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EIGENVALUES AND THE MAX-CUT PROBLEM

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1. INTRODUCTION

The *max-cut problem* is one of the well known and most studied hard optimization problems. It can be stated as follows. Given an undirected graph $G = (V, E)$ with edge weights $c_{ij} \in R$ for all $ij \in E$, find a cut $\delta(S)$ in G for which $c(\delta(S)) := \sum_{ij \in \delta(S)} c_{ij}$ is maximum. Here $\delta(S)$ denotes the set of edges $\{ij \mid i \in S, j \notin S\}$ for $S \subset V$.

The max-cut problem is known to be polynomially solvable for some classes of graphs: for planar graphs ([OD] and [H]), for graphs not contractible to K_5 ([B]), for weakly bipartite graphs ([GP]), for graphs without long odd cycles ([GN]). (The latter classes only foı nonnegative weights.) The max-cut problem is NP-complete even for the cardinality version, called the *maximum bipartite subgraph* problem, where all weights $c_{ij} = 1$ (see [Ka]).

A practical algorithm for solving large instances of the max-cut problem has been developed in [BGJR], where also some applications are presented. Another application has been given in [NP]. A simple polynomial time heuristic that guarantees a probabilistic lower bound appeared in [PT1].

In this paper we present an easily computable upper bound on the max-cut based on the maximum eigenvalue of an associated matrix. The connection between eigenvalues and cuts in graphs has been first discovered by Fiedler [F]. The eigenvalue based methods have proved to be useful also for some other problems, e.g. expanding properties of graphs ([A], [AM], [T]), isoperimetric numbers of graphs [M1], etc. For a survey of these results see [M2].

The basic result, an inequality for the max-cut, is derived in Section 2. In Section 3 we compare the eigenvalue upper bound with the actual size of maximum bipartite subgraph in two classes, Kneser graphs and circulants, where the exact solution is known. It appears that the gap between the optimum value and the upper bound can be arbitrarily large. On the other hand, the bound is exact for some other classes which are presented in Section 4.

2. BASIC RESULTS ON THE LAPLACIAN EIGENVALUES

Let $G = (V, E)$ be a graph of order n without loops and multiple edges. The *difference Laplacian matrix* $Q = Q(G)$ of G is an $n \times n$ matrix with entries q_{ij}

defined as follows.

$$(1)\qquad q_{ij} = \begin{cases} d_i & \text{the degree of the } i\text{-th vertex, if } i = j \\ -1 & \text{for } \; ij \in E \\ 0 & \text{otherwise} \end{cases}$$

In other words, $Q = D - A$ where A is the adjacency matrix of G and D is the diagonal matrix with vertex degrees on the main diagonal. The definition is extended to a weighted graph, with weight c_{ij} on an edge ij and $c_{ij} = 0$ otherwise, as follows.

$$(2)\qquad q_{ii} = \sum_{j=1}^{n} c_{ij}$$

$$q_{ij} = -c_{ij} \quad \text{for} \quad i \neq j\,.$$

In case all weights are nonnegative, the Laplacian Q is a positive semidefinite matrix with the smallest eigenvalue $\lambda_1 = 0$ (a corresponding eigenvector has all coordinates equal to 1). The eigenvalues of $Q(G)$ will always be enumerated in the increasing order $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ repeated according to their multiplicity. We will use the notation $\lambda_k = \lambda_k(G)$ to denote the k-th smallest eigenvalue of $Q(G)$, counting the multiplicites. Instead of λ_n we will write also λ_∞ for the maximum eigenvalue of G.

Lemma 2.1. *Let G be a weighted graph. We have*

$$(3)\qquad c(\delta(S)) \leqq \lambda_\infty \frac{|S|\,(n - |S|)}{n}$$

for any subset S of vertices.

Proof. It is well known (see e.g. [L, Theorem 3.2.1]) that

$$(4)\qquad \lambda_\infty = \max_{x \neq 0} \frac{x^T Q x}{x^T x}$$

for a symmetric matrix Q. Further, we have $x^T Q x = \sum_{ij \in E} c_{ij}(x_i - x_j)^2$ for any $x = (x_i)_{i \in V}$, since Q was defined by (2). Given a subset $S \subset V$, define x by

$$x_i := \begin{cases} n - s & \text{for} \quad i \in S \\ -s & \text{for} \quad i \notin S \end{cases}$$

where $s = |S|$. Then we have

$$(5)\qquad x^T Q x = \sum_{ij \in E} c_{ij}(x_i - x_j)^2 = \sum_{ij \in \delta(S)} c_{ij}(x_i - x_j)^2 = n^2 c(\delta(S))\,,$$

and $x^T x = s(n - s)^2 + (n - s)\, s^2 = s(n - s)\, n$. Using (4) and (5) we get

$$\lambda_\infty \geqq \frac{nc(\delta(S))}{s(n - s)}$$

and (3) follows. ∎

Let us denote by $MC(G) := \max_{S \subset V} c(\delta(S))$, the max-cut in G. Since $|S|(n - |S|) \leqq n^2/4$ for any $S \subset V$, we have

