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SOME DENSITY THEOREMS FOR TOEPLITZ OPERATORS ON BERGMAN SPACES

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The set of all Toeplitz operators on the Bergman space of the unit disc is shown to be dense in the set of all bounded linear operators, in the weak operator topology. Some results concerning norm density are also given.

1. INTRODUCTION

Let H^2 be the Hardy space on the unit circle $\partial\mathbb{D}$ and let $\varphi \in L^\infty(\partial\mathbb{D})$. The Toeplitz operator with symbol φ is the operator on H^2 sending x into $P_+\varphi x$, where P_+ is the orthogonal projection of $L^2(\partial\mathbb{D})$ onto H^2. It is easily seen that

$$T_z^* T_\varphi T_z = T_\varphi \quad \text{for any} \quad \varphi \in L^\infty(\partial\mathbb{D}) .$$

According to a classical result, the converse also holds: if some operator $T: H^2 \to H^2$ satisfies $T_z^* T T_z = T$, then $T = T_\varphi$ for some $\varphi \in L^\infty(\partial\mathbb{D})$. This result serves as a starting point for the theory of symbols of operators (cf. [6], [7], [4]). It shows that, loosely speaking, there are only few Toeplitz operators on H^2.

Consider now the space A^2, the (closed) subspace of $L^2(\mathbb{D})$ consisting of functions analytic in the unit disc $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, we can define the Toeplitz operator T_φ, acting on A^2, in the same way as above. In my paper [2] I have shown that these operators do not admit the characterization as above. More precisely, if $AT_\varphi B = T_\varphi$ for all $\varphi \in L^\infty(\mathbb{D})$, then $A = cI$, $B = c^{-1}I$ for some (nonzero) complex number c.

A natural question to ask is if this is not because there are, loosely speaking, more operators which are Toeplitz than in the classical (i.e. H^2) case. To put it precisely, we can ask if the following statements are true:

(a) the Toeplitz operators are dense (in some topology) in $\mathscr{B}(A^2)$;

(b) every finite dimensional operator is Toeplitz;

(c) for any linearly independent $f, g \in A^2$, there exists $\varphi \in L^\infty(\mathbb{D})$ such that $T_\varphi f = g$.

Clearly (b) implies (a) in strong operator topology, and either (a) or (c) implies the impossibility of the above-mentioned characterization $AT_\varphi B = T_\varphi$.

The statement (b) is easily seen not to be true. For example, there is no $\varphi \in L^\infty(\mathbb{D})$ such that $T_\varphi = \langle \cdot, \mathbf{1} \rangle \mathbf{1}$. This fact is an easy consequence of the Müntz-Szász theorem for L^2 spaces (see, for instance, [1]). In fact, this theorem yields a stronger result: if $T_\varphi = \langle \cdot, f \rangle g$ for some $\varphi \in L^\infty(\mathbb{D})$ and $f, g \in A^2$, $f(z) = \sum f_n z^n$, $g(z) = \sum g_n z^n$, then

$$\sum_{n: f_n = 0} n^{-1} < \infty \quad \text{and} \quad \sum_{n: g_n = 0} n^{-1} < \infty .$$

(Loosely speaking, only few Taylor coefficients of f and g can be zero.) It is a conjecture of author's that in fact there are no finite dimensional Toeplitz operators at all.

In this paper, (a) and (b) are considered. (c) is shown to be false (cf. Remark at the end of the second paragraph); (a) is true in the strong operator topology. Again, there is a conjecture that it holds in the norm topology as well. A proof of this statement appeared in [3], but it seems to contain a gap, so the problem remains still open. Some results from this field are presented in paragraph three.

Following notations shall prove useful. $\mathbb{D}$ is the unit disc $\{z \in \mathbb{C}: |z| < 1\}$ in the complex plane $\mathbb{C}$; $\partial\mathbb{D}$ is its boundary, i.e. the unit circle; $\mathrm{d}z$ is the (planar) Lebesgue measure on $\mathbb{C}$, normalized so that $\mathbb{D}$ has measure 1. The Bergman space A^2 is the space of all analytic and square integrable functions on $\mathbb{D}$. $\mathscr{B}(A^2)$, resp. $\mathscr{K}(A^2)$, resp. $\|\cdot\|_2$, resp. $\|\cdot\|$ denote the space of all bounded linear, resp. compact operators on A^2, resp. the norm on A^2, resp. the operator norm on $\mathscr{B}(A^2)$. The space A^2 is a reproducing kernel space, the reproducing kernel at $a \in \mathbb{D}$ being given by

$$g_a(x) = (1 - a^* x)^{-2} .$$

More generally, for $a \in \mathbb{D}$ and $m = 0, 1, 2, \ldots$, the functions

$$g_{m,a}(x) = \frac{(m+1)!\, z^m}{(1 - a^* z)^{m+2}} \tag{1}$$

belong to A^2 and, for arbitrary $f \in A^2$,

$$\langle f, g_{m,a} \rangle = f^{(m)}(a) ,$$

the m-th derivative of f at a.

