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Roman Frič; Fabio Zanolin

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COARSE SEQUENTIAL CONVERGENCE IN GROUPS, ETC.

ROMAN FRIČ, Košice and FABIO ZANOLIN, Udine

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As shown in [JAK], [COZ], [DFZ], [FRI] and [FGE], even decent algebras equipped with extreme compatible convergences with unique limits can have pathological properties. Coarse (sequential) convergence groups and minimal (filter) convergence groups, as the convergence counterparts of minimal Hausdorff topological groups, have been introduced in [FZB] and [FZT], respectively. In the present paper we establish some basic facts about coarse convergence semigroups, rings and fields. We show that coarseness sometimes implies the continuity of an additional algebraic operation. E.g., a coarse ring convergence in a field is a field convergence. This in turn implies the existence of a convergence field having no ring completion, a fact interesting in connection with some recent results concerning the completion of convergence rings (cf. [KNO]). The first example of a sequential convergence ring having no completion can be found in [COZ]. In the last section we introduce minimal (filter) convergence rings and prove that the first countable modification (introduced by R. Beattie and H.-P. Butzmann, cf. [BBC]), applied to a coarse (sequential) ring convergence for rational numbers (coarser than the usual metric one) and combined with the Choquet modification results in a minimal ring convergence.

1.

As a rule, R, Q, Z and N denote the real numbers, rational numbers, integers and natural numbers (positive integers), respectively, MON denotes the strictly monotone maps of N into N, if $S = \langle S(n) \rangle \in X^N$ is a sequence of points of X, then $S * s = \langle S(s(n)) \rangle$ denotes the corresponding subsequence of S and if X is equipped with an algebraic structure, then operations in X^N are defined pointwise. In Section 1 and Section 2 we shall deal solely with sequential convergences.

By a sequential convergence, or simply convergence, we understand (if not explicitly stated otherwise) an FSH-convergence, where (F) means that if S converges to x then each of its subsequences converges to x as well, (S) means that each constant sequence $\langle x \rangle$ converges to x, (H) means the uniqueness of limits. In general, we do not assume the Urysohn axiom (U). If the underlying set X of a convergence is a semigroup, group, ring, field, vector space over a scalar field F, respectively, then (L) means that the convergence and algebraic structure of X are compatibile. More precisely, if L is a convergence for X and if X is a semigroup, then (L) stands for

(Ls) If $(S, x), (T, y) \in L$, then $(ST, xy) \in L$; if X is a group, then (L) stands for

(Lg) If $(S, x), (T, y) \in L$, then $(ST^{-1}, xy^{-1}) \in L$; if X is a ring, then (L) stands for

(Lr) If $(S, x), (T, y) \in L$, then $(S - T, x - y) \in L$ and $(ST, xy) \in L$;
if X is a field, then (L) stands for

(Lf) If $(S, x), (T, y) \in L$, then $(S - T, x - y) \in L$, $(ST, xy) \in L$ and, moreover, if $x \neq 0$ and $S(n) \neq 0$ for all $n \in N$, then $(S^{-1}, x^{-1}) \in L$;
if X is a vector space over a scalar field F equipped with a sequential convergence M, then (L) stands for

(Lv) If $(S, x), (T, y) \in L$, $(A, a) \in M$, then $(S - T, x - y) \in L$ and $(AS, ax) \in L$.
If L satisfies the corresponding version of the axiom (L), then L is said to be *a semigroup*, *group*, *ring*, *field* and *vector convergence*, respectively, and X equipped with L is said to be an *L-semigroup*, *L-group*, *L-ring*, *L-field* and *vector L-space*, respectively. Sometimes weaker forms of (Lv) are considered as well. Namely, if X is a vector space over a scalar field F equipped with a sequential convergence M then L satisfying

(Lvg) If $(S, x), (T, y) \in L$, $a \in F$, then $(S - T, x - y) \in L$ and $(\langle a \rangle S, ax) \in L$;
is said to be a *vector group convergence* and L *satisfying*

(Lvw) If $(S, x), (T, y) \in L$, $(A, a) \in M$, $z \in X$, $b \in F$, then $(S - T, x - y) \in L$, $(\langle b \rangle S, bx) \in L$ and $(A\langle z \rangle, az) \in L$;
is said to be a *weak vector convergence*. Accordingly, X equipped with L is said to be a *vector L-group* and a *weak vector L-space*, respectively. If the additive notation is used, then we tacitly assume that the semigroup or group is abelian.