Theorem 2.2. *Let G be a weighted graph. Then*

$$MC(G) \leqq \lambda_\infty \frac{n}{4}. \quad \blacksquare$$

We notice that for odd n the above bound can be sharpened slightly:

$$MC(G) \leqq \frac{\lambda_\infty}{n} \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor.$$

The consequences of this result will be exploited in the subsequent sections. Let us present now few other results on the Laplacian eigenvalues of a graph. It has been shown in [F] and [AnM] that $\lambda_\infty(G) = \lambda_2(\bar{G})$ for an unweighted graph G and its complement $\bar{G}$. We extend this equality for nonnegatively weighted graphs.

Let G be a weighted graph with a weight function c. We define the *complement* of G as the weighted graph $\bar{G}$ on the same vertex set and with the weight function $\bar{c}$ where

$$\bar{c}_{ij} := \begin{cases} 1 - c_{ij} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j \,. \end{cases}$$

Clearly, $Q(G) + Q(\bar{G}) = nI - J$ where I is the identity matrix and J is the matrix with all entries equal 1. Let us denote by $\mu(B, x)$ the characteristic polynomial of B, and let, for a graph G, $\mu(G, x) := \mu(Q(G), x)$.

Lemma 2.3.

$$\mu(\bar{G}, x) = (-1)^{n+1} \frac{x}{n - x} \mu(G, n - x)\,.$$

Proof. $\mu(\bar{G}, x) = \det(xI - Q(\bar{G})) = \det(xI - nI + J + Q(G)) = (-1)^n \det((n - x)I - J - Q(G)) = (-1)^n \mu(Q(G) + J, n - x)$. But in general $\mu(Q + J, t)$ is nicely related to $\mu(Q, t)$ in the case when the column sums of Q are all equal to 0. In the matrix $tI - Q - J$ replace the first row by the sum of all rows. Each entry in this row becomes equal to $t - n$. Subtracting this row divided by $t - n$ from all the remaining rows does not change the determinant. Denote the obtained matrix by B_t. On the other hand, if we replace the first row of $tI - Q$ by the sum of all rows in this matrix we get a matrix which has exactly the same entries as B_t except for the first row where, instead of $t - n$, we have t. Consequently,

$$\frac{\mu(Q + J, t)}{t - n} = \frac{\mu(Q, t)}{t}.$$

From this and the calculation at the beginning of the proof, our lemma follows trivially. $\blacksquare$

Corollary 2.4. *Let $\lambda_1(G) \leqq \lambda_2(G) \leqq \ldots \leqq \lambda_n(G)$ be the Laplacian eigenvalues of a nonnegatively weighted graph G. Then $\lambda_\infty(\bar{G}) = n - \lambda_2(G)$.* ■

The *Cartesian product* $G \times H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ and edges $(u, v)(u', v')$ if $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.

Proposition 2.5 (cf. [F]). *The Laplacian eigenvalues of the Cartesian product $G \times H$ are precisely all sums*

$$\lambda_i(G) + \lambda_j(H), \quad i = 1, \ldots, |G|, \quad j = 1, \ldots, |H|.$$

In particular,

$$\lambda_2(G \times H) = \min\{\lambda_2(G), \lambda_2(H)\}, \quad \text{and} \quad \lambda_\infty(G \times H) = \lambda_\infty(G) + \lambda_\infty(H)$$

for nonnegatively weighted graphs G and H. ■

Let us mention some known bounds on λ_∞. First [AnM]

$$\lambda_\infty \leqq \max\{d(u) + d(v) \mid uv \in E(G)\}$$

where $d(u)$ is the degree of the vertex u (the sum of the weights of edges incident with u in the weighted case). If G is connected then, in the above inequality, there is equality if and only if G is bipartite semiregular. Also [Ke], $\lambda_\infty \leqq n$ with equality if and only if the complement of G is not connected. Let us mention two other relations about λ_∞:

$$\sum_{i=1}^{n} \lambda_i = 2|E(G)| = \sum_{v \in V} d(v)$$

and [F]

$$\lambda_\infty \geqq \frac{n}{n-1} \max\{d(v) \mid v \in V(G)\}.$$

The Laplacian spectrum can directly be obtained from the adjacency spectrum in case G is an (unweighted) d-regular graph. Let A be the adjacency matrix of G and $\mu_1 \leqq \mu_2 \leqq \ldots \leqq \mu_n$ its eigenvalues. Then $d - \mu_n \leqq \ldots \leqq d - \mu_1$ are the Laplacian eigenvalues of G (see [CDS, p. 30]). We will use this relation in the next two sections, since most graphs considered there will be regular unweighted. The adjacency spectrum has been more studied so far, and all the facts we need may be found in [CDS].

3. MAX-CUT AND EIGENVALUES IN SPECIAL CLASSES

In this section we examine two classes of graphs for for which the value of max-cut is known, and where the eigenvalue upper bound is not optimum. For Kneser graphs $K(n, r)$ the eigenvalue upper bound agrees with an upper bound obtained from the size of maximum clique, and it is quite satisfactory. On the contrary, the bound is poor for some circulants. We exhibit a sequence $\{G_n\}$ of graphs of order n where $|E(G_n)| - MC(G_n)$ is increasing while $|E(G_n)| - \frac{1}{4}\lambda_\infty n$ is bounded.

Kneser graphs. Kneser graph $K(n, r)$ is the graph whose vertex set is formed by all r-subsets of an n-set, and two r-subsets form an edge if they are disjoint. We will consider only case $r = 2$. Since $K(n, 2) = \overline{L(K_n)}$, the complement of the line graph of K_n, we have

$$\lambda_\infty K(n, 2) = \lambda_\infty \overline{L(K_n)} = \binom{n}{2} - n .$$

The exact value of max-cut in $K(n, 2)$ has been found in [PT]. The max-cut is formed by $\delta(S)$ for $S = \{\{i, j\} \mid \min(i, j) \leqq p_n\}$ where $p_n = [(2 - \sqrt{2})\, n/2]$. It is more informative to look at the *bipartite density* instead of the value of max-cut. (The bipartite density of a graph G is defined as the ratio $MC(G)/|E(G)|$.) Since each edge belongs to the same number of maximum cliques, the bipartite density of a Kneser graph is bounded by the bipartite density of its maximum clique. The upper bound on the bipartite density of $K(n, 2)$ derived from λ_∞ is

$$\frac{1}{2} + \frac{1}{n - 2} .$$

It is compared with the bound obtained from the size of maximum clique in the following table.

Kneser graph	bound from max. clique	eigenvalue u.b.
$K(4n, 2)$	$\frac{1}{2} + \frac{1}{4n - 2}$	$\frac{1}{2} + \frac{1}{4n - 2}$
$K(4n + 1, 2)$	$\frac{1}{2} + \frac{1}{4n - 2}$	$\frac{1}{2} + \frac{1}{4n - 1}$
$K(4n + 2, 2)$	$\frac{1}{2} + \frac{1}{4n + 2}$	$\frac{1}{2} + \frac{1}{4n}$
$K(4n + 3, 2)$	$\frac{1}{2} + \frac{1}{4n + 2}$	$\frac{1}{2} + \frac{1}{4n + 1}$

For a general Kneser graph $K(n, r)$, the maximum eigenvalue

$$\lambda_\infty(K(n, r)) = \binom{n - r}{r} + \binom{n - r - 1}{r - 1}$$

has been determined by L. Lovász in [Lo]. This gives an upper bound $\frac{1}{2}(1 + r/(n - r))$ on the bipartite density of $K(n, r)$, and the bound is very close to that derived from the size of the maximum clique. The exact solution is known only for $n \leqq (4.3 + o(1))\, r$ (see [PT]).

Circulants. Let $w = (w_1, w_2, \ldots, w_{n-1})$ be a real vector such that $w_i = w_{n-i}$ for all i. The w-circulant is the weighted graph C_w with vertices $0\ 1, \ldots, n-1$ and the weights w_{j-1} on the edge ij, $i < j$. For example, cycles are w-circulants with $w_1 = w_{n-1} = 1$ and $w_i = 0$ otherwise.