In the third paragraph, we are going to use the Hilbert space ℓ^2 of all square-summable sequences $\{a_n\}_{n=0}^\infty$ with the usual inner product. The ℓ^2 norm of a sequence $\{a_n\}_{n=0}^\infty \in \ell^2$ will be denoted $\|a\|_2$ (there is no danger of confusion with the A^2 norm). Finally, if X is a subset of $\mathscr{B}(A^2)$, clos X denotes the closure of X in the norm topology of $\mathscr{B}(A^2)$.

2. AN INTERPOLATION PROPERTY

Theorem 1. *Let $T \in \mathscr{B}(A^2)$, $F_k, G_k \in A^2$ $(k = 1, 2, \ldots, N)$. Then there exists $\varphi \in L^\infty(\mathbb{D})$ such that*

$$\langle T_\varphi F_k, G_k \rangle = \langle TF_k, G_k \rangle , \quad k = 1, 2, \ldots, N .$$

Proof. Let $f_1, f_2, \ldots, f_n$, resp. $g_1, g_2, \ldots, g_m$ be a basis of the (finite-dimensional) subspace of A^2 generated by $F_1, \ldots, F_N$, resp. $G_1, \ldots, G_N$. Clearly it's sufficient to find $\varphi \in L^\infty(\mathbb{D})$ such that

$$\langle T_\varphi f_i, g_j \rangle = \langle T f_i, g_j \rangle$$

for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Consider the operator $R\colon L^\infty(\mathbb{D}) \to \mathbb{C}^{n \times m}$, defined by the formula

$$(R_\varphi)_{ij} = \int_{\mathbb{D}} \varphi(z) f_i(z)\, g_j(z)^*\, \mathrm{d}z = \langle T_\varphi f_i, g_j \rangle .$$

Suppose some $u \in \mathbb{C}^{n \times m}$ is orthogonal to the range of R, i.e.

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (R\varphi)_{ij}\, u_{ij}{}^* = 0 \quad \text{for all} \quad \varphi \in L^\infty(\mathbb{D}) .$$

This means that

(2) $$\int_{\mathbb{D}} \varphi(z) \sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij}^* f_i(z)\, g_j(z)^*\, \mathrm{d}z = 0$$

for all $\varphi \in L^\infty(\mathbb{D})$, which implies

(3) $$\sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij}^* f_i(z)\, g_j(z)^* = 0$$

$\mathrm{d}z$-almost everywhere in $\mathbb{D}$. Since the left-hand side is obviously continuous in $\mathbb{D}$, this equality holds, in fact, on the whole of $\mathbb{D}$. Consequently, the function

$$F(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij}^* f_i(x)\, g_j(y^*)^* ,$$

which is analytic in $\mathbb{D} \times \mathbb{D}$, equals zero whenever $x = y^*$. By the well-known uniqueness theorem, this implies that F is identically zero on $\mathbb{D} \times \mathbb{D}$. Because the functions f_i, $i = 1, \ldots, n$, are linearly independent, we have

$$\sum_{j=1}^{m} u_{ij}\, g_j(y^*) = 0 \quad \text{for all} \quad y \in \mathbb{D}\,, \quad i = 1, 2, \ldots, n\,;$$

but g_j, $j = 1, 2, \ldots, m$, are also linearly independent, and so $u_{ij} = 0$ for all i, j, i.e. $u = 0$. This means that the range of R is all of $\mathbb{C}^{n \times m}$, which immediately yields the desired conclusion. □

Corollary. *The set* $\mathscr{T} = \{T_\varphi\colon \varphi \in L^\infty(\mathbb{D})\}$ *is dense in* $\mathscr{B}(A^2)$ *in SOT (the strong operator topology).*

Proof. In view of the preceding theorem, it is certainly dense in WOT (the weak operator topology). Because $\mathscr{T}$ is a subspace, i.e. a convex set, its WOT- and SOT-closures coincide. □

Note that the crucial step in the proof of Theorem 1 was the implication (2) ⇒ (3). Thus, the theorem remains in force if we have (2) only for $\varphi \in C(\mathrm{cl}\, \mathbb{D})$, or even $\mathscr{D}(\mathbb{D})$ (the set of all infinitely differentiable functions on $\mathbb{D}$, whose support is a compact subset of $\mathbb{D}$) – any weak-star dense subset of $L^\infty(\mathbb{D})$ will do. As a consequence, we get the following theorem.

Theorem 1′. *The set* $\mathcal{T}_1 = \{T_\varphi : \varphi \in \mathcal{D}(\mathbb{D})\}$ *is SOT-dense in* $\mathcal{B}(A^2)$.

