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INVARIANT REGIONS ASSOCIATED WITH QUASILINEAR DAMPED WAVE EQUATIONS

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1. INTRODUCTION

It is well-known that a great number of the existence results related to nonlinear evolution equations leans heavily on certain a priori estimates resulting from the constraints represented by the equation itself as well as by some boundary or initial conditions.

The classical maximum principle for linear parabolic problems found its generalization in the concept of invariant regions connected with nonlinear parabolic systems arising frequently as singular perturbations of strictly hyperbolic conservation laws (see Chueh, Conley, Smoller [1]).

While the many applications in themselves justified the widespread interest in L_∞-bounds for solutions of parabolic problems, the advantage of the method became even more transparent in conjunction with the successful treatment of certain nonlinear hyperbolic systems via the compensated compactness method (cf. DiPerna [3], Serre [7], Rascle [6] etc.).

The present paper attempts to answer similar questions associated with a nonhomogeneous weakly damped wave equation of the form

$$U_{tt} + dU_t - \sigma(U_x)_x + g(U) = f(x, t)\,, \quad d > 0\,, \tag{1.1}$$

the unknown function $U = U(x, t)$ of $x \in [0, l]$, $t \in I \subset R^1$ obeying the Dirichlet boundary conditions

$$U(0, t) = U(l, t) = 0\,, \quad t \in I\,. \tag{1.2}$$

The need of L_∞-estimates arises, for instance, when looking for time-periodic or, more generally, bounded global solutions of (1.1), (1.2) with help of the method of *vanishing viscosity* (see [4], [5]).

Let us remark that the existence of invariant regions for the corresponding parabolic regularization provides the desired estimates which are *uniform in* $t \in I = R^1$.

Note in passing that the results of Dafermos [2] ensure similar bounds on the compact time-intervals I only.

2. GENERAL CONSIDERATIONS

As to the function $\sigma: R^1 \to R^1$, we suppose that

$$(2.1) \qquad \sigma'(u) \geqq \sigma_0 > 0 \quad \text{for all} \quad u \in R^1 \,,$$

$$(2.2) \qquad \sigma''(u)\, u > 0 \quad \text{for all} \quad u \neq 0 \,,$$

σ having all prerequisite properties concerning smoothness for the analysis to be valid.

Strangely enough, even the simplest case $g \equiv 0$ brings forth unexpectable difficulties. Indeed, setting (as usual)

$$U_t = v \,, \quad U_x = u$$

we are led to the system

$$(2.3)_1 \qquad u_t - v_x = 0 \,,$$

$$(2.3)_2 \qquad v_t - \sigma(u)_x + dv - f(x, t) = 0$$

with the parabolic regularization

$$(2.4)_1 \qquad u_t - v_x = \varepsilon u_{xx} \,,$$

$$(2.4)_2 \qquad v_t - \sigma(u)_x + dv - f(x, t) = \varepsilon v_{xx} \,, \quad \varepsilon > 0 \,.$$

Following the line of ideas from [2] we consider a general system

$$(2.5)_1 \qquad u_t - v_x + \varphi(u, v, x, t) = \varepsilon u_{xx} \,,$$

$$(2.5)_2 \qquad v_t - \sigma(u)_x + \psi(u, v, x, t) = \varepsilon v_{xx}$$

along with the Riemann invariants

$$r = r(u, v) = v + \textstyle\int_0^u \surd(\sigma')(z)\, \mathrm{d}z \,,$$

$$s = s(u, v) = v - \textstyle\int_0^u \surd(\sigma')(z)\, \mathrm{d}z \,.$$

Now, the results of Chueh, Conley, Smoller [1] imply that the set

$$M_c = \{(u, v) \mid -c \leqq r(u, v),\ s(u, v) \leqq c\} \,, \quad c > 0$$

forms an invariant region for the system (2.5) (for the precise definition see Section 3) whenever the inequality

$$(2.6) \qquad \operatorname{sgn}(u) \surd(\sigma')(u)\, \varphi(u, v, x, t) + \operatorname{sgn}(v)\, \psi(u, v, x, t) > 0$$

holds for all x, t, $[u, v] \in \partial M_c$ (cf. Dafermos [2, Formula (1.4)].