Denote by $d(w) := \sum_{i=1}^{n-1} w_i$. Let $w^{(i)}$ be the vector with i-th and $(n-i)$-th entry equal 1 and equal 0 otherwise. Denote by A_i the $n \times n$ matrix with entries 0 except $(A_i)_{jk} = 1$ if $k - j = i \pmod n$. It is well known (see e.g. [CDS]) that the eigenvalues of A_1 are the n-th roots of unity, and that $A_i = A_1^i$. We have

Lemma 3.1. *The Laplacian matrix of C_w is given by*

$$Q(C_w) = d(w)\, I - \sum_{j=1}^{n-1} w_j A_1^j$$

and its spectrum consists of numberss

$$\nu_p = d(w) - \sum_{j=1}^{n-1} w_j \exp\left(\frac{2\pi \mathrm{i}}{n} jp\right), \quad p = 0, 1, \ldots, n-1 .$$

where i *is the imaginary unit.* ■

Notice that $\nu_0 = 0$ and some ν_p may have the same value. We will consider only a subclass of circulants in the sequel. We denote by $C_{n,r}$ the circulant given by $w_1 = w_{n-1} = w_r = w_{n-r} = 1$ and $w_i = 0$ otherwise. Mention that, for $r < n/2$, $C_{n,r}$ is a 4-regular graph consisting of a cycle of length n and all chords connecting vertices of distance r on the cycle. The exact value of max-cut in $C_{n,r}$ has been found in [PT2].

Proposition 3.2 ([PT2]) *The max-cut of $C_{n,r}$, $r < n/2$, is given by*

$$MC(C_{n,r}) = 2n - d$$

where $d = \min(p + |tn - pr|)$ and the minimum is taken over pairs p, t of non-negative integers satisfying $p \equiv n \pmod 2$ and $t \not\equiv r \pmod 2$. ■

Mention that one can compute d by examining all values of $t = 0, 1, \ldots, n$ and taking the best p for each t. The case $r = n/2$ is not so interesting since the circulant $C_{2r,r}$ is either bipartite or becomes bipartite after deleting two edges.

It follows from Lemma 3.1 that

$$\lambda_\infty(C_{n,r}) = 4 - 2 \min_{0 \leqq p \leqq n-1} \left(\cos\frac{2\pi p}{n} + \cos\frac{2\pi pr}{n}\right) \tag{6}$$

for $r < n/2$. Using (6) and Proposition 3.2 we compare the eigenvalue upper bound with the exact value of $MC(C_{n,r})$. The results, for some small values of n and r, are in the following table. We excluded $C_{5,2}$ $(= K_5)$, and the pairs n even, r odd since

$C_{n,r}$ is bipartite for such parameters. It will be shown in the next section that the upper bound is exact for complete and bipartite regular graphs.

n	r	$MC(C_{n,r})$	$\lambda_\infty \cdot n/4$	
6	2	8	9	
7	2	10	10·9	
7	3	10	10·9	
8	2	12	12	exact value
9	2	12	13·5	
9	3	14	15·5	
9	4	12	13·5	
10	2	14	15·6	
10	4	16	18·1	
11	2	16	16·5	
11	3	18	19·8	
11	4	18	19·8	
11	5	16	16·5	
12	2	18	18	exact value
12	4	18	20·2	
13	2	18	20·2	
13	3	22	24·2	
13	4	22	24·2	
13	5	20	21·6	
13	6	18	20·2	
14	2	20	21·8	
14	4	22	24·7	
14	6	24	26·6	
15	2	22	23·1	
15	3	26	28·4	
15	4	24	27·1	
15	5	24	26·1	
15	6	24	27·1	
15	7	22	23·1	
16	2	24	24·7	
16	4	28	29·6	

Since it seems difficult to find an explicit formula for $\lambda_\infty(C_{n,r})$, we will investigate two special classes.

Circulants $C_{n,2}$. Using Proposition 3.2 we get

$$MC(C_{4k,2}) = 6k\,, \quad \text{and} \quad MC(C_{4k+i,2}) = 6k + 2(i-1)\,, \quad i = 1, 2, 3\ .$$

For k large we have $\lambda_\infty C_{4k,2} \doteq 6.25$, which gives an upper bound $6.25k$ (while the actual value is $6k$).

Circulants $C_{r^2+1,r}$, *r even*. Using Proposition 3.2 we get $MC(C_{r^2+1,r}) = 2n - 2r =$

$= 2(r^2 - r + 1)$. The maximum eigenvalue can be estimated

$$\lambda_\infty(C_{r^2+1,r}) = 8 - cr^{-2} + O(r^{-3}) \quad (c \sim 2\pi^2).$$

Hence the eigenvalue upper bound tends to $2n - \pi^2/2$.