Definition 1. Let X be a semigroup (group, ring, vector space over a scalar field F, respectively) and let L be an FLSH-convergence for X. Then L (and also X) is said to be *coarse* if there is no FLSH-convergence for X strictly larger than L.

Observe that if L is an FLSH-convergence for X, then the Urysohn modification L^* of L is a FLUSH-convergence for X. Consequently, every coarse FLSH-convergence satisfies the Urysohn axiom (U) and hence L is a coarse FLSH-convergence iff it is a coarse FLUSH-convergence.

Let X be a group and let L be an FLgUSH-convergence for X. Then L is coarse iff the following criterion holds (cf. [FZB]):

(C) For each $S \in X^N$ etiher

(C1) $(S * s, e) \in L$ for some $s \in \mathrm{MON}$;

or

(C2) $\langle p \rangle = \prod_{i=1}^{k} T(i)$, where $p \in X$, $p \neq e$, $k \in N$, and for each i, $i = 1, \dots, k$, either $T(i)^{a(i)} = \langle x(i) \rangle S * s(i) \langle x(i)^{-1} \rangle$, $a(i) = \pm 1$, $x(i) \in X$, $s(i) \in \mathrm{MON}$, or $(T(i), e) \in L$.

If L is assumed to be only an FLgSH-convergence, then (C1) has to be replaced by

(C1*) $(S, e) \in L$.

Coarse compatible FSH-convergences in vector spaces are characterized in [JAK].

Remark 1. Let X be a semigroup. For $A \subset X^N \times X$ the smallest FLsS-convergence L for X such that $A \subset L$ is defined in the obvious way: $(S, x) \in L$ iff $S = \prod_{i=1}^{k} S(i)$ and $x = \prod_{i=1}^{k} x(i)$, where $k \in N$, and either $S(i)$ is a constant sequence generated by $x(i) \in X$, or $(T(i), x(i)) \in A$ and $S(i)$ is a subsequence of $T(i)$, $i = 1, \ldots, k$.

Theorem 1. *Let X be a semigroup and let L be an* FLSH-*convergence for X. Then L is coarse iff*

(CS) *For each $S \in X^N$ and each $x \in X$ any of the following three conditions holds:*

(CS1) $(S, x) \in L$;

(CS2) $[(\prod_{i=1}^{k} S(i), y) \in L] \wedge [y \neq \prod_{i=1}^{k} x(i)]$, *where $k \in N$, $S(i)$ is either a subsequence of S or an L-convergent sequence and $x(i) = x$ if $S(i)$ is a subsequence of S nad $x(i) = L\text{-lim } S(i)$ otherwise, $i = 1, \ldots, k$;*

(CS3) $[\prod_{i=1}^{k} S(i) = \prod_{i=1}^{m} T(i)] \wedge [\prod_{i=1}^{k} x(i) \neq \prod_{i=1}^{m} y(i)]$, *where $k, m \in N$, $S(i)$, resp. $T(i)$, is either a subsequence of S or an L-convergent sequence and $x(i) = x$, resp. $y(j) = x$, if $S(i)$, resp. $T(j)$, is a subsequence of S and $x(i) = L\text{-lim } S(i)$, resp. $y(j) = L\text{-lim } T(j)$, otherwise, $i = 1, \ldots, k$, $j = 1, \ldots, m$.*

Proof. Sufficiency. Assume that (CS) is satisfied. Let L' be an FLsSH-convergence for X such that $L \subset L'$. If $(S, x) \in L' \setminus L$, then (S, x) does not satisfy any of the conaitions (CS1), (CS2), (CS3). Hence L is coarse.