Remark. Theorem 1 suggests the following question: given $f_i, g_i \in A^2$ ($i = 1, 2, \ldots, n$), does there exist $\varphi \in L^\infty(\mathbb{D})$ such that

$$T_\varphi f_i = g_i \quad (i = 1, 2, \ldots, n)\,?$$

This result would immediately imply (c) from the introduction. Unfortunately, it is not true. To see this, it's sufficient to consider $n = 1, f_1 = \mathbf{1}$. If there were, for every $g \in A^2$, some $\varphi \in L^\infty(\mathbb{D})$ such that $g = T_\varphi \mathbf{1}(= P_+\varphi)$, then the mapping (here A^{2d} stands for the dual space of A^2)

$$A\colon L^\infty(\mathbb{D}) \to A^{2d}\,, \quad A\varphi =\cdot \langle \cdot, P_+\varphi^* \rangle = \langle \cdot, \varphi^* \rangle_{L^2(\mathbb{D})}\,,$$

would be onto. Let B be the operator of inclusion of A^2 into $L^1(\mathbb{D})$:

$$B\colon A^2 \to L^1(\mathbb{D})\,, \quad B\psi = \psi\,.$$

This is a continuous operator (by the Schwarz inequality) and has A as its adjoint: $B^d = A$. By the Hausdorff normal solvability theorem (cf. [8], chapter VII, § 5), Ran A is closed if and only if Ran B is closed. Because B is injective, Ran B is closed if and only if B is bounded below (just use the open mapping theorem). But the norm of z^n in $L^1(\mathbb{D})$ is

$$\int_{\mathbb{D}} |z^n| \, dz = \int_0^1 r^n \, 2r \, dr = \frac{2}{n+2}\,.$$

whereas the norm of z^n in A^2 is

$$\|z^n\|_2 = \left(\int_{\mathbb{D}} |z^n|^2 \, dz\right)^{1/2} = (n+1)^{-1/2}\,.$$

Consequently, B is not bounded below, so Ran A is not closed, and A cannot be surjective. □

(The last part of the argument can be avoided by evoking directly the fact that the closure of A^2 in $L^1(\mathbb{D})$ is A^1, the space of all integrable analytic functions on $\mathbb{D}$. Our method is more elementary.)

3. UNIFORM APPROXIMATION

As we have seen, the set

$$\mathcal{T}_1 = \{T_\varphi : \varphi \in \mathcal{D}(\mathbb{D})\}$$

in SOT-dense in $\mathcal{B}(A^2)$. In this paragraph we determine its closure in the uniform (i.e. norm) topology on $\mathcal{B}(A^2)$. It turns out to be $\mathcal{K}(A^2)$, the space of all compact operators on A^2.

Denote by $T_{(m,n,a)}$ the operator on A^2 given by

$$f \mapsto \langle f, g_{m,a} \rangle \, g_{n,a}\,,$$

where $a \in \mathbb{D}$, m and n are non-negative integers, and $g_{m,a}$ is given by the formula (1). One has

$$\langle T_{(m,n,a)} f, g \rangle = f^{(m)}(a)\, g^{(n)}(a)^*$$

for arbitrary $f, g \in A^2$.

Lemma 2. *Let M, N be non-negative integers, $a \in \mathbb{D}$, and denote*

$$R_{(M,N,a,t)} = \frac{T_{(M,N,a+t)} - T_{(M,N,a)}}{2t} - \mathrm{i}\,\frac{T_{(M,N,a+\mathrm{i}t)} - T_{(M,N,a)}}{2t}\,.$$

Then $R_{(M,N,a,t)}$ tends to $T_{(M+1,N,a)}$ (in norm) as the real number t tends to zero:

$$\lim_{\mathbb{R}\ni t\to 0} \left\| R_{(M,N,a,t)} - T_{(M+1,N,a)} \right\| = 0\,.$$

Similarly,

$$\lim_{\mathbb{R}\ni t\to 0} \left\| R'_{(M,N,a,t)} - T_{(M,N+1,a)} \right\| = 0\,,$$

where

$$R'_{(M,N,a,t)} = \frac{T_{(M,N,a+t)} - T_{(M,N,a)}}{2t} + \mathrm{i}\,\frac{T_{(M,N,a+\mathrm{i}t)} - T_{(M,N,a)}}{2t}\,.$$

Proof. Let $F, G \in A^2$ and denote, for a while, $f = F^{(M)}$ and $g = G^{(N)}$. Then

$$\langle (R_{(M,n,a,t)} - T_{(M+1,N,a)})\, F, G \rangle = \tag{4}$$

$$= \frac{f(a+t)\, g(a+t)^* - f(a)\, g(a)^*}{2t} - \mathrm{i}\,\frac{f(a+\mathrm{i}t)\, g(a+\mathrm{i}t)^* - f(a)\, g(a)^*}{2t} -$$

$$- f'(a)\, g(a)^*\,.$$

Let

$$f(x) = \sum_0^\infty f_n (x-a)^n\,, \quad g(x) = \sum_0^\infty g_n (x-a)^n$$

be the Taylor expansions of f, resp. g at a. These series are locally uniformly convergent in the disc $|x - a| < 1 - |a|$. Consequently, for $|t| < 1 - |a|$ the right-hand side of (4) equals to

$$\frac{1}{2t}\Big(\sum_{\substack{m,n\geqq 0\\ (m,n)\neq(0,0)}} f_m t^m g_n^* t^n \Big) - \frac{\mathrm{i}}{2t}\Big(\sum_{\substack{m,n\geqq 0\\ (m,n)\neq(0,0)}} f_m (\mathrm{i}t)^m\, g_n^*(-\mathrm{i}t)^n \Big) - f_1 g_0^*\,.$$

