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GLOBAL BEHAVIOUR OF SOLUTIONS TO SOME NONLINEAR DIFFUSION EQUATIONS

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0. INTRODUCTION

The present paper deals with the large time behaviour of solutions to the problem

$$\text{(I)} \qquad \begin{aligned} & u_t = \Delta u^m + u^p - au && x \in D\,, \quad t > 0\,,\\ & u(x,t) = 0 && x \in D\,, \quad t > 0\,,\\ & u(x,0) = u_0(x)\ (\geqq 0) && x \in D\,, \end{aligned}$$

where $D \subset R^N$ is a smoothly bounded domain, $a \geqq 0$, $m > 0$, $p > 1$ and $pm^{-1} < (N + 2)(N - 2)^{-1}$ if $N \geqq 3$. The equation in (I) without the reaction term $u^p - au$ is well known for $0 < m < 1$ as the plasma or fast diffusion equation, for $m = 1$ as the heat conduction equation and for $m > 1$ as the porous medium or slow diffusion equation.

Problems related to Problem (I) have been studied by many authors (e.g. Alikakos [1], Ball [3], Fila and Filo [6], Galaktionov [9], Levine and Sacks [11], Lions [12], Nakao [13], [14], Ni, Sacks and Tavantzis [15], Payne and Sattinger [16], Sacks [17], [18], Tsutsumi [19], understanding that the present list of authors is not complete).

It is known that Problem (I) does not admit a global solution for every u_0 if $m < p$ or if $m = p$, $a = 0$ and D is "large enough". For $m < p$ it is shown in [6] that a solution of a (slightly) more general problem blows up in a finite time if the function u_0^m belongs to a certain unstable set B (for the definition see Section 2). Here we prove a corresponding blow-up result in a case which is not included in [6], namely if $m = p$, $a > 0$ and D is "large enough".

Global existence and decay to zero in L^∞-norm of solutions to Problem (I) with $u_0^m \in W$ (W is the potential well, for the definition see Section 2) was proved by Nakao in [14] for $1 \leqq m < p$ and $a = 0$. We extend his results to $0 < m < 1$, $a = 0$ and $0 < m \leqq p$, $a > 0$. In the case of $0 < m < 1$ it is demonstrated that the solution vanishes in a finite time if $u_0^m \in W$. To prove this we first show that the solution is bounded in L^∞-norm in a similar way as Nakao in [14], and using the "potential well" method we derive its convergence to zero in L^{m+1}-norm. The existence of the extinction time follows then by comparison with a solution of the fast diffusion equation, which is known to vanish in a finite time. As is expected, for $a > 0$ the absorptive term $-au$ causes that the corresponding set W is larger than for $a = 0$, therefore our result does not follow from [14] by obvious comparison arguments. As concerns the case $m = p$, as far as we know, it has not been studied by the "potential well" method.

If $p < m$ all solutions are global and bounded (see [18]). Stabilization of solutions to Problem (I) for this case was studied in [5] in one space dimension.

1. PRELIMINARIES

Let us first introduce some notation: $Q_T = D \times (0, T)$, $S_T = \partial D \times (0, T)$, $|D|$-Lebesgue measure of the set D, $|u|_q = \|u\|_{L^q(D)}$, $1 \leqq q \leqq \infty$, $|u|_q^q = (|u|_q)^q$, ${}^+H_0^1 = \{u \in H_0^1(D): u \geqq 0 \text{ a.e. in } D, \; u \not\equiv 0\}$, $\|u\| = (\int_D |\nabla u|^2 \, dx)^{1/2}$ $\int_D h(t) = \int_D h(x, t) \, dx$, $\iint_{Q_T} h = \iint_{Q_T} h(x, t) \, dx \, dt$ and $(u(t), v(t)) = \int_D u(t) \, v(t)$.

Definition 1. By a solution of Problem (I) on $[0, T]$ we mean a nonnegative function u such that

$$u \in C([0, T]; L^2(D)) \cap L^\infty(Q_T), \quad u^m \in L^\infty(0, T; H_0^1(D)),$$

and u satisfies

$$(1.1) \qquad (u(t), \varphi(t)) - \iint_{Q_t} (u\varphi_t - \nabla u^m \nabla \varphi + f(u) \varphi) = (u_0, \varphi(0))$$

for all $t \in [0, T]$ and $\varphi \in H^1(0, T; L^2(D)) \cap L^\infty(0, T; H_0^1(D))$, where $f(u) = u^p - au$.

A subsolution (supersolution) of Problem (I) is defined as above with equality in (1.1) replaced by $\leqq$ ($\geqq$) whenever $\varphi \geqq 0$.

By E we shall denote the set of all nontrivial nonnegative stationary solutions of Problem (I).

In the sequel we shall often denote the solution $u(=u(x, t))$ of Problem (I) by $u(t, u_0)$.

Throughout this paper we shall use the following hypotheses about the data D and u_0:

(H1) D is a bounded domain in R^N whose boundary ∂D is of class C^3,

(H2) $u_0^m \in L^\infty(D) \cap H_0^1(D)$ and $u_0 \geqq 0$ a.e. in D.

We shall refer to these hypotheses collectively by (H). Afterwards we shall need the following basic results.

Proposition 1 (Comparison principle). *Suppose that D satisfies* (H1) *and that u_0 and v_0 both satisfy* (H2). *If u is a subsolution and v is a supersolution of Problem* (I) *on $[0, T]$ with $u_0 \leqq v_0$ then $u \leqq v$ a.e. in Q_T.*

For the proof of this proposition for $m \geqq 1$ we refer to [2] and for $0 < m < 1$ to [7].