Necessity. Assume that (CS) does not hold. Then there are $S \in X^N$ and $x \in X$ violating all three conditions (CS1), (CS2) and (CS3). Let L' be the smallest FLsS-convergence for X containing L and such that $(S, x) \in L'$. From the negation of (CS2) and (CS3) it follows that L' satisfies axiom (H). The negation of (CS1) implies $(S, x) \notin L$. Thus $L \subset L'$ and $L \neq L'$. This completes the proof.

Remark 2. If X is an abelian semigroup, then condition (CS2), resp. (CS3), in Theorem 1 has the following form:

(CS2′) $[(\sum_{i=1}^{k} S(i) + \sum_{i=1}^{m} T(i), y + \sum_{i=1}^{m} y(i)) \in L] \wedge [kx + \sum_{i=1}^{m} y(i) \neq y + \sum_{i=1}^{m} y(i)]$, where $k \in N$, $m \in N \cup \{0\}$, $S(i)$ is a subsequence of S, $i = 1, \ldots, k$, $y \in X$, $(T(j), y(j)) \in L$, $j = 1, \ldots, m$;

resp.

(CS3′) $[\sum_{i=1}^{k(1)} S(1, i) + \sum_{i=1}^{m(1)} T(1, i) = \sum_{i=1}^{k(2)} S(2, i) + \sum_{i=1}^{m(2)} T(2, i)] \wedge [k(1)\, x + \sum_{i=1}^{m(1)} y(i) \neq k(2)\, x + \sum_{i=1}^{m(2)} y(i)]$, where $j = 1, 2$, $k(j) \in N$, $m(j) \in N \cup \{0\}$, $S(j, i)$ is a subsequence of S, $i = 1, \ldots, k(j)$, $(T(j, i), y(j, i)) \in L$, $i = 1, \ldots, m(j)$.

As a rule, $\sum_{i=1}^{m} a(i)$ for $m = 0$ represents an empty symbol.

Theorem 2. *Let X be an abelian group and let L be a convergence for X. Then the* *following are equivalent:*

(i) *L is a coarse semigroup* (i.e. FLsSH-) *convergence;*

(ii) *L is a coarse group* (i.e. FLgSH-) *convergence.*

Proof. (i) implies (ii). In fact, it suffices to prove that if $(-S, -x) \in L$, then $(S, x) \in L$. Suppose that, on the contrary, $(-S, -x) \in L$ and $(S, x) \notin L$. Since L satisfies (CS) and (S, x) violates (CS1), (S, x) has to satisfy either (CS2′) or (CS3′). Suppose that $(\sum_{i=1}^{m} S(i) + \sum_{i=1}^{m} T(i),\ y + \sum_{i=1}^{m} y(i)) \in L$, where $k \in N$, $m \in N \cup \cup \{0\}$, $S(i)$ is a subsequence of S, $i = 1, \ldots, k$, $y \in X$, $(T(j), y(j)) \in L$, $j = 1, \ldots, m$. Since $(-S(i), -x) \in L$, $i = 1, \ldots, k$, we have $(\sum_{i=1}^{m} T(i),\ y - kx + \sum_{i=1}^{m} y(i)) \in L$ and hence $y = kx$. Thus (S, x) violates (CS2′). In a similar way it can be shown that (S, x) violates (CS3′). This is a contradiction.

Remark 3. Observe that in Theorem 2 the assumption that L is coarse cannot be left out. Indeed, using the free group technique, it is not difficult to construct an abelian group X, an FLsUSH-convergence L for X and a pair (S, x) such that $(S, x) \in L$ and $(-S, -x) \notin L$.

2.

Now, let us turn to rings. Recall that all algebraic operations are assumed to be associative. If X is a ring, then X^N is the ring of all mappings of N into X. The proofs of the next two lemmas are omitted.