Rearranging all terms into one series, the terms corresponding to $(m, n) = (1, 0)$ and $(0, 1)$ cancel, and we get

$$\tfrac{1}{2} \sum_{\substack{m,n\geqq 0\\ m+n\geqq 2}} f_m g_n^* t^{m+n-1} (1 - \mathrm{i}^{m-n+1})\,. \tag{5}$$

Let F_n, resp. G_n be the coefficients of the Taylor expansion of the function F, resp. G at a. Because $f = F^{(M)}$, we have

$$f_m = \frac{1}{m!} f^{(m)}(a) = \frac{1}{m!} F^{(M+m)}(a) = \frac{(M+m)!}{m!} F_{M+m}\,,$$

and similarly for g_n. It follows that (5) is equal to

$$(6)\qquad \frac{1}{2}\sum_{\substack{m,n\geqq 0\\ m+n\geqq 2}} \frac{(M+m)!}{m!} F_{m+M} \frac{(N+n)!}{n!} G^*_{n+N} t^{m+n-1}(1-\mathrm{i}^{m-n+1}).$$

We are going to estimate the absolute value of the last expression in terms of $\|F\|_2$ and $\|G\|_2$. One has

$$\|F\|_2^2 = \int_D |F(z)|^2 \,\mathrm{d}z \geqq \int_{|z-a|<1-|a|} |F(z)|^2 \,\mathrm{d}z =$$

$$= \int_0^{1-|a|} \int_0^{2\pi} \sum_{j,k\geqq 0} F_j F_k^* r^{j+k} \mathrm{e}^{(j-k)it} \frac{r}{\pi} \,\mathrm{d}t\,\mathrm{d}r$$

(we have passed to polar coordinates). Since the Taylor series

$$F(z) = \sum_0^\infty F_n(z-a)^n$$

is locally uniformly convergent on the disc $|z-a| < 1-|a|$, we can interchange the integration and summation signs and get

$$(7)\qquad \|F\|_2^2 \geqq \sum_{j,k\geqq 0} \int_0^{1-|a|} \int_0^{2\pi} F_j F_k^* r^{j+k} \mathrm{e}^{(j-k)\mathrm{i}t} \frac{r}{\pi} \,\mathrm{d}t\,\mathrm{d}r =$$

$$= \sum_{k=0}^\infty \frac{(1-|a|)^{2k+2}}{k+1} |F_k|^2 .$$

Similar estimate holds for G. Denote, for a while,

$$\alpha_k = \frac{(1-|a|)^{k+1}}{(k+1)^{1/2}} |F_k|, \quad \beta_k = \frac{(1-|a|)^{k+1}}{(k+1)^{1/2}} |G_k| .$$

Then, according to (7), α and β belong to ℓ^2 and

$$\|\alpha\|_2 \leqq \|F\|_2, \quad \|\beta\|_2 \leqq \|G\|_2 .$$

Returning to our previous calculations, we see that the absolute value of (6) is not greater than

$$\frac{1}{2}\sum_{\substack{m,n\geqq 0\\ m+n\geqq 2}} \frac{(M+m)!}{m!} |F_{m+M}| \frac{(N+n)!}{n!} |G_{n+N}|\, t^{m+n-1} .2 =$$

$$= \sum_{\substack{m,n\geqq 0\\ m+n\geqq 2}} \frac{(M+m)!}{m!} \frac{(M+m+1)^{1/2}\, t^{m-1/2}}{(1-|a|)^{M+m+1}} \alpha_{M+m} \,.$$

$$.\ \frac{(N+n)!}{n!} \frac{(N+n+1)^{1/2}\, t^{n-1/2}}{(1-|a|)^{N+n+1}} \beta_{N+n} .$$

Break this sum into three parts, namely

$$\sum_{m,n\geqq 1} + \sum_{\substack{m=0\\ n\geqq 2}} + \sum_{\substack{n=0\\ m\geqq 2}} =^{\mathrm{def}} \sigma_1 + \sigma_2 + \sigma_3 .$$

Let's first consider σ_1. We have, obviously,

$$\sigma_1 = \left(\sum_{m=1}^{\infty} \frac{(M+m)!}{m!} \frac{(M+m+1)^{1/2}\, t^{m-1/2}}{(1-|a|)^{M+m+1}} \alpha_{M+m}\right) \cdot \tag{8}$$

$$\cdot \left(\sum_{n=1}^{\infty} \frac{(N+n)!}{n!} \frac{(N+n+1)^{1/2}\, t^{m-1/2}}{(1-|a|)^{N+n+1}} \beta_{N+n}\right).$$