Proposition 2 (Existence). *Suppose that* (H) *holds. Then there exists a time* t_{max}, $0 < t_{max} \leqq \infty$ (*which depends on the data* D, m, f *and* u_0) *such that Problem* (I) *possesses a unique solution* u *on* $[0, T]$ *for any* $T \in (0, t_{max})$. *If* $t_{max} < \infty$ *then*

$$(1.2) \qquad \lim_{t \to t_{max}^-} |u(t, u_0)|_\infty = \infty .$$

Moreover, for $0 \leqq s < t < t_{max}$ u *satisfies*

$$(1.3) \qquad \frac{4m}{(m+1)^2} \int_s^t |(u^{(m+1)/2})_t|_2^2 + J(u^m(t, u_0)) \leqq J(u^m(s, u_0)),$$

where

$$(1.4) \qquad J(w) = \tfrac{1}{2}\|w\|^2 - \int_D \int_0^w f(r^{1/m})\, dr .$$

For the proof of Proposition 2 for $m \geqq 1$ we refer to [11] and for $0 < m < 1$ to [7].

2. THE CASE $0 < m < p$

Throughout this section we shall always use the following assumptions about the parameters m and p:

$$(2.1) \qquad \begin{aligned} &0 < m < p, \quad 1 < p \quad \text{for} \quad N = 1, 2 \quad \text{and} \\ &0 < m < p < (N+2)\, m/(N-2), \quad 1 < p \quad \text{for} \quad N \geqq 3 . \end{aligned}$$

In the same way as in [6] put

$$(2.2) \qquad d = k \inf_{w \in {}^+H^1_0} \left(\frac{(\|w\|^2 + a|w|_{1+1/m}^{1+1/m})^{1/2}}{|w|_{1+p/m}} \right)^{2(p+m)/(p-m)},$$

where $k = \min(1/2,\ m/(m + \operatorname{sign} a)) - m/(m+p)$. By the Sobolev embedding theorem, $|w|_{1+p/m} \leqq C_s \|w\|$, $C_s > 0$, and it is easy to see that d is positive. Using the notation

$$(2.3) \qquad K(w) = \|w\|^2 + a|w|_{1+1/m}^{1+1/m} - |w|_{1+p/m}^{1+p/m},$$

we set

$$(2.4) \qquad W = \{w \in {}^+H^1_0 : J(w) < d \text{ and } K(w) > 0\} \cup \{0\}$$

and

$$(5.) \qquad B = \{w \in {}^+H^1_0 : J(w) < d \text{ and } K(w) < 0\} .$$

We shall call the sets W and B a *stable set* (potential well) and an *unstable set*, respectively. The number d given by (2.2) is a modification of the "depth of the potential well", which was introduced by Payne and Sattinger in [16] for semilinear parabolic equations that cover our Problem (I) for $m = 1$ and $a = 0$.

Remark. If $a = 0$ or $m = 1$ then $d = \inf_{w \in {}^+H^1_0} (\sup_{0 \leqq \lambda < \infty} J(\lambda w))$ (see e.g. [13], [19]) and it is not difficult to verify that in this case

$$W = \{w \in {}^+H^1_0 \cup \{0\} : 0 \leqq J(\lambda w) < d \text{ for } 0 \leqq \lambda \leqq 1\}$$

and

$$B = \{w \in {}^+H_0^1 \colon J(\lambda w) < d \text{ for } 1 \leqq \lambda < \infty\}.$$

Moreover,

$$d = \min_{v \in E} J(v^m)$$

(see e.g. [6]).

Theorem 2.1. *Assume that D and u_0 satisfy* (H) *and let* (2.1) *hold. Suppose further that $u_0^m \in W$. Then there exists a global solution $u(t, u_0)$ of Problem* (I), *$u^m(t, u_0) \in W$ for $0 \leqq t < \infty$, and it satisfies the following decay property*:

(i) *If $0 < m < 1$ then there exists a time T_e, $0 \leqq T_e < \infty$ such that*

$$u(t, u_0) \equiv 0 \quad \text{for} \quad T_e \leqq t < \infty . \tag{2.6}$$

(ii) *If $m = 1$ then there exist positive constants C, α such that*

$$|u(t, u_0)|_\infty \leqq C \exp(-\alpha t) \quad \text{for} \quad 0 \leqq t < \infty .$$

(iii) *If $m > 1$ then there exists a positive constant C such that*

$$|u(t, u_0)|_\infty \leqq C(t + 1)^{-1/(m-1)} \quad \text{for} \quad 0 \leqq t < \infty .$$

Remark. The rate of convergence to zero in (iii) is "optimal" only for the case $a = 0$ as for $a > 0$ we deduce from a simple comparison argument that all solutions of Problem (I) which decay to zero in L^∞-norm decay at least as const. $\exp(-\alpha' t)$ for some $\alpha' > 0$.

To make the description of the flow given by Problem (I) by the "energy" method more complete, let us recall the following result (for the proof see [6]).

Theorem 2.2. *Assume that D, u_0 satisfy* (H) *and let* (2.1) *hold. If $u_0^m \in B$ then $u^m(t, u_0) \in B$ for $0 \leqq t < t_{\max}$ and*

$$t_{\max} \leqq ((|D|^{-1} |u_0|_{m+1}^{m+1})^{(p-1)/(m+1)} (p-1)(1-C))^{-1},$$

where the constant $C \in (0, 1)$ depends on d, u_0, m and p, i.e. the solution blows up in a finite time in L^∞-norm for $u_0^m \in B$.