Lemma 1. *Let X be a ring and let L be an* FLrS-*convergence for X. Then $A = L^{\leftarrow}(0)$ has the following properties:*

(i) *A is a subring of the ring X^N;*

(ii) *If $S \in A$, then $S * s \in A$ for each $s \in$* MON;

(iii) *$\langle x \rangle A \subset A$ and $A\langle x \rangle \subset A$ for each $x \in X$;*

(iv) *$L^{\leftarrow}(x) = A + \langle x \rangle$.*

Further, L satisfies (H) iff

(v) *$\langle x \rangle \notin A$ whenever $x \neq 0$.*

Lemma 2. *Let X be a ring and let A be a subset of X^N satisfying conditions* (i), (ii) *and* (iii) *in Lemma* 1. *Then there is an* FLrS-*convergence L for X such that $A = L^{\leftarrow}(0)$.*

Remark 4. Let X be a ring. Let B be a subset of X^N, let $Z(B)$ be the set of all subsets A of X^N satisfying conditions (i), (ii) and (iii) in Lemma 1 such that $B \subset A$, let $P(B)$ be the set of all sequences of the form $\pm S * s$, $\langle x \rangle S * s$, $S * s\langle y \rangle$, $\langle x \rangle S * * s\langle y \rangle$, where $S \in B$, $s \in$ MON, $x, y \in X$ and let $R(B)$ be the set of all sequences of the form $\sum_{i=1}^{m} T(i, 1) \ldots T(i, k(i))$, where $m, k(i) \in N$ and $T(i, j) \in P(B)$, $i = 1, \ldots, m$, $j = 1, \ldots, k(i)$. Clearly, $R(B) \in Z(B)$ and $R(B) \subset A$ for all $A \in Z(B)$. Hence there is an FLrS-convergence L for X such that $R(B) = L^{\leftarrow}(0)$ and if K is an FLrS-convergence for X such that $B \subset K^{\leftarrow}(0)$, then $L \subset K$.

Theorem 3. *Let X be a ring and let L be an* FLrSH-*convergence for X. Then L is coarse iff*

(CR) *For each $S \in X^N$ either:*

(CR1) $(S, 0) \in L$,

or

(CR2) $[\langle p \rangle = \sum_{i=1}^{m} T(i, 1) \ldots T(i, k(i))]$,
where $p \in X$, $p \neq 0$, $m, k(i) \in N$, $T(i, j) \in L^{\leftarrow}(0) \cup P(\{S\})$, $i = 1, \ldots, m$, $j = 1, \ldots, k(i)$.

Proof. Necessity. Assume that L does not satisfy (CR). Let S be a sequence of points of X which satisfies neither (CR1) nor (CR2). Let L' be the smallest FLrS-convergence for X such that $L \subset L'$ and $(S, 0) \in L'$. Clearly, L' satisfies axiom (H) and hence L fails to be coarse.

Sufficiency. Assume that L satisfies (CR). Let L' be an FLrSH-convergence for X such that $L \subset L'$. If $(T, x) \in L' \setminus L$, then $S = T - \langle x \rangle$ does not satisfy (CR1) and hence satisfies (CR2). But $(\sum_{i=1}^{m} T(i, 1) \ldots T(i, k(i)), p) \in L$, $p \neq 0$, contradicts $(\sum_{i=1}^{m} T(i, 1) \ldots T(i, k(i)), 0) \in L'$ and hence $L' = L$. Thus L is coarse and the proof is complete.

It follows from Proposition 2 in [FR1] that in any ring convergence for Q coarser than the metric one no unbounded sequence can converge to 0. In Theorem 3, in this case, it suffices to check whether or not all bounded sequences satisfy conditions (CR1) and (CR2). This leads to the following generalization.

Definition 2. Let X be a ring and let L be an FLrSH-convergence for X. A sequence S is said to be *bounded* (more precisely *L-bounded*) whenever for each sequence T converging to 0 the sequence ST converges to 0 as well. If S is not bounded, then it is said to be *unbounded*. The convergence L is said to be *balanced* if for each unbounded sequence S there exists a sequence T converging to 0 such that ST converges to some $x \neq 0$.

Remark 5. In order to avoid some obvious pathologies concerning bounded sequences it suffices to assume the Urysohn axiom or some suitable weaker axiom of convergence. Since coarse convergences are always Urysohn, the assumption of (U) in connection with condition (CR) is not restrictive. Observe that if X is a ring of real-valued functions equipped with the pointwise convergence, then we cannot replace in the definition of a balanced convergence "$x \neq 0$" by "x is the unit element".