Due to the fact that $\varphi \equiv 0$ in $(2.4)_1$, the condition (2.6) does not hold for the system (2.4) no matter how large the number c may be chosen.

Another choice

$$U_t + dU = v \,, \quad U_x = u$$

gives rise to the system

$$(2.7)_1 \qquad u_t - v_x + du = \varepsilon u_{xx}\,,$$

$$(2.7)_2 \qquad v_t - \sigma(u)_x - f(x, t) = \varepsilon v_{xx}\,.$$

One observes easily that (2.6) is not satisfied again.

Fortunately, the third possibility seems to guarantee the desirable result. At this point, we pause in our rigour to make the main ideas clear. The exact proof of a more general assertion will be given in Section 4.

For $\delta > 0$ small, constants a_1, a_2, $0 < a_1 \leqq a_2$ can be found such that

$$d = a_1 + a_2\,, \quad \delta = a_1 a_2\,.$$

Adding the term δU to both sides of (1.1) and letting

$$(2.8) \qquad U_t + a_1 U = v\,, \quad U_x = u$$

we get the regularized system of the form

$$(2.9)_1 \qquad u_t - v_x + a_1 u = \varepsilon u_{xx}\,,$$

$$(2.9)_2 \qquad v_t - \sigma(u)_x + a_2 v - \delta U - f(x, t) = \varepsilon v_{xx}$$

where, according to (1.2),

$$(2.10) \qquad U(x, t) = \textstyle\int_0^x u(z, t)\,\mathrm{d}z\,.$$

If we assume

$$(2.11) \qquad |f(x, t)| \leqq f_0 \quad \text{for all} \quad x, t\,,$$

the condition (2.6) takes the form

$$(2.12) \qquad a_1|u|\,\surd(\sigma')(u) + a_2|v| > \delta\left|\textstyle\int_0^x u(z, t)\,\mathrm{d}z\right| + f_0$$

for $u, v \in \partial M_c$, $[u(x, t), v(x, t)] \in M_c$ for all x, t.

In view of (2.2) we immediately obtain

Lemma 1. *Let σ satisfy* (2.1), (2.2).
Then

$$(2.13) \qquad |u|\,\surd(\sigma')(u) + |v| \geqq c$$

whenever $[u, v] \in \partial M_c$.

Taking Lemma 1 into account we have

$$a_1|u|\,\surd(\sigma')(u) + a_2|v| \geqq a_1 c$$

for any $[u, v] \in \partial M_c$.

On the other hand, if $[u(x, t), v(x, t)] \in M_c$ we deduce

$$l\,F^{-1}(-c) \leqq \textstyle\int_0^x u(z, t)\,\mathrm{d}z \leqq l\,F^{-1}(c)$$

where

$$(2.14) \qquad F(u) = \textstyle\int_0^u \surd(\sigma')(z)\,\mathrm{d}z\,.$$

Thus (2.12) reduces to

$$a_1 c > \delta l \max\{-F^{-1}(-c), F^{-1}(c)\} + f_0 . \tag{2.15}$$

Suppose, for example, that

$$\lim_{|z| \to \infty} \sqrt{(\sigma')}\,(z) = +\infty . \tag{2.16}$$

Then F^{-1} is sublinear and, consequently, (2.15) holds for $c > 0$ sufficiently large. In other words, the set M_c is an invariant region for the system (2.9).