The proof of Theorem 2.1 will be preceded by some useful lemmas.

Lemma 2.3. *Let $u_0^m \in W$ and $\nu = (J(u_0^m)/d)^{(p-m)/(p+m)}$. Then $u^m(t, u_0) \in W$ for $0 \leqq t < t_{\max}$ and u satisfies*

$$|u(t, u_0)|_{m+p}^{m+p} \leqq \nu(\|u^m(t, u_0)\|^2 + a|u(t, u_0)|_{m+1}^{m+1}) \tag{2.7}$$

and

$$k(\|u^m(t, u_0)\|^2 + a|u(t, u_0)|_{m+1}^{m+1}) < J(u^m(t, u_0)) < d \tag{2.8}$$

for $0 \leqq t < t_{\max}$.

Proof of Lemma 2.3. To see that W is nonempty and invariant we can proceed in the same way as in the proof of Theorem 1 of [6], and we omit it here. The estimate

(2.8) follows immediately from (1.3) and (2.4). Now (1.3) and (2.2) yield

$$(2.9) \qquad J(u^m(t)) \leqq d^{-1} J(u_0^m)\, k(\|u^m(t)\|^2 + \\ + a|u(t)|_{m+1}^{m+1})^{(p+m)/(p-m)} (|u(t)|_{m+p}^{m+p})^{2m/(m-p)}$$

for $0 \leqq t < t_{\max}$. As $0 < K(u^m(t))$, (2.9) gives

$$(2.10) \qquad |u(t)|_{m+p}^{m+p} \leqq d^{-1} J(u_0^m) (\|u^m(t)\|^2 + \\ + a|u(t)|_{m+1}^{m+1})^{(p+m)/(p-m)} (|u(t)|_{m+p}^{m+p})^{1-(p+m)/(p-m)}$$

which implies (2.7).

Lemma 2.4. *Let $u_0^m \in W$. Then u satisfies*

$$(2.11) \qquad |u(t, u_0)|_{m+1} \leqq |u_0|_{m+1}(1 + Ct)^{-1/(p-1)}, \quad 0 \leqq t < t_{\max},$$

where $C = (v^{-1} - 1)(p - 1)(|D|^{-1} |u_0|_{m+1}^{m+1})^{(p-1)/(m+1)}$.

Proof of Lemma 2.4. Inserting $u^m(t)$ into (1.1) we obtain using (2.7)

$$(2.12) \qquad \frac{\mathrm{d}}{\mathrm{d}t} |u(t)|_{m+1}^{m+1} \leqq (m + 1)(1 - v^{-1}) |u(t)|_{m+p}^{m+p} \quad \text{for a.e.} \quad t \in [0, t_{\max}).$$

Now, using the Hölder inequality, (2.12) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} |u(t)|_{m+1}^{m+1} + (m + 1)(v^{-1} - 1) |D|^{(1-p)/(m+1)} |u(t)|_{m+1}^{m+p} \leqq 0$$

for a.e. $t \in [0, t_{\max})$. Hence (2.11) follows by the standard comparison theorem for ordinary differential equations.

Lemma 2.5. *Let $|u(t, u_0)|_{m+p}$ be bounded on $[0, t_{\max})$. Then $t_{\max} = \infty$ and*

$$(2.13) \qquad |u(t, u_0)|_\infty \leqq C(|u_0|_\infty, \sup_{0 \leqq t < \infty} |u(t, u_0)|_{m+p})$$

for $0 \leqq t < \infty$.

Proof of Lemma 2.5. We use Moser's technique just like Nakao in [14] (see also Alikakos [1]). As the case of $0 < m < 1$ is not considered there, let us outline the proof for the sake of completeness.

Let $r > m$ and $0 < T < t_{\max}$. Inserting $\varphi = u^r$ into (1.1) and performing obvious manipulations we obtain

$$(2.14) \qquad \frac{\mathrm{d}}{\mathrm{d}t} |u(t)|_{r+1}^{r+1} + \frac{4mr(r + 1)}{(m + r)^2} \|u^{(m+r)/2}(t)\|^2 = \\ = (r + 1) |u(t)|_{p+r}^{p+r} - a(r + 1) |u(t)|_{r+1}^{r+1}$$

for a.e. $t \in [0, T]$. If $N \geqq 3$ the first term on the right hand side of (2.14) may be estimated as follows,

$$(2.15) \qquad \int_D u^{p+r} \leqq \left(\int_D u^{r+1}\right)^{P_1} \left(\int_D u^{m+p}\right)^{P_2} \left(\int_D u^{(r+m)N/(N-2)}\right)^{P_3},$$

where

$$P_1 = (2(m+p) - N(p-m))/(2(m+p) - N(1-m)),$$
$$P_2 = 2(p-1)/(2(m+p) - N(1-m)),$$
$$P_3 = (N-2)(p-1)/(2(m+p) - N(1-m))$$

and if $N = 2$,

$$\int_D u^{p+r} \leqq (\int_D u^{r+1})^{Q_1} (\int_D u^{m+p})^{Q_2} (\int_D u^{(r+m)(m+p)/m})^{Q_3}, \tag{2.16}$$

where

$$Q_1 = m/(p-1+m), \quad Q_2 = p(p-1)/(m+p),$$
$$Q_3 = m(p-1)/(m+p)(p-1+m).$$

Now using the Sobolev embedding theorem, the last term of (2.15)((2.16)) may be estimated by the gradient of $u^{(m+r)/2}$ and then using Young's inequality we have

$$(r+1)|u|_{p+r}^{p+r} \leqq \varepsilon \|u^{(m+r)/2}\|^2 + C(\varepsilon)(r+1)^Q (|u|_{m+p}^{m+p})^R |u|_{r+1}^{r+1} \tag{2.17}$$

where

$$Q = (2(m+p) - N(1-m))/(2(m+p) - N(p-m)),$$
$$R = 2(p-1)/(2(m+p) - N(p-m))$$

if $N \geqq 3$ and $Q = Q_1^{-1}$, $R = QQ_2$ if $N = 2$.