Corollary 1. *Let X be a ring and let L be a balanced* FLrSH-*convergence for X. Then L is coarse iff*

(CR*) *For each bounded sequence $S \in X^N$ either of conditions* (CR1) *or* (CR2) *in Theorem* 3 *holds true.*

Let X be a field. If L is a field (i.e. FLfSH-) convergence for X, then L is a ring (i.e. FLrSH-) convergence. The converse implication is not true in general.

Example 1. Consider the field Q of rational numbers equipped with the usual metric convergence M. As shown in [FRI], M is not a coarse ring convergence and can be enlarged to a coarse ring (i.e. FLrUSH-) convergence R. Let $(T, x) \in R \setminus M$, $x \in Q \setminus \{0\}$. According to Proposition 5 in [FRI], there are $t \in \text{MON}$ and a transcendental number y such that $U = T * t$ converges in the real line to y and $U(n) \neq 0$ for all $n \in N$. Using Lemma 1 and Lemma 2 we can construct the smallest FLrS-convergence for Q which is coarser than M and in which $T * t$ converges to x. Denote it by L. Since R satisfies axiom (H) and $M \subset L \subset R$, it follows that L is an FLrSH-convergence for Q. We shall prove that L does not satisfy axiom (Lf). Clearly, it suffices to prove that $(U^{-1}, x^{-1}) \notin L$. Contrariwise, assume that $(U^{-1}, x^{-1}) \in L$. Since $L^{\leftarrow}(0)$ consists of all sequences of the form $\sum_{i=1}^{m} T(i, 1) \ldots \ldots T(i, k(i))$, where m, $k(i) \in N$, $T(i, j) \in M^{\leftarrow}(0) \cup P(\{U - \langle x \rangle\})$, $i = 1, \ldots, m$, $j = 1, \ldots, k(i)$, for some $\sum_{i=1}^{m} T(i, 1) \ldots T(i, k(i)) \in L^{\leftarrow}(0)$ we have $U^{-1} - \langle x^{-1} \rangle = = \sum_{i=1}^{m} T(i, 1) \ldots T(i, k(i))$. Multiplying both sides by U and passing to the limits in the real line, we get that y is a solution of a nontrivial polynomial equation with rational coefficients. Since y is a transcendental number, we have a contradiction.

Theorem 4. *Let X be a field and let L be a coarse ring (i.e.* FLrSH-) *convergence for X. Then L is a coarse field (i.e.* FLfSH-) *convergence.*

Proof. We are to prove that L satisfies axiom (Lf). Let $(T, x) \in L$, $x \neq 0$ and $T(n) \neq 0$ for $n \in N$. Since L satisfies axiom (Lr), it suffices to prove that $(T^{-1}, x^{-1}) \in \in L$. Contrariwise, suppose that $(T^{-1}, x^{-1}) \notin L$. Put $S = T^{-1} - \langle x^{-1} \rangle$. Then $(S, 0) \notin L$ and, by Theorem 3, there are $p \in X$, $p \neq 0$, $m \in N$, $k(i) \in N$, $T(i, j) \in \in L^{\leftarrow}(0) \cup P(\{S\})$, $i = 1, \ldots, m$, $j = 1, \ldots, k(i)$, such that $\langle p \rangle = \sum_{i=1}^{m} T(i, 1) \ldots \ldots T(i, k(i))$. Multiplying both sides by $T * s(i, j)$ whenever for some $y(i, j) \in X$ we have $T(i, j) = \langle y(i, j) \rangle S * s(i, j) = \langle y(i, j) \rangle (T^{-1} * s(i, j) - \langle x^{-1} \rangle)$ (i.e. $T(i, j) \in \in P(\{S_j\})$) and passing to L-limits we obtain $px^k = 0$, where $k \in N$ is the total number of multiplications by $T * s(i, j)$. Since X is a field, this contradicts $p \neq 0$ and $x \neq 0$.

As shown by M. Contessa and F. Zanolin ([COZ]), a convergence ring need not have a completion. In view of the recent results by V. Koutník and J. Novák (certain class of fields equipped with a sequential ring convergence admits a ring completion, cf. [KNO]), the following result is interesting.