Ramanujan graphs are r-regular graphs for which

$$\lambda_2(G) \geqq r - 2\sqrt{(r-1)} \quad \text{and} \quad \lambda_\infty \leqq r + 2\sqrt{(r-1)}.$$

This interesting class of graphs was introduced by Lubotzky, Phillips and Sarnak [LPS], and for any $r = p + 1$, where p is a prime congruent to 1 mod 4, an infinite family was constructed. We have $MC(G) \leqq \frac{1}{4}nr + \frac{1}{2}n\sqrt{(r-1)}$ for a Ramanujan graph G.

4. EXACT GRAPHS

The eigenvalue upper bound of Theorem 2.2 can be tight only in case that we have large cuts separating two large sets of vertices (each close to half of the vertices). Examples of such graphs are complete graphs and their Cartesian products, or tensor (categorical) products.

In this section we describe some classes for which the upper bound is best possible. For simplicity, we restrict ourselves to graphs of even order only. Let us call a graph G *exact* if $MC(G) = \lambda_\infty n/4$. We show that the following graphs are exact (with possible restriction on even parity of some parameters): complete graphs and their categorical and cartesian products, bipartite regular graphs, line graphs of semi-regular bipartite graphs, line graph $L(K_{4k+1})$, complement of $L(K_{m,n})$. We also show that exact graphs are closed under the cartesian product. The maximal cuts in these graphs are easily found, and Theorem 2.2 provides a proof of their optimality. The used facts on λ_∞ may be found in [CDS] (cf. remark in the end of Section 2).

Proposition 4.1. *The cartesian product $G \times H$ of two exact graphs is exact.*

Proof. We have $\lambda_\infty(G \times H) = \lambda_\infty(G) + \lambda_\infty(H)$ by Proposition 2.5. Conversely, let $\delta(V_0)$ and $\delta(W_0)$ be the maximal cut of G and H, respectively. Then $|V_0| = \frac{1}{2}|V(G)|$, $W_0 = \frac{1}{2}|V(H)|$, and $\delta(V_0 \times W_0 \cup (V(G) \setminus V_0) \times (V(H) \setminus W_0))$ is the maximal cut in the product. ■

Complete graphs. We have $\lambda_\infty K_n = n$. The max-cut in K_n is obviously

$$\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$$

which agrees with the upper bound. Hence the cartesian product of complete graphs of even order is exact. We show that also complements of these products are exact.

Categorical product. $G \otimes H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and edges $(u, v)(u', v')$ if $uu' \in E(G)$ and $vv' \in E(H)$.

The product $K_m \otimes K_n$ of complete graphs equals $\overline{K_m \times K_n}$, the complement of their cartesian product. Since

$$\lambda_\infty(\overline{G \times H}) = mn - \lambda_2(G \times H) = mn - \min\{\lambda_2(G), \lambda_2(H)\},$$

where $n = |G|$ and $m = |H|$, we have $\lambda_\infty(K_n \otimes K_m) = mn - \min(m, n)$.

Proposition 4.2. *Let $n \leqq m$, n even. Then $K_n \otimes K_m$ is exact.*

Proof. The maximal cut is $\delta(\{1, \ldots, \frac{1}{2}n\} \times \{1, \ldots, m\})$. ■

The results easily generalize to products of greater number of complete graphs. In particular, the max-cut in the complement of d-dimensional cartesian cube is $2^{d-1}(2^{d-1} - 1)$.

Bipartite regular graphs are exact, since $\lambda_\infty(G) = 2r$ for an r-regular bipartite graph G.

Line graphs of bipartite graphs and their complements. A bipartite graph G is (r, s)-*semiregular* if r and s are the degrees in either bipartite class. If $r \neq 1$ and $s \neq 1$, we have

$$\lambda_\infty L(G) = r + s, \quad \text{and}$$
$$\lambda_\infty \overline{L(G)} = |E(G)| - r - s + \lambda_{n-1}(G), \quad n = |V(G)|.$$

In particular, for $m, n \neq 1$, we have

$$\lambda_\infty \overline{L(K_{m,n})} = mn - \min(m, n).$$

Proposition 4.3. *Let G be a bipartite (r, s)-semiregular graph where both r and s are even. Then $L(G)$ is exact.*

Proof. The edge set $E(G)$ can be decomposed into two $(\frac{1}{2}r, \frac{1}{2}s)$-semiregular subgraphs which form the optimum bipartition of $L(G)$. The eixstence of such decomposition of $E(G)$ is well known. ■

Proposition 4.4. *Let $n \leqq m$, n even. Then $\overline{L(K_{m,n})}$ is exact.*

Proof. We have $\lambda_\infty = n(m - 1)$. The max-cut is obtained by $\delta(\{ij \mid i = 1, \ldots, \frac{1}{2}n, j = 1, \ldots, m\})$. ■

Line graphs of complete graphs. We have $\lambda_\infty L(K_n) = 2(n - 1)$.