According to the Cauchy-Schwarz inequality, the first factor on the right-hand side is less than or equal to

$$\|\alpha\|_2 \left(\sum_{m=1}^{\infty} \frac{(M+m)!^2}{m!^2} \frac{(M+m+1)\, t^{2m-1}}{(1-|a|)^{2M+2m+2}}\right)^{1/2} = \tag{9}$$

$$= \|\alpha\|_2 \left(t \sum_{m=0}^{\infty} \frac{(M+m+1)!^2}{(m+1)!^2} \frac{(M+m+2)\, t^{2m}}{(1-|a|)^{2M+2m+4}}\right)^{1/2}.$$

Since

$$\lim_{m\to\infty} \frac{(M+m+1)!^2}{(m+1)!^2} (M+m+2) \binom{m+2M+1}{2M+1}^{-1} =$$

$$= (2M+1)! < +\infty\,,$$

there exists a number $c_1(M) > 0$, depending only on M, such that

$$\frac{(m+M+1)!^2}{(m+1)!^2} (M+m+2) \leqq c_1(M) \binom{m+2M+1}{2M+1}.$$

Thus the right-hand side of (9) is less than or equal to

$$\|\alpha\|_2 \left(c_1(M)\, t \sum_{m=0}^{\infty} \binom{m+2M+1}{2M+1} t^{2m} \frac{1}{(1-|a|)^{2M+2m+4}}\right)^{1/2} =$$

$$= \frac{\|\alpha\|_2\, t^{1/2}\, c_1(M)^{1/2}}{(1-|a|)^{M+2}} \left[1 - \frac{t^2}{(1-|a|)^2}\right]^{-1-M}.$$

A similar estimate holds for the second factor in (8). Putting these two estimates together, we see that

$$\sigma_1 \leqq \|\alpha\|_2\, \|\beta\|_2\, tc_2(M, N, a, t) \leqq \|F\|_2\, \|G\|_2\, tc_2(M, N, a, t)\,, \tag{10}$$

where

$$c_2(M, N, a, t) = \frac{c_1(M)^{1/2}\, c_1(N)^{1/2}}{(1-|a|)^{M+N+4}} \left[1 - \frac{t^2}{(1-|a|)^2}\right]^{-2-M-N}$$

tends to a finite limit as $t \to 0$.

Now let's turn our attention to σ_2. We have

$$\sigma_2 \leqq \frac{(M+1)^{1/2}}{(1-|a|)^{M+1}} \|\alpha\|_2 \left[\sum_{n=2}^{\infty} \frac{(N+n)!}{n!} \frac{(N+n+1)^{1/2}}{(1-|a|)^{N+n+1}} t^{n-1} \beta_{N+n}\right].$$

Using the Cauchy-Schwarz inequality shows that the bracketed term is not greater than

$$\|\beta\|_2 \left(\sum_{n=2}^{\infty} \frac{(N+n)!^2}{n!^2} \frac{(N+n+1)}{(1-|a|)^{2N+2n+2}} t^{2n-2} \right)^{1/2},$$

and, going through the same calculations as above, this is seen to be less than or equal to

$$\|\beta\|_2 \left[c_1(N) \frac{1}{(1-|a|)^{2N+4}} \sum_{n=1}^{\infty} \binom{n+2N+1}{2N+1} \frac{t^{2n}}{(1-|a|)^{2n}} \right]^{1/2} \leqq$$

$$\leqq \|\beta\|_2 \, t c_3(N, a, t),$$

where

$$c_3(N, a, t) = \frac{c_1(N)^{1/2}}{(1-|a|)^{N+2}} \left[\frac{(1 - t^2(1-|a|)^{-2})^{-2N-2} - 1}{t^2} \right]^{1/2}$$

tends to a finite limit as $t \to 0$. Consequently,

$$\sigma_2 \leqq c_4(M, N, a, t) \|\alpha\|_2 \|\beta\|_2 \, t \leqq c_4(M, N, a, t) \|F\|_2 \|G\|_2 \, t,$$

with

$$c_4(M, N, a, t) = \frac{(M+1)^{1/2}}{(1-|a|)^{M+1}} c_3(N, a, t)$$

tending to a finite limit as $t \to 0$.

Similar estimate, of course, can be obtained for σ_3. Summing up, we see that

$$|\langle (R_{(M,N,a,t)} - T_{(M+1,N,a)}) F, G \rangle| \leqq c_5(M, N, a, t) \, t \|F\|_2 \|G\|_2$$

for all $F, G \in A^2$, where $c_5(M, N, a, t)$ tends to a finite limit as $t \to 0$. Consequently

$$\|R_{(M,N,a,t)} - T_{(M+1,N,a)}\| \leqq t c_5(M, N, a, t)$$

and the first part of the lemma follows. The assertion concerning $R'_{(M,N,a,t)}$ can be proved in the same way. □

We shall need one more lemma, the proof of which is (fortunately) a little shorter. Remember that the symbol „clos" denotes the closure in the norm topology of $\mathscr{B}(A^2)$.