Corollary 2. *There is a* (FLfUSH-) *convergence field having no ring completion.*

Proof. Consider Q equipped with a coarse ring (i.e. FLrUSH-) convergence L coarser than the usual metric convergence M. According to Theorem 4, L is a coarse field (i.e. FLfSH-) convergence for Q. As shown in [FRI], there is a sequence S of rational numbers such that $(S, 0) \in L$ and S converges in the real line to an irrational number y. Clearly, we can assume that $S(n) \neq 0$ for all $n \in N$. Then S^{-1} is an L-Cauchy sequence and $\langle 1 \rangle = SS^{-1}$ cannot L-converge to zero. Hence Q equipped with L is a (FLfUSH-) convergence field having no ring completion.

3.

In this section we introduce minimal (filter) convergence rings, i.e. rings equipped with a compatible convergence of filters admitting no coarser compatible one, and investigate how coarse and minimal convergences are related in the ring of rational numbers. For the background information on filter convergence structures the reader is referred to [GAH], [BBH], [BBC] and [BBS]. Minimal group convergences were introduced in [FZT], where their relationship to coarse sequential groups is also investigated.

It is known (cf. [GAH], [FZT]), that compatibile filter convergences in a group, ring, etc., can be introduced by defining suitable systems of filters converging to the neutral element of the underlying group. In the same way as in [FZT], it can be shown that each ring convergence of filters with unique limits can be enlarged to a coarse (i.e. a minimal) one.

Let X be a set and let λ be a filter convergence for X. Denote by $\chi(\lambda)$ the Choquet modification of λ (a filter $\mathscr{F}$ $\chi(\lambda)$-converges to x iff each ultrafilter finer than $\mathscr{F}$ λ-converges to x). Observe that if X is a group (ring, field, etc.) and λ is compatible with the algebraic structure of X, then $\chi(\lambda)$ is compatible as well (cf. [BBS]). Thus for each minimal ring convergence we have $\chi(\lambda) = \lambda$, i.e. λ is a Choquet convergence.

For rings the following minimality criterion can be proved (the proof is analogous as for minimal groups, see [FZT], and it is omitted).

Theorem 5. *Let X be a ring and let λ be a ring convergence for X. Then λ is minimal iff the following condition holds true*:

(MR) *For each filter $\mathscr{F}$ on X either*

(MR1) *$\mathscr{F}$ λ-converges to* 0,

or

(MR2) *There are $p \in X$, $p \neq 0$, $m \in N$, $k(i) \in N$, $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} \mathscr{F}(i, 1) \ldots \mathscr{F}(i, k(i)) \subset \dot{p}$, where $\dot{p}$ denotes the fixed ultrafilter generated by p, $\mathscr{F}(i, j)$ is either a filter finer than any of the filters $\pm\mathscr{F}, x\mathscr{F}, \mathscr{F}y, x\mathscr{F}y$, $x, y \in X$, or $\mathscr{F}(i, j)$ is a filter λ-converging to* 0, *$i = 1, \ldots, m$, $j = 1, \ldots, k(i)$.*

Remark 6. It is easy to see that if X is compact and Choquet, then it is minimal. Further, to prove that λ is minimal, it suffices to replace condition (MR2) by (MR2') in which "$\sum_{i=1}^{m} \mathscr{F}(i, 1) \ldots \mathscr{F}(i, k(i)) \subset \dot{p}$" is replaced by "$\sum_{i=1}^{m} \mathscr{F}(i, 1) \ldots \mathscr{F}(i, k(i))$ λ-converges to p $(\neq 0)$."