Proposition 4.5. *$L(K_{4r+1})$ is exact.*

Proof. It is well-known that K_{4r+1} has a $2r$-regular factor, which is the maximal cut in the line graph. ■

Let us remark that also the circulants $C_{8,2}$ and $C_{12,2}$ are exact.

References

[A] *N. Alon:* Eigenvalues and expanders, Combinatorica *6*, (1986) 83—96.

[AM] *N. Alon, V. D. Milman:* λ_1, isoperimetric inequalities for graphs and superconcentrators, J. Comb. Th. B *38* (1985) 73—88.

[AnM] *W. N. Anderson, T. D. Morley:* Eigenvalues of the Laplacian of a graph, Lin. Multilin. Algebra *18* (1985) 141—145.

[B] *F. Barahona:* The max-cut problem in graphs not contractible to K_5, Oper. Res. Lett. *2* (1983) 107—111.

[BGJR] *F. Barahona, M. Grötschel, M. Jünger, G. Reinelt:* An application of combinatorial optimization to statistical physcis and circuit layout design, Oper. Res. *36* (1988) 493—513.

[CDS] *D. M. Cvetkovic, M. Doob,* and *H. Sachs:* Spectra of graphs — Theory and applications, VEB Deutscher Verlag der Wiss. Berlin 1979/Acad. Press, N.Y. 1979.

[F] *M. Fiedler:* Algebraic connectivity of graphs, Czechoslovak Math. J. *23* (*98*) (1973) 298—305.

[GN] *M. Grötschel* and *G. L. Nemhauser:* A polynomial algorithm for the max-cut problem on graphs without long odd cycles, Math. Programming *29* (1984) 28—40.

[GP] *M. Grötschel* and *W. R. Pulleyblank:* Weakly bipartite graphs and the max-cut problem, Oper. Res. Lett. *1* (1981) 23—27.

[H] *F. Hadlock:* Finding a maximum cut of a planar graph in polynomial time, SIAM J. on Comp. *4* (1975) 221—225.

[Ka] *R. M. Karp:* Reducibility among combinatorial problems, in Complexity of Computer Computations, R. E. Miller, J. W. Thatcher (eds.), Plenum Press, New York 1972, pp. 85—103.

[Ke] *A. K. Kel'mans:* Properties of the characteristic polynomial of a graph, "Kibernetiky — na službu kommunizmu" vol. *4*, Energija, Moskva-Leningrad, 1967, pp. 27—41 (in Russian).

[L] *P. Lankaster:* Theory of Matrices, Acad. Press, New York-London 1969.

[Lo] *L. Lovász:* On the Shannon capacity of a graph, IEEE Trans. Inform. Theory *25* (1979) 1—7.

[LPS] *A. Lubotzky, R. Phillips* and *P. Sarnak:* Ramanujan graphs, Combinatorica, *8* (1988) 261—277.

[M1] *B. Mohar:* Isoperimetric numbers of graphs, J. Comb. Th. B, to appear.

[M2] *B. Mohar:* The Laplacian spectrum of graphs, submitted.

[NP] *J. Nešetřil* and *S. Poljak:* A remark on max-cut problem with an application to digital-analogue convertors, Oper. Res. Lett. *4* (1986) 289—291.

[OD] *G. I. Orlova* and *Y. G. Dorfman:* Finding the maximal cut in a graph, Engrg. Cybernetics *10* (1972) 502—506.

[PT] *S. Poljak* and *Z. Tuza:* Maximum bipartite subgraphs of Kneser graphs, Graphs and Combinatorics, *3* (1987) 191—199.

[PT1] *S. Poljak* and *D. Turzik:* A polynomial heuristic for certain subgraph optimization problems with guaranteed lower bound, Discrete Math. *58* (1986) 99—104.

[PT2] *S. Poljak* and *D. Turzik:* Maximum bipartite subgraphs of circulants, submitted.

[T] *R. M. Tanner:* Explicit concentrators from generalized *n*-gons, SIAM J. Alg. Disc. Meth. *5* (1984) 287—293.

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