Putting $\varepsilon = 2mr(r+1)/(m+r)^2$, (2.14) and (2.17) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|_{r+1}^{r+1} + \frac{m}{2}\|u^{(m+r)/2}(t)\|^2 \leqq \bar{C}(r+1)^Q (|u(t)|_{m+p}^{m+p})^R |u(t)|_{r+1}^{r+1} \tag{2.18}$$

where

$$\bar{C} = Q^{-1}((m+r)^2 CNP_2/4mr(r+1))^{RN/2} \quad \text{if} \quad N \geqq 3 \quad \text{and}$$
$$\bar{C} = Q^{-1}((m+r)^2 C(p-1)/2m(p-1+m)\, r(r+1))^{(p-1)/m}$$
$$\text{if} \quad N = 2.$$

As $|u(t, u_0)|_{m+p}$ is bounded on $[0, t_{\max})$, (2.18) can be rewritten into

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|_{r+1}^{r+1} + C_0\|u^{(m+r)/2}(t)\|^2 \leqq C_1(r+1)^Q |u(t)|_{r+1}^{r+1} \tag{2.19}$$

for any $r > m$ and a.e. $t \in [0, T]$. At this step we need the following proposition which for $m \geqq 1$ is a special case of Lemma 3.1 of [14]. As for $0 < m < 1$ the arguments of [14] need some modifications we shall outline the proof at the end of this section.

Proposition 2.6. *Let $u(t)$ be a function defined on $D \times [0, T]$, $0 < T \leqq \infty$ (appropriately smooth) satisfying (2.19) for any $r > m$ with some constants $C_0(>0)$, $C_1(>0)$ and $Q(\geqq 1)$. Suppose that $u_0 = u(0) \in L^\infty(D)$, $\sup_{0 \leqq t \leqq T} |u(t)|_{m+1} < \infty$ and in the case $N \geqq 3$, $m(N+2) > N-2$. Then*

$$\sup_{0 \leqq t \leqq T} |u(t)|_\infty \leqq C(|u_0|_\infty, \sup_{0 \leqq t \leqq T} |u(t)|_{m+1}, C_1). \tag{2.20}$$

Now the constant C in (2.20) does not depend on T, hence $t_{\max} = \infty$ and the proof of Lemma 2.5 is complete.

Proof of Theorem 2.1. We emphasize the proof of the assertion (i) as the assertions (ii) and (iii) may be obtained using our definition of d and repeating Nakao's arguments of [13], [14], hence we only sketch their proofs.

(i) By Lemmas 2.3–5 we know that $t_{\max} = \infty$ and $|u(\cdot, u_0)|_\infty$ is bounded on $[0, \infty)$. Put

$$L = \max(0, M^{p-1} - a), \quad M = \sup_{0 \leqq t < \infty} |u(t, u_0)|_\infty, \tag{2.21}$$

and consider for a while the problem

$$\begin{aligned} &v_t = \Delta v^m + Lv && x \in D, \quad t > 0, \\ &v(x, t) = 0 && x \in \partial D, \quad t < 0, \\ &v(x, 0) = v_0(= u(T, u_0)) && x \in D, \end{aligned} \tag{2.22}$$

where T, sufficiently large, will be chosen later. We shall consider the case $L > 0$ as the case $L = 0$ follows easily. Putting $v = w \exp(Lt)$ and changing the time scale to $t = -c^{-1} \ln(1 - cs)$, $c = L(1 - m)$, (2.22) may be rewritten into

$$\begin{aligned} &z_s = \Delta z^m && x \in D, \quad 0 < s < T_c = c^{-1}, \\ &z(x, s) = 0 && x \in \partial D, \quad 0 < s < T_c, \\ &z(x, 0) = v_0 && x \in D, \end{aligned} \tag{2.23}$$

where $z(x, s) = w(x, t(s))$ and $s(t) = c^{-1}(1 - \exp(-ct))$. Now it is well known that any solution of Problem (2.23) considered on $(0, \infty)$ has a finite extinction time $t_e = t_e(v_0)$, i.e. $z \equiv 0$ for $s \geqq t_e$ (see e.g. [4], [7]). As (2.1) holds we easily obtain

$$t_e(v_0) \leqq |v_0|_{m+1}^{1-m} / C(1 - m). \tag{2.24}$$

As concerns (2.23), if v_0 is so small that $t_e(v_0) < T_c$, then $z \equiv 0$ for $s \geqq t_e$, but then also $v \equiv 0$ for $t \geqq -c^{-1} \ln(1 - ct_e)$. Hence we can choose by (2.11) T so large that $t_e(u(T, u_0)) < T_c$, and using simple comparison arguments we have

$$u(x, t) \equiv 0 \quad \text{for} \quad t \geqq T_e = T - c^{-1} \ln(1 - ct_e(u(T, u_0))), \quad \text{i.e. (2.6)}.$$