Lemma 3. *Denote* $\mathscr{T}_1 = \{T_\varphi : \varphi \in \mathscr{D}(\mathbb{D})\}$. *Then*

$$T_{(0,0,a)} \in \operatorname{clos} \mathscr{T}_1 \quad \textit{for every} \quad a \in \mathbb{D}.$$

Proof. For each δ in the interval $(0, 1 - |a|)$, pick a function $f_\delta \in \mathscr{D}(\mathbb{D})$ such that

$$f_\delta(z) = 0 \quad \text{if} \quad |z - a| \geqq \delta + \delta^2,$$

$$f_\delta(z) = \delta^{-2} \quad \text{if} \quad |z - a| \leqq \delta,$$

and

$$0 \leqq f_\delta(z) \leqq \delta^{-2} \quad \text{if} \quad \delta < |z - a| < \delta + \delta^2.$$

Let $f, g \in A^2$. Then

$$\langle (T_{f_\delta} - T_{(0,0,a)}) f, g \rangle = \int_D f_\delta(z) f(z)\, g(z)^* \,\mathrm{d}z - f(a)\, g(a)^* = \tag{11}$$
$$= \left[\delta^{-2} \int_{|z-a| \leqq \delta} f(z)\, g(z)^* \,\mathrm{d}z - f(a)\, g(a)^*\right] +$$
$$+ \int_{\delta < |z-a| < \delta + \delta^2} f_\delta(z) f(z)\, g(z)^* \,\mathrm{d}z =^{\mathrm{def}} \varrho_1 + \varrho_2 \,.$$

Let f_n, resp. g_n be the coefficients of the Taylor expansion of f, resp. g at a:

$$f(x) = \sum_0^\infty f_n(x-a)^n \,, \quad g(x) = \sum_0^\infty g_n(x-a)^n \,.$$

Substituting these formulas into the expression for ϱ_1 gives

$$\varrho_1 = \delta^{-2} \int_{|z-a| \leqq \delta} \sum_{m,n=0}^\infty f_n(z-a)^n\, g_m^*(z-a)^{*m} \,\mathrm{d}z - f_0 g_0^* =$$
$$= \delta^{-2} \int_0^\delta \int_0^{2\pi} \sum_{m,n=0}^\infty f_n g_m^* r^{n+m} \mathrm{e}^{(n-m)it} \frac{r}{\pi} \,\mathrm{d}t \,\mathrm{d}r - f_0 g_0^* =$$
$$= \delta^{-2} \sum_{n=0}^\infty f_n g_n^* \frac{\delta^{2n+2}}{n+1} - f_0 g_0^* = \sum_{n=1}^\infty f_n g_n^* \frac{\delta^{2n}}{n+1} \,.$$

Denote again, for a little while,

$$\alpha_n = \frac{(1-|a|)^{n+1}}{(n+1)^{1/2}} |f_n| \,, \quad \beta_n = \frac{(1-|a|)^{n+1}}{(n+1)^{1/2}} |g_n| \,.$$

In course of the proof of Lemma 2, we have seen that α and β belong to ℓ^2 and

$$\|\alpha\|_2 \leqq \|f\|_2 \,, \quad \|\beta\|_2 \leqq \|g\|_2 \,.$$

Now

$$\sum_{n=1}^\infty |f_n g_n^*| \frac{\delta^{2n}}{n+1} = \sum_{n=1}^\infty \alpha_n \beta_n \frac{\delta^{2n}}{(1-|a|)^{2n+2}} =$$
$$= \frac{\delta^2}{(1-|a|)^4} \sum_{n=1}^\infty \alpha_n \beta_n \left[\frac{\delta}{1-|a|}\right]^{2(n-1)} \leqq$$
$$\leqq \frac{\delta^2}{(1-|a|)^4} \sum_{n=1}^\infty \alpha_n \beta_n \leqq \frac{\delta^2}{(1-|a|)^4} \|\alpha\|_2 \|\beta\|_2 \leqq \frac{\delta^2}{(1-|a|)^4} \|f\|_2 \|g\|_2 \,,$$

which implies that

$$|\varrho_1| \leqq \frac{\delta^2}{(1-|a|)^4} \|f\|_2 \|g\|_2 \,. \tag{12}$$

As for ϱ_2, we have

$$|\varrho_2| \leqq \sup_{\delta < |z-a| < \delta + \delta^2} |f_\delta(z) f(z)\, g(z)^*| \int_{\delta < |z-a| < \delta + \delta^2} \mathrm{d}z \leqq$$
$$\leqq \delta^{-2} \sup_{\delta < |z-a| < \delta + \delta^2} |f(z)\, g(z)^*|\, \delta^3(\delta + 2) \,.$$

Because

$$|f(z)| = |\langle f, g_{0,z}\rangle| \leqq \|f\|_2 \, \|g_{0,z}\|_2 = \frac{\|f\|_2}{1 - |z|^2}$$

(and similarly for g), the supremum does not exceed

$$\frac{\|f\|_2 \, \|g\|_2}{[1 - (|a| + \delta + \delta^2)^2]^2} .$$

Summing up, we obtain

$$(13) \qquad |\varrho_2| \leqq \delta \|f\|_2 \, \|g\|_2 \, c_6(a, \delta) ,$$

where

$$c_6(a, \delta) = \frac{\delta + 2}{[1 - (|a| + \delta + \delta^2)^2]^2}$$

tends to a finite limit as $\delta \to 0+$.