3. MAIN RESULTS

Repeating the procedure from Section 2 we can transform the equation (1.1) to the system

$$u_t - v_x + a_1 u = \varepsilon u_{xx} , \tag*{$(3.1)_1$}$$

$$v_t - \sigma(u)_x + a_2 v - \delta U + g(U) - f(x. t) = \varepsilon v_{xx} \tag*{$(3.1)_2$}$$

where the function U is determined by (2.10).

For later purposes, it seems convenient to work with the functions u, v defined (and smooth) on the whole real line. With the boundary conditions (1.2) in mind, the suitable way to achieve this is to consider periodic functions belonging to certain symmetry classes, namely, we postulate

$$u(x + 2l, t) = u(x, t) , \quad u(-x, t) = u(x, t) , \tag{3.2}$$

$$v(x + 2l, t) = v(x, t) , \quad v(-x, t) = -v(x, t) \tag{3.3}$$

for all $x \in R^1$, $t \in I = [0, t_0)$.

The Cauchy data

$$u(x, 0) = u^0(x) , \quad v(x, 0) = v^0(x) , \quad x \in R^1 \tag{3.4}$$

where (of course)

$$\begin{aligned} &u^0(x + 2l) = u^0(x) , \quad u^0(-x) = u^0(x) , \\ &v^0(x + 2l) = v^0(x) , \quad v^0(-x) = -v^0(-x) , \quad x \in R^1 \end{aligned} \tag{3.5}$$

complete the problem (3.1)–(3.4) to be well posed on condition that

$$f(x + 2l, t) = f(x, t) , \quad f(-x, t) = -f(x, t) , \tag{3.6}$$

$$g: R^1 \to R^1 \text{ is smooth with } g(-U) = -g(U) , \tag{3.7}$$

and, since $U(l, t) = 0$,

$$\textstyle\int_0^l u^0(z)\,\mathrm{d}z = 0 \tag{3.8}$$

(cf. [5]).

As to the solution pair (u, v), we will be interested exclusively in classical solutions, i.e.

$$u, v \in C(R^1 \times [0, t_0))\,, \quad u_t, v_t, u_x, v_x, u_{xx}, v_{xx} \in C(R^1 \times (0, t_0))$$

satisfying the equations (3.1) together with (3.2)–(3.4) pointwise.

Definition 1. A domain $M \subset R^2$ is called an *invariant region* related to the problem (3.1)–(3.4) if the solution (u, v) satisfying the condition

$$(3.9) \qquad [u^0(x), v^0(x)] \in M \quad \text{for all} \quad x \in R^1$$

is bound to remain in M; more specifically,

$$(3.10) \qquad [u(x, t), v(x, t)] \in M \quad \text{for all} \quad x \in R^1\,, \quad t \in [0, t_0)\,.$$

In the present paper, our aim is to establish the following theorem while the applications of the result to the time-periodic solutions of a quasilinear damped wave equation will be given in [5].

Theorem 1. *Let the data satisfy the following conditions: As to the function* σ *we require* (2.1), (2.2), (2.16), f *satisfies* (2.11), (3.6), g *is as in* (3.7), *and finally, the conditions* (3.5), (3.8) *hold for* u^0, v^0.

Moreover, let

$$(3.11) \qquad \lim_{|z| \to \infty} \frac{g'(z)}{\sqrt{(\sigma')(z)}} = 0$$

hold.

Then the set M_c *defined in Section 2 is an invariant region of the system* (3.1)–(3.4) *whenever the number* c *is large. The sufficient magnitude of* c *does not depend on* $\varepsilon > 0$.

4. THE PROOF OF THEOREM 1

(A) To begin with, we are going to show that the function U given by (2.10) satisfies (3.3).

To see this, we only have to prove

$$(4.1) \qquad \int_0^l u(z, t)\, \mathrm{d}z = 0 \quad \text{for all} \quad t \in [0, t_0)\,.$$

In view of (3.2), (3.3), we are allowed to integrate $(3.1)_1$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{a_1 t} \int_0^l u(z, t)\, \mathrm{d}z = 0\,, \quad t > 0$$

which, combined with (3.8), yields (4.1).

