q} - \frac{\delta}{p} + \frac{1}{q} - \frac{1}{p} < 0 .$

16.3. Example. Suppose $\Omega \in \mathbf{G1}$, $1 \leqq q < p < \infty$, $\beta \neq 0$. Then the following five conditions are equivalent:

(i) $W_0^{1,p}(\Omega; e^{\beta|x|}, e^{\beta|x|}) \hookrightarrow\hookrightarrow L^q(\Omega; e^{\alpha|x|})$,

(ii) $W^{1,p}(\Omega, e^{\beta|x|}, e^{\beta|x|}) \hookrightarrow\hookrightarrow L^q(\Omega; e^{\alpha|x|})$,

(iii) $W_0^{1,p}(\Omega; e^{\beta|x|}, e^{\beta|x|}) \hookrightarrow L^q(\Omega; e^{\alpha|x|})$,

(iv) $W^{1,p}(\Omega; e^{\beta|x|}, e^{\beta|x|}) \hookrightarrow L^q(\Omega; e^{\alpha|x|})$,

(v) $\frac{\alpha}{q} - \frac{\beta}{q} < 0 .$

16.4. Remark. The conditions (i), (iii), (v) from Example 16.3 are equivalent even under the weaker assumption $\Omega \in \mathbf{D1}$.

17. N-DIMENSIONAL HARDY INEQUALITY ON UNBOUNDED DOMAINS

As a consequence of the imbedding theorems and theorems on equivalent norms on the spaces $W_0^{1,p}(\Omega; v_0, v_1)$ or $W^{1,p}(\Omega; v_0, v_1)$ we obtain conditions for the validity of the N-dimensional Hardy inequality. The corresponding result if formulated in the following proposition.

17.1. Proposition. *Let the expressions* $\|\cdot\|_{1,p,\Omega,v_0,v_1}$ *and* $|||\cdot|||_{1,p,\Omega,v_1}$, *where*

$$|||u|||_{1,p,\Omega,v_1} = \left(\sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i}(x) \right|^p v_1(x) \, dx \right)^{1/p} ,$$

be equivalent norms on the space $W_0^{1,p}(\Omega; v_0, v_1)$ (*or* $W^{1,p}(\Omega; v_0, v_1)$). *Then*

$$W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$$

(or

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w))$$

if and only if there exists a positive constant C such that the N-dimensional Hardy inequality

$$(17.1) \qquad \left(\int_\Omega |u(x)|^q \, w(x) \, dx\right)^{1/q} \leqq C \left(\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^p v_1(x) \, dx\right)^{1/p}$$

holds for all $u \in \mathcal{T}(\Omega)$ *with* $\mathcal{T}(\Omega) = W_0^{1,p}(\Omega; v_0, v_1)$ (*or* $\mathcal{T}(\Omega) = W^{1,p}(\Omega; v_0, v_1)$, *respectively*).

Proof is trivial.

We shall write $\Omega \in \mathbf{G1}^\#$ *if* $\Omega \in \mathbf{G1} \cap \mathbf{D1}^\#$.

Proposition 17.1 and the examples from Sections 13, 14 and 16 imply

17.2. Example. Suppose $1 \leqq p, q < \infty$, $\alpha, \beta \in \mathbb{R}$. For $x \in \Omega$ we define

$$w(x) = |x|^\alpha, \quad v_0(x) = |x|^{\beta - p}, \quad v_1(x) = |x|^\beta .$$

I. Let $\Omega \in \mathbf{D1}$ (or $\Omega \in \mathbf{G1}^\#$), $_*a > 0$, $\beta \neq p - N$ (or $\beta > p - N$). Then the inequality (17.1) holds with $\mathcal{T}(\Omega) = W_0^{1,p}(\Omega; v_0, v_1)$ (or $\mathcal{T}(\Omega) = W^{1,p}(\Omega; v_0, v_1)$, respectively) if and only if

$$\text{(i)} \qquad 1 \leqq p \leqq q < \infty , \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0 , \quad \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 \leqq 0$$

or

$$\text{(ii)} \qquad 1 \leqq q < p < \infty , \quad \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0 .$$