If $a = 0$ the assertion (iii) of Theorem 2.1 is proved in [14] and (ii) follows e.g. from [13] and [14]. As we have already mentioned, for $a > 0$ our results do not follow from [14] by comparison arguments, but thanks to our definition of d for $a > 0$ (cf. (2.2)) we can obtain the same results. First, by the same way as in Theorem 3.1 of [13] we may obtain the estimates

$$\begin{aligned} &J(u^m(t, u_0)) \leqq C(1 - d^{-1} J(u_0^m)) (t + 1)^{-2m/(m-1)} \quad \text{if} \quad m > 1 \quad \text{and} \\ &J(u(t, u_0)) \leqq C(1 - d^{-1} J(u_0)) \exp(-\lambda t), \quad \lambda > 0 \quad \text{if} \quad m = 1, \end{aligned} \tag{2.25}$$

and we omit it here. Then using the Sobolev embedding theorem, (2.8) and (2.25), we have

$$(2.26)\qquad |u(t,u_0)|_{m+p} \leqq (C_s\|u^m(t,u_0)\|)^{1/m} \leqq C(t+1)^{-1/(m-1)} \quad \text{if} \quad m > 1\,,$$
$$|u(t,u_0)|_{1+p} \leqq C\exp(-\lambda' t)\,, \quad \lambda' > 0 \quad \text{if} \quad m = 1\,.$$

Now put $w(t) = (t+1)^{1/(m-1)}\,u(t)$ if $m > 1$ and $w(t) = \exp(\lambda' t)\,u(t)$ if $m = 1$. Then $w(t)$ satisfies, after changing the time scale,

$$(2.27)\qquad w_s = \Delta w^m + \exp((m-p)\,s/(m-1))\,w^p + ((m-1)^{-1} - a\exp(s))\,w \quad \text{if} \quad m > 1\,,$$
$$w_t = \Delta w + \exp((1-p)\,\lambda t)\,w^p + (\lambda - a)\,w \quad \text{if} \quad m = 1\,.$$

From (2.26) and (2.27) we can obtain the boundedness of w in the L^∞-norm in the same way as in Theorem 3.1 of [14], hence the conclusion.

Proof of Proposition 2.6. Put $r_k = 2^k + m - 1$, $d_k = C_1(r_k+1)^Q$, $q_k = (r_k+1)/(r_{k-1}+1/2)$ and $v = u^{r_{k-1}+1/2}$ for $k = 1, 2, 3, \ldots$. Then (2.19) takes the form

$$(2.28)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_D v^{q_k}(t) \leqq -C_0\|v(t)\|^2 + d_k\int_D v^{q_k}(t)\,.$$

Now we use the Nirenberg-Gagliardo inequality ([8, p. 27, Theorem 10.1]) in the form

$$(2.29)\qquad \int_D v^{q_k} \leqq C_F^{q_k}\|v\|^{bq_k}\left(\int_D v^{s_k}\right)^{q_k(1-b)/s_k}\,,$$

where $s_k = (r_{k-1}+1)/(r_{k-1}+1/2)$ and $b = 2N(q_k - s_k)/q_k(2N - s_k(N-2))$. Let us note that $q_k > 2$, $q_k \to 2$ as $k \to \infty$ and $v^{s_k} = u^{r_{k-1}+1}$. As we have supposed $N - m(N+2) < 2$, for $N \geqq 3$, we can apply the Young inequality and (2.29) then yields

$$(2.30)\qquad \int_D u^{r_k+1} \leqq \varepsilon_k\|u^{r_{k-1}+1/2}\|^2 + C(\varepsilon_k, k)\left(\int_D u^{r_{k-1}+1}\right)^{p_k}\,,$$

where $0 < \varepsilon_k < 1$ will be given later, $p_k = (2^{k+1} - e)/(2^k - e)$, $e = N - m(N+2)$ and $C(\varepsilon_k, k)$ may be estimated by $C\varepsilon_k^{-N/(2-e)}$. Now choosing $\varepsilon_k = 2^{-Qk-\mu}$ for μ so large that $d_k\varepsilon_k + \varepsilon_k^2 \leqq C_0$ it follows from (2.28) and (2.30) that

$$(2.31)\qquad \frac{1}{|D|}\frac{\mathrm{d}}{\mathrm{d}t}\int_D u^{r_k+1} \leqq -\varepsilon_k\frac{1}{|D|}\int_D u^{r_k+1} + \varepsilon_k\delta_k\left(\sup_{0\leqq t\leqq T}\frac{1}{|D|}\int_D u^{r_{k-1}+1}\right)^{p_k}$$

for $k = 1, 2, \ldots$, where $\delta_k = (d_k + \varepsilon_k)\,C(\varepsilon_k, k)\,|D|^{p_k-1}/\varepsilon_k$, hence

$$(2.32)\qquad \frac{1}{|D|}\int_D u^{r_k+1} \leqq \max\left(\delta_k\left(\sup_{0\leqq t\leqq T}\frac{1}{|D|}\int_D u^{r_{k-1}+1}\right)^{p_k}, \frac{1}{|D|}\int_D u_0^{r_k+1}\right)$$

for $k = 1, 2, \ldots$. Now we can take μ so large that $\delta_k > 1$ and δ_k may be then estimated by $c2^{Q'k}$ for some $c = c(C_1) > 0$ and $'Q = 'Q(Q, \mu, N, e) > 0$. Thus, if we denote

$$K = \max 1,\ |u_0|_\infty^{m+2},\ \left(\sup_{0\leqq t\leqq T}\frac{1}{|D|}\int_D u^{m+1}(t)\right)^{p_1},$$

from (2.32) we can obtain inductively

$$\frac{1}{|D|}\int_D u^{r_k+1} \leqq \delta_k \delta_{k-1}^{p_k} \ldots \delta_1^{p_2 p_3 \ldots p_k} K^{p_2 p_3 \ldots p_k}. \tag{2.33}$$