Putting together (11), (12) and (13) yields

$$\|T_{f_\delta} - T_{(0,0,a)}\| \leqq c_7(a, \delta) \, \delta ,$$

where $c_7(a, \delta)$ tends to a finite limit as $\delta \to 0+$. The lemma now follows immediately. □

Following assertion drops out easily from Lemma 2 and 3.

Proposition 4. *Let* $\mathscr{T}_1 = \{T_\varphi : \varphi \in \mathscr{D}(\mathbb{D})\}$. *Then*

$$\operatorname{clos} \mathscr{T}_1 \supset \mathscr{K}(A^2) .$$

Proof. Note that the mapping $\varphi \mapsto T_\varphi$ is linear, so $\mathscr{T}_1$ and clos $\mathscr{T}_1$ are linear subsets (i.e. subspaces) of $\mathscr{B}(A^2)$. In view of Lemma 3, $T_{(0,0,a)} \in \operatorname{clos} \mathscr{T}_1$ for each $a \in \mathbb{D}$. By linearity, $R_{(0,0,a,t)}$ and $R'_{(0,0,a,t)}$ belong to clos $\mathscr{T}_1$ whenever $a \in \mathbb{D}$ and $|t| < 1 - |a|$; by Lemma 2, this implies $T_{(1,0,a)}$ and $T_{(0,1,a)}$ belong to clos $\mathscr{T}_1$. Proceeding by induction, we conclude that $T_{(m,n,a)} \in \operatorname{clos} \mathscr{T}_1$ for every $a \in \mathbb{D}$ and $m, n = 0, 1, \ldots$. Taking $a = 0$ shows that, in particular, $\langle \cdot, z^m \rangle z^n \in \operatorname{clos} \mathscr{T}_1$. By linearity, $\langle \cdot, p \rangle q \in \operatorname{clos} \mathscr{T}_1$ whenever p, q are polynomials. Because polynomials are dense in A^2, necessarily $\langle \cdot, f \rangle g \in \operatorname{clos} \mathscr{T}_1$ for all $f, g \in A^2$, i.e. all one-dimensional operators are in clos $\mathscr{T}_1$. Using the linearity of clos $\mathscr{T}_1$ for the third time shows that all finite rank operators belong to clos $\mathscr{T}_1$; since these are dense in $\mathscr{K}(A^2)$, Proposition 4 follows. □

Remark. Because $\mathscr{K}(A^2)$ is SOT-dense in $\mathscr{B}(A^2)$ and norm convergence implies SOT-convergence, Proposition 4 yields another proof of Theorem 1'.

Proposition 4 can be somewhat sharpened; to do this, we first prove a lemma.

Lemma 5. *If the support of* $\varphi \in L^\infty(\mathbb{D})$ *is a compact subset of* $\mathbb{D}$, *then* T_φ *is a compact operator. In particular,* T_φ *is compact if* $\varphi \in \mathscr{D}(\mathbb{D})$.

Proof. Let $R \in (0, 1)$ be such that $\varphi(z) = 0$ if $|z| \geqq R$. Suppose $f_n \in A^2$, $f_n \to 0$

weakly. We have

$$(14) \qquad \|T_\varphi f_n\|_2^2 \leqq \|\varphi f_n\|^2_{L^2(\mathbb{D})} = \int_{|z|\leqq R} |\varphi(z) f_n(z)|^2 \, dz .$$

Since a weakly convergent sequence is bounded, thre exists $c > 0$ such that $\|f_n\|_2 < c$ for all n. Consequently,

$$|\varphi(z) f_n(z)| \leqq \|\varphi\|_\infty |\langle f_n, g_{0,z}\rangle| \leqq \|\varphi\|_\infty \|f_n\|_2 \|g_{0,z}\|_2 \leqq$$
$$\leqq \|\varphi\|_\infty c_\infty (1 - |z|^2)^{-1} ,$$

and if $|z| \leqq R$, this is less than or equal to

$$\frac{\|\varphi\|_\infty c}{1 - R^2} .$$

Also,

$$f_n(z) = \langle f_n, g_{0,z}\rangle \to \langle 0, g_{0,z}\rangle = 0 \quad \text{as} \quad n \to \infty$$

for all $z \in \mathbb{D}$. This allows us to apply the Lebesgue dominated convergence theorem to the integrals (14); consequently, $\|T_\varphi f_n\|_2 \to 0$ as $n \to \infty$. Since this is true for every sequence weakly convergent to zero, T_φ must be a compact operator. □