II. Let $\Omega = \mathbb{R}^N \setminus \{0\}$, $\beta \neq p - N$ (or $\beta > p - N$). Then the inequality (17.1) holds with $\mathcal{T}(\Omega) = W_0^{1,p}(\Omega; v_0, v_1)$ (or $\mathcal{T}(\Omega) = W^{1,p}(\Omega; v_0, v_1)$, respectively) if and only if

$$(17.2) \qquad 1 \leqq p \leqq q < \infty , \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0 , \quad \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0 .$$

III. Let $\Omega = \mathbb{R}^N$, $p - N < \beta < N(p - 1)$. Then the inequality (17.1) holds with $\mathcal{T}(\Omega) = W^{1,p}(\Omega; v_0, v_1)$ or $\mathcal{T}(\Omega) = W_0^{1,p}(\Omega; v_0, v_1)$ if and only if the condition (17.2) is fulfilled.

17.3. Example. Suppose $\Omega \in \mathbf{D1}$ (or $\Omega \in \mathbf{G1}^\#$), $_*a > 1$, $1 \leqq p$, $q < \infty$, $\beta \neq p - N$ (or $\beta > p - N$). For $x \in \Omega$ we define

$$w(x) = |x|^\alpha \log^\gamma |x| , \quad v_0(x) = |x|^{\beta - p} \log^\delta |x| , \quad v_1(x) = |x|^\beta \log^\delta |x| .$$

Then the inequality (17.1) holds with $\mathcal{T}(\Omega) = W_0^{1,p}(\Omega; v_0, v_1)$ (or $\mathcal{T}(\Omega) =$

$= W^{1,p}(\Omega; v_0, v_1)$, respectively) if and only if

(i) $$1 \leqq p \leqq q < \infty\,, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0 \text{ and } \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0$$

$$\text{or } \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0\,, \quad \frac{\gamma}{q} - \frac{\delta}{p} \leqq 0\,;$$

or

(ii) $$1 \leqq q < p < \infty\,, \quad \text{and}$$

$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0$$

$$\text{or } \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 = 0\,, \quad \frac{\gamma}{q} - \frac{\delta}{p} + \frac{1}{q} - \frac{1}{p} < 0\,.$$

17.4. Example. Suppose $\Omega \in \mathbf{D1}$, $1 \leqq p, q < \infty$, $\alpha, \beta \in \mathbb{R}$. For $x \in \Omega$ we define

$$w(x) = e^{\alpha|x|}\,, \quad v_0(x) = v_1(x) = e^{\beta|x|}\,.$$

I. Let one of the following conditions be fulfilled:

(i) $${}_*a > 0\,, \quad \beta \neq 0\,;$$

(ii) $${}_*a = 0\,, \quad \beta > 0 \quad \text{or} \quad \beta < 0 \quad \text{and} \quad p > N\,.$$

Then the inequality (17.1) holds with $\mathscr{T}(\Omega) = W_0^{1,p}(\Omega; v_0, v_1)$ if and only if

$$1 \leqq p \leqq q < \infty\,, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqq 0\,, \quad \frac{\alpha}{q} - \frac{\beta}{p} \leqq 0 \tag{17.3}$$

or

$$1 \leqq q < p < \infty\,, \quad \frac{\alpha}{q} - \frac{\beta}{p} < 0\,. \tag{17.4}$$

II. Let $\Omega \in \mathbf{G1}^{\#}$, $\beta > 0$. Then the inequality (17.1) holds with $\mathscr{T}(\Omega) = W^{1,p}(\Omega; v_0, v_1)$ if and only if (17.3) or (17.4) is fulfilled.

Concluding remark. The survey of results of this paper was presented at the international conference "Summer School on Function Spaces, Differential Operators and Nonlinear Analysis" held in Sodankylä (Finland) in 1988 (see [8]).

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