Now, since $p_k \leqq n_k = (2^k - 1)/(2^{k-1} - 1)$ for $k \geqq 2$, (2.33) yields

$$\begin{aligned} \int_D u^{r_k+1} &\leqq |D|\, c^{1+n_k+\ldots+n_2 n_3 \ldots n_k} K^{n_2 n_3 \ldots n_k}\, 2^{Q'(k+(k-1)n_k+(k-2)n_k n_{k-1}+\ldots+n_2 n_3 \ldots n_k)} \\ &\leqq |D|\, c^{2^{k+1}-1}\, 2^{Q'(k+2^{k+2}-4)} K^{2^k-1}. \end{aligned} \tag{2.34}$$

Taking the $(r_k + 1)$-st root of (2.34) and letting $k \to \infty$ we obtain

$$|u(t)|_\infty \leqq c^2\, 2^{4Q'} K,$$

hence (2.20).

3. THE CASE $m = p > 1$

In this section we shall discuss Problem (I) for $m = p > 1$, i.e.

$$\begin{aligned} &u_t = \Delta u^m + u^m - au && x \in D, \quad t > 0, \\ &u(x, t) = 0 && x \in \partial D, \quad t > 0, \\ &u(x, 0) = u_0(x)\ (\geqq 0) && x \in D, \end{aligned} \tag{3.1}$$

where $a \geqq 0$. Before we introduce our result, let us collect some known facts.

Theorem 3.1. *Let λ_1 denote the first eigenvalue and φ_1 the corresponding eigenfunction of the Dirichlet problem $\Delta\varphi + \lambda\varphi = 0$ in D, $\varphi = 0$ on ∂D, and let* (H) *hold.*

(i) *If $\lambda_1 > 1$, $a \geqq 0$ then* $\lim_{t\to\infty} |u(t, u_0)|_\infty = 0$.

(ii) *If $\lambda_1 = 1$, $a = 0$ then* $\lim_{t\to\infty} |u(t, u_0) - C\varphi_1^{1/m}|_\infty = 0$, *where* $C = (u_0, \varphi_1)/|\varphi_1|_{1+1/m}^{1+1/m}$.

(iii) *If $\lambda_1 = 1$, $a > 0$ then* $\lim_{t\to\infty} |u(t, u_0)|_\infty = 0$.

(iv) *If $\lambda_1 < 1$, $a = 0$, $u_0 \not\equiv 0$ then $t_{\max}(u_0) < \infty$, i.e. any solution $u(t, u_0)$ blows up in a finite time in L^∞-norm.*

Some comments to the proof of Theorem 3.1 will be given later.

Now we shall treat the case $\lambda_1 < 1$ and $a > 0$. In order to describe our result let us define

$$d = \inf_{w \in {}^+H^1{}_0} \Big(\sup_{0 \leqq \lambda < \infty} J(\lambda w) \Big).$$

In [6] we have demonstrated that

$$0 < d = \frac{m-1}{2(m+1)}\, a^{2m/(m-1)} \inf_{w \in Q} \left(\frac{|w|_{1+1/m}}{(|w|_2^2 - \|w\|^2)^{1/2}} \right)^{2(m+1)/(m-1)} < \infty, \tag{3.2}$$

where $Q = \{w \in {}^+H_0^1\colon |w|_2^2 > \|w\|^2\}$, and we can introduce the stable set W and the unstable set B as follows:

(3.3) $$W = \{w \in {}^+H_0^1 \colon J(w) < d \text{ and } K(w) > 0\} \cup \{0\}\,,$$

(3.4) $$B = \{w \in {}^+H_0^1 \colon J(w) < d \text{ and } K(w) < 0\}\,.$$

Theorem 3.2. *Assume that D and u_0 satisfy* (H), $m = p > 1$, $\lambda_1 < 1$ *and* $a > 0$.

(i) *If $u_0^m \in W$ then there exists a constant $C = C(u_0) \geqq 0$ such that*

(3.5) $$|u(t, u_0)|_\infty \leqq C \exp(-a(1 - v)\, t)\,, \quad 0 \leqq t < \infty\,,$$

where $v = (J(u_0^m)/d)^{(m-1)/2m}$.

(ii) *If $u_0^m \in B$ then $t_{\max}(u_0) < \infty$, i.e. the solution $u(t, u_0)$ blows up in a finite time. Moreover,*

(3.6) $$d = \min_{v \in E} J(v^m)$$

hence E *is nonempty.*

Proof of Theorem 3.1. The assertions (i), (ii) have been proved by Sacks in [18] and (iv) by Galaktionov in [9]. To prove (iii), let us note that there exists no nonnegative nontrivial stationary solution to (3.1). Really, if v were such solution, it would hold

$$-\|v^m\|^2 + |v^m|_2^2 = a|v|_{m+1}^{m+1} > 0\,,$$

which is a contradiction to the fact that $\lambda_1 = 1$, i.e. E is empty. The assertion (ii) yields by a comparison argument that $u(t, u_0)$ remains bounded in L^∞-norm as $t \to \infty$, so the semi-orbit $\{u^m(t, u_0) \colon t \geqq 0\}$ is relatively compact in $C(\bar{D})$, $\omega(u_0)$ is nonempty and $\omega(u_0) \subset E \cup \{0\} = \{0\}$ (see [10, Theorem 2.5]), hence the conclusion.