Theorem 6. *Let $\mathcal{T}_1 = \{T_\varphi : \varphi \in \mathcal{D}(\mathbb{D})\}$. Then clos $\mathcal{T}_1 = \mathcal{K}(A^2)$.*

Proof. Immediate from Proposition 4 and Lemma 5. □

4. SOME OTHER FACTS ABOUT UNIFORM APPROXIMATION

This section contains some more results about uniform closures. Denote

$$\mathcal{T}_1 = \{T_\varphi : \varphi \in \mathcal{D}(\mathbb{D})\} ,$$
$$\mathcal{T}_2 = \{T_\varphi : \varphi \in C(\text{cl}\, \mathbb{D})\} ,$$
$$\mathcal{T} \;\; = \{T_\varphi : \varphi \in L^\infty(\mathbb{D})\} .$$

Here $C(\text{cl}\, \mathbb{D})$ stands for the space of all functions continuous on the closed unit disc cl $\mathbb{D}$. We have shown that clos $\mathcal{T}_1 = \mathcal{K}(A^2)$. Because $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}$, clos $\mathcal{T}$ and clos $\mathcal{T}_2$ contain $\mathcal{K}(A^2)$ as well. As has already been mentioned in the introduction, very probably clos $\mathcal{T}$ is the whole of $\mathcal{B}(A^2)$. A proof of this assertion was published by H.-F. Gautrin [3] but it seems to contain a gap: on page 180, the sixth line from above ("G is dense in G'") should read "G_1 is dense in G_1'" in the topology of E'" (it need not be a priori dense in the topology of G_1', which is what's needed in the proof).

A natural question is, what's clos $\mathcal{T}_2$? It certainly contains $\mathcal{T}_2$, and, also, $\mathcal{K}(A^2)$ (cf. Theorem 6). The next proposition shows it doesn't contain much more.

Denote alg $\mathcal{T}_2$ the norm-closed subalgebra of $\mathcal{B}(A^2)$ generated by $\mathcal{T}_2$. It is easy to see that this is, in fact, a C^*-algebra and coincides with the C^*-algebra generated by T_z. The following theorem, based on the work of Bunce, is due to Olin and Thomson.

Theorem. *The commutator ideal of* $\operatorname{alg}\mathcal{T}_2$ *is* $\mathcal{K}(A^2)$. *The quotient algebra* $\operatorname{alg}\mathcal{T}_2/\mathcal{K}(A^2)$ *is isometrically* $*$*-isomorphic to* $C(\partial\mathbb{D})$. *This isomorphism is given by*

$$f \in C(\partial\mathbb{D}) \leftrightarrow T_F + \mathcal{K}(A^2) \in \operatorname{alg}\mathcal{T}_2/\mathcal{K}(A^2),$$

where F *is the harmonic extension of* f *inside* $\mathbb{D}$ (*the Poisson integral of* f) (*in fact, any continuous extension of* f *into* $\mathbb{D}$ *would do here* – cf. Lemma 5). *Moreover, for* $\varphi \in C(\mathrm{cl}\,\mathbb{D})$, *the essential spectrum of* T_φ *is* $\sigma_e(T_\varphi) = \varphi(\partial\mathbb{D})$.

Proof. See [5], Theorem 1 and its Corollary. □

Proposition 7. $\operatorname{clos}\mathcal{T}_2 = \operatorname{alg}\mathcal{T}_2 = \mathcal{T}_2 + \mathcal{K}(A^2)$.

Proof. Since $\mathcal{T}_2 + \mathcal{K}(A^2) \subset \operatorname{clos}\mathcal{T}_2 \subset \operatorname{alg}\mathcal{T}_2$, it suffices to show that $\operatorname{alg}\mathcal{T}_2 \subset \mathcal{T}_2 + \mathcal{K}(A^2)$. But this is immediate from the preceding theorem, because $F \in C(\mathrm{cl}\,\mathbb{D})$ for $f \in C(\partial\mathbb{D})$. □

Corollary. $\operatorname{clos}\mathcal{T}_2$ *is not all of* $\mathcal{B}(A^2)$.

Proof. If $T \in \operatorname{clos}\mathcal{T}_2$, then $T = T_f + K$ for some $f \in C(\mathrm{cl}\,\mathbb{D})$ and $K \in \mathcal{K}(A^2)$, and, according to the last sentence in the Olin-Thomson theorem, $\sigma_e(T) = \sigma_e(T_f) = f(\partial\mathbb{D})$. Since $f \in C(\mathrm{cl}\,\mathbb{D})$, this is a connected set. Consequently, operators with disconnected essential spectrum cannot belong to $\operatorname{clos}\mathcal{T}_2$. (As an example, take a projection Q with both range and kernel infinite-dimensional; then $\sigma_e(Q) = \{0, 1\}$.)

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