Let L be a FUSH-convergence for X. The first countable filter modification $\gamma(t)$ of L is a filter convergence for X defined as follows (cf. [BBC]): a filter $\mathscr{F}$ $\gamma(L)$-converges to x iff there exists a filter $\mathscr{G}$ with a countable basis such that $\mathscr{G} \subset \mathscr{F}$ and whenever $\mathscr{G} \subset \mathscr{F}(S)$, then S L-converges to x (by $\mathscr{F}(S)$ we denote the elementary filter generated by the sequence S). It is known that γ induces a functor from sequential convergence spaces into spaces with convergence of filters and the functor has many nice properties ([BBH], [BBC]). In particular, if X is a ring (semigroup, group, field, vector space) and L is compatible with the algebraic structure of X, then $\gamma(L)$ is compatible with the algebraic structure of X, too. Since L and $\gamma(L)$ induce the same closure (adherence) operator, many counterexamples from the sequential realm can be transformed into the filter realm by applying γ, see e.g. [FRI], [FGE]. In [FZT] it has been shown that in some groups $\gamma(L)$ is minimal if L is coarse and that this cannot be proved in general (in ZFC). Since groups are zero-rings, analogous statements hold for rings as well. We close with a quest into nonzero-rings, in particular, we investigate the minimality of the γ-modification of a coarse ring convergence for rational numbers.

Let X be the ring of rational numbers, let d be the usual metric for the real line, let L be a coarse FLrSH-convergence for X coarser than the one induced by d and let $\lambda = \gamma(L)$. For each real number r, let $\mathscr{N}_r$ be the trace of the d-neighborhood filter of r onto X (i.e. sets $\{g \in X;\ d(r, g) < 1/n\}$, $n = 1, 2, \dots$, form a base of $\mathscr{N}_r$).

Lemma 3. *Let $\mathscr{F}$ be a bounded ultrafilter on X. Then $\mathscr{F}$ satisfies either condition* (MR1) *or condition* (MR2').

Proof. Let r be the real number to which $\mathscr{F}$ d-converges. Let $\mathscr{S}$ be the set of all sequences of rational numbers d-converging to r. It follows from Proposition 5 in [FRI] that there are two possibilities.

1. Foı some $x \in X$, each sequence in $\mathscr{S}$ L-convergences to x. Then the filter $\mathscr{S}_r$ λ-converges to x. Since $\mathscr{S}_r \subset \mathscr{F}$, $\mathscr{F}$ λ-converges to x as well and hence satisfies either (MR1) or (MR2').

2. No sequence in $\mathscr{S}$ L-converges in X. Fix $S \in \mathscr{S}$. Since L is coarse, the sequence S satisfies condition (CR2), i.e., for some $p \in X$, $p \neq 0$, we have $\langle p \rangle = \sum_{i=1}^{m} T(i, 1) \dots$ $\dots T(i, k(i))$, where $m \in N$, $k(i) \in N$, $i = 1, \dots. k$, and each $T(i, j)$ is either a product of a subsequence $S * s$ of S and a constant sequence, or a sequence L-converging to 0. Since each L-convergent sequence contains a subsequence d-converging in the real line (cf. Proposition 5 in [FRI]), we can and do assume that if $T(i, j)$ L-converges to 0, then it d-converges to a real number $r(i, j)$; in this case put $\mathscr{F}(i, j) = \mathscr{N}_{r(i,j)}$ and let $\mathscr{F}(i, j)$ be $\mathscr{N}_r$ multiplied by the corresponding fixed ultrafilter otherwise. Since the filter $\sum_{i=1}^{m} \mathscr{F}(i, 1) \dots \mathscr{F}(i, k(i))$ λ-converges to p and if we in this expression replace $\mathscr{F}(i, j)$ by $\mathscr{F}$ multiplied by the fixed ultrafilter whenever $\mathscr{F}(i, j)$ involves $\mathscr{N}_r$, the resulting finer filter λ-converges to p, too Thus $\mathscr{F}$ satisfies condition (MR2') and the proof is complete.

Corollary 3. *Let L be a coarse ring convergence for the rational numbers. Then $\chi(\gamma(L))$ is a minimal ring convergence.*

Proof. Since $\gamma(L)$ is a ring convergence (of filters) coarser than the metric one, no $\gamma(L)$-divergent unbounded ultrafilter can converge in any ring convergence coarser than $\gamma(L)$. Since by Lemma 3 no $\gamma(L)$-divergent bounded ultrafilter can converge either, $\chi(\gamma(L))$ is minimal.

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Author's address: R. Frič, Matematický ústav SAV, 040 01 Košice, Grešákova 6, Czechoslovakia; F. Zanolin, Dipartimento di Scienze Matematiche, Università di Trieste, P. le Europa 1, 34100 Trieste, Italy.