Before we give the proof of Theorem 3.2 let us introduce two lemmas.

Lemma 3.3. *Let $|u(\cdot, u_0)|_{m+1}$ be bounded on $[0, t_{\max})$. Then $t_{\max} = \infty$ and*

(3.7) $$|u(t, u_0)|_\infty \leqq C(|u_0|_\infty, \sup_{0 \leqq t < \infty} |u(t, u_0)|_{m+1})\,, \quad 0 \leqq t < \infty\,.$$

Proof of Lemma 3.3. Putting $\varphi = u^r$, $r > m$ in (1.1) and performing standard manipulations we get

(3.8) $$\frac{\mathrm{d}}{\mathrm{d}t} |u|_{r+1}^{r+1} + \frac{4mr(r+1)}{(r+m)^2} \|u^{(m+r)/2}\|^2 = (r+1)\, |u|_{r+m}^{r+m} - a(r+1)\, |u|_{r+1}^{r+1}\,.$$

The right hand side of (3.8) may be estimated by the Nirenber-Gagliardo inequality and Young's inequality as follows:

(3.9) $$|u|_{r+m}^{r+m} \leqq C_F^2(\varepsilon\|u^{(m+r)/2}\|^2 + C(\varepsilon)\,(\textstyle\int_D u^{(m+r)(m+1)/2m})^{2m/(m+1)})$$

for $0 < \varepsilon < \infty$. As $m + 1 < (2m)^{-1}(m + r)(m + 1) < r + 1$, (3.8) and (3.9) yield (putting $\varepsilon = 2rm/(m + r)^2 C_F^2$ and computing $C(\varepsilon)$)

(3.10) $$\frac{\mathrm{d}}{\mathrm{d}t} |u|_{r+1}^{r+1} + \frac{2mr(r+1)}{(m+r)^2} \|u^{(m+r)/2}\|^2 \leqq$$
$$\leqq C|u|_{m+1}^{2m(m-1)}(r+1)^{1+N(m-1)/2(m+1)}\, |u|_{r+1}^{r+1}$$

for $0 \leqq t < t_{\max}$ and $r > m$. As $|u|_{m+1}$ is bounded on $[0, t_{\max})$, we can apply Proposition 2.6 to obtain

$$|u(t, u_0)|_\infty \leqq C(|u_0|)_\infty, \sup_{0 \leqq t < t_{\max}} |u(t, u_0)|_{m+1}) \quad \text{for} \quad 0 \leqq t < t_{\max},$$

hence $t_{\max} = \infty$ by (1.2).

Lemma 3.4. *Let $u_0^m \in W$. Then $u^m(t, u_0) \in W$ for $0 \leqq t < t_{\max}$ and*

$$(3.11) \qquad |u(t, u_0)|_{m+1} \exp(a(1 - v) t) \leqq |u_0|_{m+1}$$

for $0 \leqq t < t_{\max}$.

Proof of Lemma 3.4. The fact that the set W is invariant may be proved like in [6] and we omit it here. Now let us suppose that $u^m(t, u_0) \in Q$ (cf. (3.2)). Then according to (1.3), (3.2) and (3.4) we have

$$(3.12) \qquad J(u^m(t)) \leqq J(u_0^m)(m - 1)(a|u(t)|_{m+1}^{m+1})^{2m/(m-1)}/(2d(m + 1)(|u^m(t)|_2^2 - \\ - \|u^m(t)\|^2)^{(m+1)/(m-1)}).$$

As $K(u^m(t)) > 0$, (3.12) yields

$$(3.13) \qquad |u^m(t)|_2^2 - \|u^m(t)\|^2 \leqq va|u(t)|_{m+1}^{m+1} \quad \text{for} \quad 0 \leqq t < t_{\max}.$$

Here we can omit the assumption that $u^m(t) \in Q$ because if it does not hold, (3.13) is satisfied automatically. So, using the estimate (3.13), (1.1) for $\varphi = u^m$ gives the differential inequality

$$\frac{d}{dt} |u(t)|_{m+1}^{m+1} + (m + 1)(1 - v) a|u(t)|_{m+1}^{m+1} \leqq 0,$$

which yields (3.11).

Proof of Theorem 3.2. (i) Set $w = u \exp(a(1 - v) t)$. Then it is not difficult to verify that w satisfies

$$w_t \exp(a(1 - v)(m - 1) t) = \Delta w^m + w^m.$$

Changing the scale to $s = c^{-1}(1 - \exp(ct))$, $c = a(1 - v)(m - 1)$ and putting $v(x, s) = w(x, t(s))$, v satisfies

$$\begin{aligned} v_s &= \Delta v^m + v^m & x \in D, \quad & s \in (0, s_{\max}), \\ v(x, s) &= 0 & x \in \partial D, \quad & s \in (0, s_{\max}), \\ v(x, 0) &= u_0(x) & x \in D. \quad & \end{aligned}$$

As Lemma 3.4 implies $|v(s, u_0)|_{m+1} \leqq |u_0|_{m+1}$ for $0 \leqq s < s_{\max}$, we can apply Lemma 3.3 to obtain that $s_{\max} = \infty$ and

$$|v(s, u_0)|_\infty \leqq C(|u_0|_\infty, \ |u_0|_{m+1}),$$

hence the conclusion.