(B) Assume that M_c is not invariant. Thus, there exists a solution (u, v) of (3.1)–(3.4) satisfying (3.9) and, for certain (x_1, t_1), $\varrho > 0$, we have

$$(4.2) \qquad [u^1, v^1] \in \partial M_{c+\varrho} \quad \text{with} \quad u^1 = u(x_1, t_1)\,, \quad v^1 = v(x_1, t_1)\,.$$

Moreover, $t_1 > 0$ may be found such that (4.2) holds along with

$$(4.3) \qquad [u(x, t), v(x, t)] \in M_{c+\varrho} \quad \text{for all} \quad x \in R^1\,, \quad t \in [0, t_1]\,.$$

Seeing that the situation exhibits certain symmetry we are allowed to restrict ourselves to the case $u^1, v^1 \geqq 0$. Consequently,

$$r(u^1, v^1) = c + \varrho$$

and, according to (4.3),

$$r_x\big|_{(x_1,t_1)} = 0\,, \quad r_{xx}\big|_{(x_1,t_1)} \leqq 0\,, \tag{4.4}$$

$$r_t\big|_{(x_1,t_1)} \geqq 0\,, \tag{4.5}$$

$r = r(u, v)$ being viewed as a function of x, t.

Our goal is to show that (4.5) is not possible provided $c > 0$ is large enough.

Since (u, v) solves (3.1), one obtains

$$\begin{aligned} r_t = r_u u_t + r_v v_t = \sqrt{(\sigma')}(u)\,(v_x + \sqrt{(\sigma')}(u)\,u_x) + \\ + \varepsilon(v_{xx} + \sqrt{(\sigma')}(u)\,u_{xx}) - a_2 v - a_1 \sqrt{(\sigma')}(u)\,u + \\ + \delta U - g(U) + f(x, t)\,. \end{aligned}$$

Now, the relation (4.4) may be rewritten as

$$r_x = v_x + \sqrt{(\sigma')}(u)\,u_x\big|_{(x_1,t_1)} = 0\,,$$

$$r_{xx} = v_{xx} + \sqrt{(\sigma')}(u)\,u_{xx} + \frac{1}{2}\,\frac{\sigma''(u)}{\sqrt{(\sigma')}(u)}\,u_x^2\big|_{(x_1,t_1)} \leqq 0\,.$$

Hence, by virtue of (2.2), we conclude

$$r_t\big|_{(x_1,t_1)} \leqq -a_2 v^1 - a_1 \sqrt{(\sigma')}(u^1)\,u^1 + \delta U(x_1, t_1) - g(U(x_1, t_1)) + f_0\,. \tag{4.6}$$

Lemma 1 together with (4.2) imply

$$a_2 v^1 + a_1 \sqrt{(\sigma')}(u^1)\,u^1 \geqq a_1(c + \varrho)\,. \tag{4.7}$$

As a consequence of (4.1), (4.3), we have

$$lF^{-1}(-c - \varrho) \leqq U(x_1, t_1) \leqq lF^{-1}(c + \varrho)$$

(F is determined by (2.14)).

With the desirable relation $r_t|_{(x_1,t_1)} < 0$ in mind, we only need to estimate the term $g(U(x_1, t_1))$, the sum $\delta U(x_1, t_1) + f_0$ being treated analogously as in Section 2.

We get

$$|g(U(x_1, t_1))| \leqq \max\left\{|g(l\,F^{-1}(z))| \mid z \in [-c - \varrho,\ c + \varrho]\right\}.$$

Taking (4.7) into account we need to show

$$\lim_{|z| \to \infty} \frac{g(l\,F^{-1}(z))}{z} = 0\,. \tag{4.8}$$

With help of the standard L'Hospital rule, the relation (4.8) follows from (3.11).

Thus, we have completed the proof of Theorem 1.

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