To prove the assertion (ii) of Theorem 3.2 we note that in a similar way as in the proof of Lemma 3.4 we may obtain the estimate

$$|u(t, u_0)|_{m+1} \geqq |u_0|_{m+1} \exp(a(1 - v)(m - 1)t/(m + 1))$$

for $0 \leqq t < t_{\max}$ if $u_0^m \in B$. The next lemma completes then the proof of (ii).

Lemma 3.5. *Let the hypotheses of Theorem 3.2 be satisfied. Then there exists no global solution u of Problem (3.1) for which $|u(t, u_0)|_{m+1} \to \infty$ as $t \to \infty$.*

Proof of Lemma 3.5. Following an idea from [16] we proceed by contradiction. Suppose that $t_{\max} = \infty$ and denote

$$M(t) = \int_0^t |u|_{m+1}^{m+1}.$$

Then we have

$$M'(t) = |u_0|_{m+1}^{m+1} + \int_0^t \int_D (u^{m+1})_t =$$

$$= |u_0|_{m+1}^{m+1} + (m + 1) \int_0^t (-\|u^m\|^2 + |u^m|_2^2 - a|u|_{m+1}^{m+1}),$$

and further,

$$M''(t) = (m + 1)(-2J(u^m(t)) + (m + 1)^{-1}(m - 1) a|u(t)|_{m+1}^{m+1}).$$

Now (1.3) yields the inequality

$$(3.14) \qquad MM'' - 2m(m + 1)^{-1} M'^2 \geqq 2m(m + 1)^{-1} |u_0|_{m+1}^{2(m+1)} + 8m(m + 1)^{-1} (\int_0^t \int_D u^{m+1} \int_0^t \int_D (u^{(m+1)/2})_t^2 - (\int_0^t \int_D u^{(m+1)/2} (u^{(m+1)/2})_t)^2) + (m + 1)^{-1}(m - 1) aMM' - 2(m + 1) J(u_0^m) M - 4m(m + 1)^{-1} |u_0|_{m+1}^{m+1} M'.$$

It is not difficult to see that there exists a $t_0 > 0$ such that the right hand side of (3.14) is positive for $t \geqq t_0$, therefore

$$(M^{-\lambda})'' < 0 \quad \text{for} \quad t \geqq t_0 \quad \text{where} \quad \lambda = (m - 1)/(m + 1).$$

Since $M^{-\lambda}$ is decreasing, it must have a root $t_1 > 0$, which is a contradiction.

For the proof of (3.6) we refer to the proof of the analogous result in Theorem 2 of [6].

References

[1] *Alikakos, N. D.:* L^p bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations *4* (1979), 827—868.

[2] *Aronson, D. G., Crandall, M. G., Peletier, L. A.:* Stabilization of solutions of a degenerate nonlinear diffusion problem, Nonlinear Analysis *6* (1982), 1001—1022.

[3] *Ball, J. M.:* Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, Quart. J. Math. Oxford *28* (1977), 473—486.

[4] *Benilan, P., Crandall, M. G.:* The continuous dependence on ϕ of solutions of $u_t - \Delta\phi(u) = 0$, Indiana Univ. Math. J. *30* (1978), 161—177.

[5] *Fila, M., Filo, J.:* Stabilization of solutions of certain one-dimensional degenerate diffusion equations, Mathematica Slovaca *37* (1987), 217—229.

[6] *Fila, M., Filo, J.:* A blow-up result for nonlinear diffusion equations, Mathematica Slovaca *39* (1989), 331—346.
[7] *Filo, J.:* On solutions of perturbed fast diffusion equation, Aplikace Matematiky *32* (1987), 364—380.
[8] *Friedman, A.:* Partial Differential Equations, Holt, Rinehart and Winston, New York 1969.
[9] *Galaktionov, V. A.:* A boundary value problem for the nonlinear parabolic equation $u_t = \Delta u^{\alpha+1} + u^\beta$, Differential Equations *17* (1981), 836—842 (Russian).
[10] *Langlais, M., Phillips, D.:* Stabilization of solutions of nonlinear and degenerate evolution equations, Nonlinear Analysis *9* (1985), 321—333.
[11] *Levine, H. A., Sacks, P. E.:* Some existence and nonexistence theorems for solutions of degenerate parabolic equations, J. Differential Equations *52* (1984), 135—161.
[12] *Lions, P. L.:* Asymptotic behavior of some nonlinear heat equations, Physica D *5* (1982), 293—306.
[13] *Nakao, M.:* Existence, nonexistence and some asymptotic behavior of global solutions of a nonlinear degenerate parabolic equation, Math. Rep., College Gen. Ed. Kyushu Univ., 1983, 1—21.
[14] *Nakao, M.:* L^p-estimates of solutions of some nonlinear degenerate diffusion equations, J. Math. Soc. Japan *37* (1985), 41—63.
[15] *Ni, W. M., Sacks, P. E., Tavantzis, J.:* On the asymptotic behavior of solutions of certain quasilinear parabolic equations, J. Differential Equations *54* (1984), 97—120.
[16] *Payne, L. E., Sattinger, D. H.:* Saddle points and instability of nonlinear hyperbolic equations, Israel J. Math. *22* (1975), 273—303.
[17] *Sacks, P. E.:* Continuity of solutions of a singular parabolic equation, Nonlinear Analysis *7* (1983), 387—409.
[18] *Sacks, P. E.:* Global behavior for a class of nonlinear evolution equations, SIAM J. Math. Anal. *16* (1985).
[19] *Tsutsumi, M.:* Existence and nonexistence of global solutions for nonlinear parabolic equations, Publ. R.I.M.S., Kyoto Univ. *8* (1972/73), 211—229.

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